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SCRIPTORES LOGARITHMICI

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A COLLECTION

SEVERAL CURIOUS TRACTS

ON THE

NATURE AND CONSTRUCTION

OF

MENTIONED IN

DR. HUTTON'S HISTORICAL INTRODUCTION TO HIS NEW EDITION OF
SHEPHERD'S MATHEMATICAL TABLES:

TOGETHER WITH

SOME TRACTS ON THE BINOMIAL THEOREM AND OTHER SUBJECTS CON-
NECTED WITH THE DOCTRINE OF LOGARITHMS.

VOLUME I

LONDON

PRINTED BY J. DAVIS

AND SOLD BY E. WHITE AND SON, STATIONERS

1755

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DR. HUTTON'S HISTORICAL INTRODUCTION TO HIS NEW EDITION OF

SHERWIN'S LOGARITHMICAL TABLES;

CONTAINING A HISTORY OF THE ART OF LOGARITHMS, AND OF THE

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AND SOLD BY A. MILLAR AND SON, WEST-CHURCH.

INDEXED.

P R E F A C E.

THE present collection of tracts relating to logarithms, and intitled "*Scriptores Logarithmici*," &c, was owing to the publication of Dr. Hutton's very curious historical introduction to his new edition of Sherwin's Mathematical Tables in the year 1785. That introduction (which I have here reprinted at the beginning of this collection, together with Dr. Hutton's preface to that edition of the Mathematical Tables) excited my admiration in a high degree; not only because it afforded a striking proof of the unwearied industry and very great extent of reading, as well as of the uncommon mathematical skill and judgment, of its learned author, but also on account of its great usefulness, in exhibiting, to persons of an inferior degree of knowledge in these sciences, a full and clear view of all the different steps that had been taken by mathematicians of all countries, both in ancient and modern times, towards the cultivation and improvement of trigonometry and the doctrine of logarithms, which are two of the most useful and important branches of the mathematicks. And, as most of the tracts that have been written on the latter of these subjects, namely, the doctrine of logarithms, since the publication of Briggs's *Arithmetica Logarithmica* and *Trigonometria Britannica*, and that are mentioned in Dr. Hutton's introduction as deserving notice, are but short,

I thought it would be by no means impracticable to collect them all together into one book; and I did not doubt that such a collection of them, ranged in the same order in which they had been originally published, and in which, for the most part, they had been mentioned in the aforesaid historical introduction, would prove an acceptable present to all the lovers of these sciences. With this view I undertook the present publication, in which I have omitted the two tracts of Briggs above mentioned, namely, the *Arithmetica Logarithmica* and the *Trigonometria Britannica*, on account of their length (they being small folio volumes), and the tract of Baron Napier, called *Canon mirificus Logarithmorum* (though it is but a short one), because it was published before the said two treatises of Briggs. But these omissions are pretty well supplied by the ample accounts which Dr. Hutton has given of the contents of these valuable pieces in his aforesaid historical introduction, which is here reprinted. The present collection begins with the two tracts published on this subject by the famous John Kepler in the years 1624 and 1625; in which (as Dr. Hutton has justly observed in his above-mentioned historical introduction) that very elegant and accurate geometrician has treated of logarithms according to the true and genuine idea of them, as being *measures of ratios, or proportions*, and has delivered his whole doctrine concerning them in a very full and scientific manner.

When this collection of tracts was first undertaken, I had thought that they might all have been comprised in one volume, quarto. But, as some of the tracts were written in a very obscure style and manner, and seemed much to stand in need of explanation;—and, as they were also founded on a supposition of the truth of the famous Binomial Theorem, both in the case of integral powers and in the case of roots, which theorem but few mathematical authors have attempted to demonstrate;—I resolved to endeavour to supply these defects as well as I was able, partly by adopting

ing and inserting a demonstration of the said binomial theorem, both in the case of integral powers and in the case of roots, that had been published by the late very learned Mr. John Landen of Walton near Peterborough in Northamptonshire, in the years 1758 and 1764, and by adopting some other hints given by other authors on the same subject, and partly by notes and tracts written by myself with a view to answer the same purpose, and on which (as the reader will easily perceive) I have bestowed no small share of attention and labour. And by these additional and explanatory tracts, together with a few other mathematical tracts of my own composition, on subjects that have for the most part a remarkable connection with the binomial theorem, this collection has been swelled to such a size that it cannot be concluded in less than three volumes, quarto; of which the two first are now presented to the publick.

The tracts in these two volumes that are of my own composition are as follows.

The first of them is intituled, “ *Remarks on the two infinite Serieses* $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \mathcal{E}c$ and $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \mathcal{E}c$, which were found by Mr. Nicholas Mercator and Dr. John Wallis in the foregoing Tracts for the Purpose of squaring the Hyperbolick Spaces *BIru* and *FIru*.”

This is a very long tract, extending from page 233 of the first volume of this collection to page 344. My design in writing it was as follows.

Mr. Mercator, after explaining in the first 13 propositions of his *Logarithmotechnia*, the methods by which he advises the computers of a table of logarithms to proceed in their calculations (which methods are purely arithmetical and derived from the nature of ratios and of

the powers of numbers, without any mention of the hyperbola, or of any other geometrical figure whatsoever), concludes his discourse with some curious propositions concerning the hyperbola, from which he derives a method of squaring an asymptotick area of a rectangular hyperbola by means of the infinite series $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$, in which the letter A is supposed to denote a portion of one of the asymptotes of an hyperbola contiguous to the central square of the hyperbola (or the square formed by the two asymptotes of it, and by two lines drawn from its vertex parallel to the said asymptotes respectively), and to be less than the side of the said square (which is supposed to be denoted by 1), and consequently the several powers of A , to wit, $A, A^2, A^3, A^4, A^5, A^6, \&c$, will form a series of decreasing quantities. For he demonstrates that, upon these suppositions, the said infinite series $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$ will be equal to the asymptotick area corresponding to the line A , or of which the line A in the asymptote forms the base. And, as this asymptotick area is the measure of the ratio of $1 + A$ to 1, it is evident that the said series $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$ must consequently be the measure of the said ratio of $1 + A$ to 1, or, if computed in numbers, must be its logarithm. And thus, in these latter propositions of his said tract, Mr. Mercator exhibits a new method of computing logarithms totally different from the former methods, which are the subject of the first 13 propositions of his book, and which he seems principally to recommend.

But, though Mr. Mercator himself, at the time of publishing his *Logarithmotechnia*, seems most to value his first method of computing logarithms, which is contained in the first 13 propositions of his said tract, the generality of mathematicians have given the preference to his second method of computing them by means of the infinite series $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$, which he has
given

given us for the quadrature of the hyperbola. And, notwithstanding the many learned tracts that have been written on the subject since his time by the most eminent mathematicians, I do not know any one method of computing logarithms that deserves to be preferred to it, though Dr. Wallis's series $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \&c \text{ ad infinitum}$ (which exhibits the logarithm of the ratio of 1 to $1 - A$, and which the Doctor found out in consequence of the perusal of Mr. Mercator's *Logarithmotechnia* and after an attentive examination of the aforesaid series $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c \text{ ad infinitum}$ invented by Mercator) may be considered as *equally* convenient for this purpose.

But both Mercator's series, $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c \text{ ad infinitum}$, and Dr. Wallis's series $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \&c \text{ ad infinitum}$, are, by their respective inventors, derived from the contemplation of the hyperbola, and are given by them rather as methods of squaring the hyperbola than as methods of computing logarithms, though they are also capable of being applied to this latter purpose with great success. This manner of deriving them, Dr. Halley thought, was a defect in the foundation of them, when considered as methods of computing logarithms; and therefore he endeavoured to supply this defect by giving other investigations of them, which were founded on the mere abstract principles of arithmetick and the nature of ratios, and particularly on Sir Isaac Newton's famous theorem for raising the powers, either integral or fractional, of a binomial quantity. And he accordingly published such investigations of them in the Philosophical Transactions of the royal Society of London in the year 1695, in a tract which he intitled "*A most compendious and facile Method for constructing Logarithms, exemplified and demonstrated from the Nature of Numbers, without any*

any Regard to the Hyperbola, with a speedy Method for finding the Number from the Logarithm given."

But this tract by no means answers to the title Dr. Halley has given it, of *a compendious and facile method for constructing logarithms*. It is indeed *compendious*: but *facile* it most certainly is not. For, if ever there was a piece of writing that was peculiarly fit to serve as an example of the danger of an author's falling into obscurity by aiming at too much brevity, which Horace has mentioned in his judicious admonition on the subject, *brevis esse laboro; obscurus fio*; it is this tract of Dr. Halley: it being so uncommonly obscure, that I scarce ever knew a mathematician that had read it (I except not the most learned and acute) that did not confess that he had found the greatest difficulty in comprehending it. And, for my part, I cannot understand some parts of it even now, after having employed so much time and pains upon the subject. And this was the consideration that induced me to write the "*Remarks on Mr. Mercator's and Dr. Wallis's Serieses*," which constitute the seventh tract of the first volume of this collection. I have endeavoured to do that *intelligibly* which Dr. Halley has done in a manner that is *almost unintelligible*; that is, I have endeavoured to give such investigations of those two logarithmick serieses, invented by Mercator and Dr. Wallis, as Dr. Halley either has given, or has attempted to give, of them (I know not which), in the aforesaid tract, namely, investigations founded entirely on arithmetical principles, and on Sir Isaac Newton's binomial theorem, without any mention of the hyperbola, or any other geometrical figure; and also without having recourse to the doctrine of indivisibles, or infinitely small quantities, or the doctrine of fluxions, or, in general, to the arithmetick of infinites in any of its modifications. These investigations I have drawn up with great care, and have set them forth in different forms and styles, namely, first at great length, and afterwards more concisely, in order to adapt them to different sorts of readers, and to render them as clear and easy as the nature of the subject (which

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is a very abstruse one) will admit: and I have subjoined no less than nine examples of the computation of logarithms by each of the said two serieses, in order to make the manner of applying them to that purpose familiar to the reader, being of opinion (as I before observed) that no better method of computing logarithms has yet been found out, or need be sought for, than the judicious application of these two serieses.

From the computations contained in these examples I have derived the logarithms of the ratios of the first 22 numbers, to wit, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, and 23, to 1, both in Napier's system and in Briggs's, to eighteen places of figures; which calculations are, I believe, enough to make the industrious reader that shall patiently go through them all, become very well acquainted with this method of computing logarithms.

The next tract of my composition in this collection is intitled "*An Appendix to the foregoing Remarks on the two Logarithmick Serieses invented by Mr. Mercator and Dr. Wallis,*" and extends from page 345 to page 383, and concludes the first volume of these tracts.

This tract also has a reference to Dr. Halley's *compendious and facile method*, &c, above mentioned. For his investigations of the two anti-logarithmick serieses which he gives us in that tract, to wit, the serieses $1 + L + \frac{L^2}{2} + \frac{L^3}{3} + \frac{L^4}{4} + \frac{L^5}{5} + \frac{L^6}{6} + \&c \text{ ad infinitum}$, and $1 - L + \frac{L^2}{2} - \frac{L^3}{3} + \frac{L^4}{4} - \frac{L^5}{5} + \frac{L^6}{6} - \&c \text{ ad infinitum}$, seemed to me almost as obscure as those of the two logarithmick serieses invented by Mercator and Wallis. I therefore resolved to give new investigations of these two serieses derived from the pure principles of Arithmetick and from the Binomial Theorem, in the same manner

manner as the investigations given of them by Dr. Halley himself in the aforesaid tract. These investigations are also drawn up with great care and pains, and set forth at considerable length, in order to make them as clear and as easy as possible. And, after the investigations of these two anti-logarithmick serieses in this appendix, I have given some examples of the application of the said serieses to the computation of the numbers corresponding to given logarithms, in order to illustrate the method of making such computations, and render it familiar to the reader. There is also in this tract, amongst other curious matters, a computation of the terms of the remarkable ratio which Mr. Cotes has called the *ratio modularis*, carried to 24 places of decimal fractions. These terms are 2.718,281,828,459,045,235,360,274, &c and 1, or 1 and 0.367, 879,441,171,442,321,595,522, &c, of which numbers all but the three last figures may be depended on as exact.

The third tract of my composition in this collection is in the second volume, and consists of notes on those parts of Dr. Halley's above-mentioned Discourse on Logarithms, which I had found extremely difficult, but yet had, after repeated perusals of them, been able to comprehend. By the help of the two tracts above mentioned, to wit, the *Remarks on the two Logarithmick Serieses invented by Mr. Mercator and Dr. Wallis*, and the *Appendix to those Remarks*, and of these explanatory notes on Dr. Halley's tract contained in the second volume of this collection, my readers will, I hope, be enabled to perceive the truth of all the propositions advanced in that celebrated discourse. These notes take up 30 pages, to wit, pages 92, 93, 94, &c 122.

The fourth tract of my composition in this collection has also some relation to Dr. Halley's above-mentioned discourse. For it contains a direct method of computing logarithms founded on the principles which Dr. Halley recommends, namely, the pure prin-
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ciples of arithmetick, without any reference to the hyperbola, or any other geometrical figure, and likewise without having recourse to the method of indivisibles, or to the doctrine of fluxions, or to the arithmetick of infinites in any of its forms: and, further, it is grounded (as well as Dr. Halley's investigations in the above-mentioned discourse) on Sir Isaac Newton's binomial theorem, though by means of a different application of it from that made by Dr. Halley. And for these reasons I have intitled it "*An Appendix to the foregoing Discourse of Dr. Edmund Halley concerning Logarithms.*" This tract, or *Appendix*, &c, takes up pages 123, 124, 125, &c. 152.

Several of the remaining tracts in the second volume of this collection are allotted to the demonstration of the Binomial Theorem itself; that theorem being so very closely connected with the subject of logarithms as to be the foundation of the best methods of computing them, and having been itself but imperfectly demonstrated in most of the books of mathematicks that have treated of it. For example, Sir Isaac Newton (who invented it, or, at least, extended it to the case of roots and other fractional powers) has never attempted to demonstrate it, even in the case of integral powers: and Professor Saunderson, in his Algebra (though written for the use of beginners), takes it also for granted, or gives no proof of it but from induction: and Dr. Halley (though he has made so much use of it in the above-mentioned discourse) does not give us the least proof of its truth. And the same omission is to be complained of in the works of many other eminent writers on mathematicks. I therefore thought that, in a collection of tracts relating to logarithms (of which that theorem is such an important foundation) it would be proper to supply this defect. And this has been the occasion of my composing many of the following tracts in the second volume of this collection, as well as of my inserting in it Mr. Landen's demonstration of the said theorem.

The fifth tract of my composition in this collection is a demonstration of the binomial theorem in the case of integral powers, which is nearly the same with one given by Mr. John Stewart, of Aberdeen, in the sixth section of his Commentary on Sir Isaac Newton's learned little tract, called *Analysís per æquationes numero terminorum infinitas*, published at the end of his commentary on Sir Isaac Newton's *Quadrature of Curves*. But the demonstration of this theorem given in the present tract is much fuller than that in Mr. Stewart's book. And in the latter part of the present tract the demonstration is extended from $\overline{a + b}^m$, or the m th power of the binomial quantity $a + b$, to $\overline{a - b}^m$, or the m th power of the residual quantity $a - b$; the letter m denoting any whole number whatsoever in both cases.

This fifth tract takes up pages 153, 154, 155, &c 169.

The next tract in the second volume of this collection is the demonstration given of the binomial theorem by Mr. John Landen, of Walton near Peterborough in Northamptonshire, which has been already mentioned. This demonstration extends to the case of the fractional powers of the binomial quantity $1 + x$, as well as to that of its integral powers, being a general demonstration of the theorem in the case of the quantity $\overline{1 + x}^{\frac{m}{n}}$, when m and n represent any whole numbers whatsoever. It is extracted partly from Mr. Landen's *Discourse concerning the Residual Analysis*, published in the year 1758, and partly from another tract of Mr. Landen, on the same subject, published in the year 1764, and intitled, *The Residual Analysis*, to which the former tract, called *A Discourse concerning the Residual Analysis*, had been only an introduction. This demonstration of Mr. Landen (though I believe it to be just) is extremely intricate and perplexed in the algebraick operations that are necessary to be performed in it, and is, upon the whole, exceeding difficult to understand: for which reason I have subjoined to it another tract of my own composition, in which I have used my best

endeavours to explain it. Mr. Landen's demonstration extends through pages 170, 171, 172, 173, 174, 175; and the subsequent explanation of it extends through pages 176, 177, 178, &c.

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The next tract in the second volume of this collection is also of my composition, and relates to the same subject of the binomial theorem. It is a very full discourse on the said theorem in the case of fractional powers, containing, first, various illustrations and exemplifications of it, that are calculated to make the reader become familiarly acquainted with it, and ready in the application of it to any proposed particular cases; and, afterwards, exhibiting two different methods of demonstrating it, of which the latter and more compleat was suggested to me by a very close and attentive perusal of Mr. Landen's demonstration above mentioned, to which it bears a considerable resemblance. It does not, however, seem to me to be quite the same with Mr. Landen's, though I acknowledge it to have been suggested by it: and it is at least more fully and clearly expressed than Mr. Landen's, and vastly easier to understand; and will, I believe, if properly attended to, give the readers of it all the satisfaction that can reasonably be expected on so subtle and abstruse a subject. This tract is intitled, "*A Discourse concerning the Binomial Theorem in the case of fractional Powers, or Powers of which the Indexes are Fractions.*" It extends from page 194 to page

344.

The next tract in the second volume of this collection is also written by myself. It is intitled, "*A Discourse concerning Sir Isaac Newton's Residual Theorem, or Theorem for raising the Powers of the Residual Quantity $1 - x$, in the case of fractional Powers, or Powers of which the Indexes are Fractions.*"

The object of this tract is to transfer all the conclusions that have been demonstrated in the foregoing long discourse concerning

the quantity $\sqrt[n]{1+x}$, or the $\frac{m}{n}$ th power of the binomial quantity $1+x$, to the quantity $\sqrt[n]{1-x}$, or the $\frac{m}{n}$ th power of the residual quantity $1-x$; and to do this without repeating over again the long trains of reasoning by which those conclusions had been obtained in the said former discourse. This tract extends from page 345 to page 378.

All the remaining tracts in the second volume of this collection are likewise of my composition. The subjects of them are as follows.

The tract which comes immediately after the last-mentioned Discourse concerning Sir Isaac Newton's Residual Theorem, is a very curious application of the binomial and residual theorems to the finding of the lesser of the two roots of the cubick equation $qy - y^3 = r$ by means of one of the expressions given by Cardan's first rule for the root of the equation $qy + y^3 = r$, or $qx + x^3 = r$. But this application, or extension of Cardan's rule, can take place only when the absolute term r of the equation $qy - y^3 = r$ is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or when $\frac{rr}{2}$ is less than $\frac{q^3}{27}$, or when $\frac{rr}{4}$ is less than $\frac{q^3}{54}$.

And, as the root of the cubick equation $x^3 - qx = r$ may always be derived from either of the roots of the cubick equation $qy - y^3 = r$ by means of a quadratick equation, when the absolute term r is of any magnitude less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$ or when $\frac{rr}{4}$ is of any magnitude less than $\frac{q^3}{27}$, it follows that, when r is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{54}$, the root of the cubick equation $x^3 - qx = r$ may be deduced from Cardan's expression for the root of the cubick equation $x^3 + qx = r$, or $y^3 + qy = r$, or $qy + y^3 = r$, by the mediation

diation of the lesser root of the cubick equation $qy - y^3 = r$, which may be derived from Cardan's said expression for the root of the equation $qy + y^3 = r$ by the method described in this tract.

The matters contained in this tract are in their nature extremely subtle and difficult; which made it necessary for me to be very full and particular in the explanation of them. And this is the cause of the length to which this discourse is carried, which is greater than I could at first have imagined. It reaches from page 378 to page 440.

The next tract in the second volume of this collection is on the same subject as the last. It is an application of the binomial and residual theorems to the purpose of extending one of the expressions given by Cardan's second rule for the root of the cubick equation $x^3 - qx = r$, in the first case of that equation, or when the absolute term r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or when $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, to the resolution of the second case of the same equation, in which the absolute term r is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{27}$; provided that r (though less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$) be greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or that $\frac{rr}{4}$ (though less than $\frac{q^3}{27}$) be greater than $\frac{q^3}{54}$.

This tract is full of abstruse and subtle matter, as well as the foregoing tract, and has been therefore treated in a very full and explanatory manner, which is the cause of its extending through no less than 135 pages, or from page 440 to page 575. It had formerly been published in the Philosophical Transactions for the year 1778. But the subject is here explained in a much fuller and better manner than before.

By means of the expressions obtained in these two tracts we may always

always find the root of the cubick equation $x^3 - qx = r$ in the second case of it, or that in which the absolute term r is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{27}$ (which has obtained the name of *the irreducible case* amongst many of the writers on Algebra) by the help of one or other of Cardan's two rules for the resolution of the equations $x^3 + qx = r$ and $x^3 - qx = r$, to wit, by the help of the former of these rules, if r is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{54}$, and by the help of the latter of them if r is greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, though less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{54}$, though less than $\frac{q^3}{27}$.

These two tracts enable us to do what is equivalent to what is called by Algebräists *finding the cube-roots of two impossible binomial quantities*. But they enable us to do this by means of reasonings conducted in an intelligible manner, and without any mention, or supposition, of either impossible or negative quantities; which, I hope, will be a recommendation of them to many of my readers, and be considered as an excuse for their extraordinary length and difficulty.

The next tract in the second volume of this collection is a conjecture concerning the method by which Cardan's two rules for resolving the cubick equations $x^3 + qx = r$ and $x^3 - qx = r$ might have been found out by *Scipio Ferreus of Bononia*, and *Nicholas Tartalea*, or whoever else were the first inventors of them.

This tract has no connection either with logarithms or the binomial theorem. But, as the two celebrated rules of Cardan to which it relates, had been so much the object of the reader's attention in the two preceeding tracts, I thought it might be agreeable to him to see a probable conjecture concerning the invention of them, more especially as it takes up only eight pages, to wit, pages 579, 586.

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This tract was formerly printed in the Philosophical Transactions for the year 1780. It is now reprinted without any additions.

The next tract in the second volume of this collection, and with which the volume concludes, contains only five pages, to wit, page 587, 588, 589, 590, and 591. It is an improvement upon a hint given by Professor Saunderson in his Algebra, relating to the old subject of the Binomial Theorem, and shews how a mathematician, that had been in pursuit of a general rule for deriving the numeral co-efficients of the terms of any integral power of a binomial quantity, $a + b$, one from another in terms of the index m (as Dr. Wallis tells us he himself had been, but without success) might have been led to suspect that they might be generated by the continual multiplication of 1 into the following series of fractions, to wit, $\frac{m}{1}$, $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, $\frac{m-6}{7}$, &c; after which suspicion he would naturally have tried these fractions in a variety of instances, or with a variety of different values of the index m ; and, having found that they always produced the true numeral co-efficients of the second, third, fourth, and other following terms of the m th power of $a + b$, he would then, of course, have concluded, by induction from the instances in which he had found this method of generating the said co-efficients to be true, that it must be true in all other cases whatsoever, or when the index m of the power to which the binomial quantity $a + b$ was to be raised, was equal to any other whole number than those in which he had tried it.

And, when this conclusion had thus been formed by an induction grounded on a great number of trials, the mathematician might have confirmed the truth of it by a general demonstration extending it from the cases in which it had been tried and found to be true, to the cases of all higher powers whatsoever, in the man-

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ner described in the former tract contained in the second volume of this collection, in pages 153, 154, 155, &c. 169.

I think therefore that these two tracts may be considered as forming a compleat and satisfactory investigation and demonstration of the Binomial and Residual Theorems in the case of integral powers: after which the reader may proceed to the two tracts above described, which extend from page 194 to page 378, in which he will find pretty full, and, I hope, satisfactory demonstrations of the truth of these important theorems; first, in the case of roots, and secondly, in the case of the powers of roots, and consequently in the cases of all fractional powers whatsoever.

INNER-TEMPLE, Aug. 16, 1791.

FRANCIS MASERES.

A
T A B L E
OF THE
C O N T E N T S

OF THE
F I R S T V O L U M E;

EXCLUSIVE OF DR. HUTTON'S HISTORICAL INTRODUCTION.

I.

JOHANNIS KEPLERI, Imperatoris Cæsaris Ferdinandi Secundi Mathematici, Chilias Logarithmorum, ad totidem numeros rotundos; præmissâ demonstratione legitimâ ortûs logarithmorum, eorûmque usûs: quibus nova traditur arithmetica, seu compendium, quo, post numerorum notitiam, nullum nec admirabilius, nec utilius, solvendi pleraque problemata calculatoria, præsertim in doctrinâ triangulorum, citrà multiplicationis, divisionis, radicûmque extractionis, in numeris prolixis, labores molestissimos. Ad illustrissimum principem et dominum, Dominum Philippum, Landgravium Hassiæ, &c; cum privilegio auctoris Cæsareo.

N. B. Prima hujus tractatûs editio impressa fuit Marpurgi, et excusa typis Casparis Chemlini, anno Domini M,DC,XXIV.

In pages 1, 2, 3, 4, &c — — — 92.

II.

Johannis Kepleri, Imperatoris Cæsaris Ferdinandi Secundi Mathematici, Supplementum Chiliadis Logarithmorum; continens præcepta de eorum usu. Ad illustrissimum principem et dominum, Dominum Philippum, Landgravium Hassiæ, &c.

N. B. Prima hujus tractatûs editio impressa fuit Marpurgi, et excusa typis Casparis Chemlini, anno Domini M,DC,XXV.

In pages 93, 94, 95, &c — — — 166.

Vol. I.

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III. Loga.

III.

Logarithmotechnia: five methodus construendi logarithmos nova, accurata, et facilis; scripto antehac communicata, anno, scilicet, 1667, nonis Augusti. Cui nunc accedit vera quadratura hyperbolæ, et inventio summæ logarithmorum.

Auctore Nicolao Mercatore, Holsato, è Societate Regiâ Londinensi.

N. B. Prima hujus tractatûs editio impressa fuit Londini, in Transactionibus Philosophicis Societatis Regiæ Londinensis, anno Domini M,DC,LXVIII.

In pages 167, 168, 169, 170, 171, — — — — 196.

IV.

Michaelis Angeli Riccii Exercitatio Geometrica de Maximis et Minimis.

N. B. Hic tractatus adjunctus fuit præcedenti tractatui Nicolai Mercatoris, cui titulus *Logarithmotechnia*, et cum eo impressa in Transactionibus Philosophicis Societatis Regiæ Londinensis, anno Domini M,DC,LXVIII, ob argumenti præstantiam et exemplarium raritatem.

In pages 197, &c — — — 212.

V.

An Extract from the third Volume of the Philosophical Transactions, published on the 13th day of April, 1668, by Mr. Henry Oldenburgh, at that time Secretary of the Royal Society; containing a method of squaring the hyperbola by an infinite series of rational numbers; together with its demonstration. By the Right Honourable the Lord Viscount Brouncker.

In pages 213, 214, 215, 216, 217, and 218.

VI.

Another Extract from the Philosophical Transactions, Number XXXVIII, published on the 17th day of August, 1668; containing, first, an account of Mr. Nicholas Mercator's tract on logarithms, called *Logarithmotechnia*, with another infinite series for the quadrature of the hyperbola, by Dr. John Wallis, Savilian Professor of Geometry in the University of Oxford, in a letter to the Lord Viscount Brouncker; and, 2dly, a method of finding the sums of logarithms, by the said Dr. Wallis, in another letter to the same learned lord; and, 3dly, an illustration of the said Mr. Nicholas Mercator's tract aforesaid, called *Logarithmotechnia*, by the said Mr. Mercator himself.

In pages 219, &c — — — 232.

VII.

Remarks on the two infinite serieses $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$ and $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \&c$, which were found by Mr. Nicholas Mercator and Dr. John Wallis, in the foregoing tracts, for the purpose of squaring the hyperbolick spaces *BI ru* and *FH ru*.

By Francis Maferes, Esq. F. R. S. Curfitor Baron of the Court of Exchequer.

In pages 233, &c — — — 344.

VIII.

An Appendix to the foregoing remarks on the two logarithmick serieses of Mr. Nicholas Mercator and Dr. John Wallis: containing investigations of two other infinite serieses, which were published by Dr. Edmund Halley, and which are related to, and derived from, the two former serieses; and by which we are enabled, when the value of either of the two former serieses is given, to discover the ratio to which it belongs, or of which it is the logarithm.

By Francis Maferes, Esq. F. R. S. Curfitor Baron of the Court of Exchequer.

In pages 345, 346, 347, &c — — — 383.

ERRATA IN VOL. I.

In page 202, in the figure to the first lemma, instead of the letter B at the end of the right hand line of the figure, insert the letter E.

In the same page, the first line of the second lemma, instead of "*dico, $AC^2 \times CB^3$,*" read "*dico, si $AC^2 \times CB^3$.*"

In page 208, line 6 from the bottom, instead of the words, "*ducto in altero segmentum A,*" read "*ducto in alterum segmentum A.*"

In page 209, line 10, instead of "*dignitatem,*" read "*dignitatum.*"

In page 210, line 19, instead of "*secet E,*" read "*secet in E,*" and instead of "*productum,*" read "*productam.*"

In page 224, line 6, instead of "*accomodari,*" read "*accommodari.*"

In page 229, line 14th from the bottom, instead of "*L 99,*" read "*L 49.*"

In page 231, in line 17th and in line 14th from the bottom, instead of "*infinitissimâ,*" read "*infinitesimâ.*"

In page 307, line 10, instead of "*will be the series,*" read "*will be to the series.*"

DR. HUTTON'S PREFACE

TO HIS

MATHEMATICAL TABLES.

THE very ample introduction, prefixed to the following collection of Mathematical Tables, has superseded the necessity of using many words here by way of preface, and has left me little more to mention than the necessity and occasion of this work, with some account of the contents and mode of execution.

The undertaking was occasioned by the extreme incorrectness of the 5th, or last, edition of Sherwin's once very useful book of tables. Finding, as well from the report of others, as from my own experience, that that edition (to say nothing of the very improper alteration in the form of the table of sines, tangents, and secants) was so very incorrectly printed, the errors being multiplied beyond all tolerable bounds, and no dependence to be placed on it for any thing of real practice, I was led to undertake the painful office of preparing a correct edition of another similar work. And I was lucky enough to meet with a bookseller of sufficient spirit to be at the great expence of printing the book, as well as to allow me what I demanded for my trouble in preparing it; which demand, however, was nothing adequate to the great labour attending it, as I was well aware that the profits of the book would not afford him the means of rewarding my pains.

I have, in the first place, therefore, used all the means in my power to render the work correct. I began by collating the 3d, or best edition of Sherwin's tables, with some others of the most perfect works of

the same kind, as Briggs's, Vlacq's, Gardiner's quarto book, &c. by which means I detected many errors in each of them, which had not before been discovered; and of these between twenty and thirty were in the two editions of Gardiner's work, printed at London in 1742, and at Avignon in 1770; the errata of which two books are here printed at the end of our tables. But besides detecting many unknown errors in my copy of the said 3d Edition of Sherwin, which was no more than what I expected, I discovered, with no small surprize, that the last figures in the table of logarithms were not uniformly true to the nearest unit, except in a very few pages at the beginning and end of the table; although Mr. Gardiner, the editor of that edition, had made the table correct in that respect in his own quarto work before-mentioned, which was also printed in the same year 1742 with the said third edition of Sherwin! The errors from this cause amounted to several thousands; and they have continued to run through all the editions of Sherwin ever since that time! But I have here corrected them. Nor have I employed less attention in correcting the press, than in previously correcting the copy; every proof having been several times read over, and compared with the best of the books before-mentioned.

But in giving this edition to the world, I was not satisfied with barely making it correct. I was aware that the materials themselves might be much improved; and I have accordingly enlarged, or otherwise greatly amended them in various respects. Among the improvements of the old materials, may be reckoned the following: namely, in the large table of logarithms, the proportional parts, near the beginning, are more conveniently arranged, being now all placed in the same opening of the book where their corresponding differences occur: The logarithms to sixty-one figures are brought to their proper place in the book, and more conveniently disposed all in one page: The large table of sines, tangents and secants, is more commodiously arranged, and rendered more distinct and convenient for use; the natural sines, tangents, secants and versed sines, being all separated from the others, and placed all together on the left-hand pages, and the logarithmic ones facing them

them on the right-hand pages; the common differences, in both, set between the two columns to which each of them answers; and the versed sines are introduced into their proper place in the same pages with the sines, tangents and secants. Besides these, there are some other alterations in the new tables here given, and the reader will find a number of very important improvements in the description and use of the whole; especially in the arithmetic of logarithms, and in the resolution of plane and spherical triangles, according to the present improved methods of calculation used by the Astronomer Royal, and other persons the most experienced in these matters.

The improvements in the tables by the introduction of new matter, are both great and numerous. The tables numbered 2, 3, and 4 are here added, being an entire new set, with their differences, for finding numbers and logarithms to twenty places. The columns of common differences in the pages of natural sines, &c, are now first introduced: As are also the tables of hyperbolic and logistick logarithms; the logarithmic sines and tangents for every second, in the first two degrees of the quadrant; together with a table of the lengths of arcs; a table to change common and hyperbolic logarithms from the one to the other; &c: the uses and exemplifications of the whole being very amply detailed.

But the greatest alteration of all, is the very extensive and new introduction here given, instead of the former inadequate and heterogeneous one, consisting of about 180 pages of new matter, on a methodical plan, containing the historical account and description of all trigonometrical writings, and the tables relating to that subject, both natural and logarithmic; besides the complete use of our own tables. Inventions are here ascribed to the proper authors, and their methods and improvements described and compared. This historical description will evidently appear to be the result of immense labour and reading. And indeed I have painfully gone over all the books which are here so minutely described; and that description with a detail in some degree adequate to their great merits; especially the works of Napier, Briggs,

Kepler, &c; which was the more necessary, as the writings and methods of those great masters had not been any where properly described and discriminated, although they are in themselves highly curious and important.

These readings and commentaries have been carried on to an extent far beyond what was at first intended. But the tables having been in the press for the space of seven or eight years, I had thereby an opportunity of collecting and examining a still greater number of books; so that I was gradually led on, and my views and plans rendered still more extensive and compleat. This delay, therefore, though in many respects it proved very inconvenient and disagreeable, has at length given the occasion of rendering these commentaries more perfect and satisfactory.

Besides what immediately relates to trigonometrical subjects, the reader will here find many other curious and uncommon articles, relating to the several authors and their discoveries, which have occurred in the course of my reading, and which appeared of too much consequence to be passed over unnoticed, in the analysis of their several compositions. Among these is the discovery of the first author of the binomial theorem, and the differential method, which are due to Mr. Henry Briggs, whose writings are replete with ingenious and original matter, and are well deserving to be more generally known and studied than they have been for some time past.

This long course of examination and description, however, having been carried on for so many years, at different intervals, and interrupted by various avocations, and by business of different kinds, it will be no wonder if this circumstance may have occasioned some inequalities in the style and composition of this history; and for which therefore, should any such appear, it is hoped the occasion will plead an apology.

I N T R O D U C T I O N.

I. OF TRIGONOMETRICAL TABLES, &c.

N ECESSITY, the fruitful mother of most useful inventions, gave birth to the various numerical tables which compose the following work. Astronomy has been cultivated from the earliest ages. The progress of that science, requiring numerous arithmetical computations of the sides and angles of triangles, both plane and spherical, gave rise to trigonometry; for those frequent calculations suggested the necessity of performing them by the property of similar triangles; and for the ready application of this property, it was necessary that certain lines described in and about circles, to a determinate radius, should be computed and disposed in tables. Navigation, and the continually improving accuracy of astronomy, have also occasioned as perpetual an increase in the accuracy and extent of those tables. And this, it is evident, must ever be the case, the improvement of trigonometry uniformly following the improvement of those other useful sciences, for the sake of which it is more especially cultivated.

The ancients performed their trigonometry by means of the chords of arcs, which with the chords of their supplemental arcs, and the constant diameter, formed all species of right-angled triangles. Beginning with the radius, and the arc whose chord is equal to the radius, they divided them both into 60 equal parts, and estimated all other arcs and chords by those parts, namely all arcs by 60ths of that arc, and all chords by 60ths of its chord or the radius: At least this method is as old as the writings of Ptolemy, who used the sexagenary arithmetic for this division of chords and arcs, and for astronomical purposes.—And this, by the bye, shews the reason why the whole circumference is divided into 360, or 6 times 60, equal parts or degrees, the whole circumference being equal to 6 times the first arc whose chord is equal to the radius: Unless perhaps we are to seek for the division of the circle in the number of days in the year; for thus, the ancient year consisting of 360 days, the sun or earth in each day described the 360th part of the orbit; and thence might arise the method of dividing every circle into 360 parts; and the radius being equal to the chord of 60 of those parts, the sexagesimal division both of the radius and of the parts might thence arise. Trigonometry however must have been cultivated long before the time of Ptolemy; and indeed Theon, in his commentary on Ptolemy's *Almagest*,

Almagest, l. 1. ch. 9, mentions a work of the philosopher Hipparchus, written about a century and a half before Christ, consisting of 12 books on the chords of circular arcs; which must have been a treatise on trigonometry. And Menelaus also, in the first century of Christ, wrote 6 books concerning subtenses or chords of arcs. He used the word *nadir* (of an arc), which he defined to be the right line subtending the double of the arc; so that his nadir of an arc, was the double of our sine of the same arc; and therefore whatever he proves of the former, may be applied to the latter, substituting the double sine for the nadir.

The radius has since been decimally divided; but the sexagesimal divisions of the arc have continued in use to this day. Indeed our countrymen Briggs and Gellibrand, having a general dislike to all sexagesimal divisions, made an attempt at some reformation of this custom, by dividing the degrees of the arcs, in their tables, into centesims or hundredth parts, instead of minutes or 60th parts. The same was also recommended by Vieta and others; and a decimal division of the whole quadrant might perhaps soon have followed, had it not been for the tables of Vlacq which came out a little after, to every ten seconds, or 6th part of a minute.—But the compleat reformation would be, to express all arcs by their real lengths, namely in equal parts of the radius decimally divided: of which more in its proper place.

It is not to be doubted that many of the ancients wrote on the subject of trigonometry, as being a necessary part of astronomy; although few of their labours on that branch have come to our knowledge, and still fewer of the writings themselves have been handed down to us.

We are in possession of the three books of Menelaus on spherical trigonometry; but the 6 books are lost which he wrote on chords, being probably a treatise on the construction of trigonometrical tables.

The trigonometry of Menelaus was much improved by Ptolemy (Claudius Ptolemæus) the celebrated philosopher and mathematician. He was born at Pelusium, taught astronomy at Alexandria in Egypt, and died in the year of Christ 147, being the 78th year of his age. In the first book of his Almagest, Ptolemy delivers a table of arcs and chords, with the method of construction. This table contains 3 columns: in the 1st are the arcs to every half degree or 30 minutes; in the second are their chords, expressed in degrees, minutes and seconds, of which degrees the radius contains 60; and in the 3d column are the differences of the chords answering to 1 minute of the arcs, or the 30th part of the differences between the chords in the 2d column. In the construction of this table, among others, Ptolemy shews, for the first time that we know of, this property of any quadrilateral figure inscribed in a circle, namely that the rectangle under the two diagonals, is equal to the sum of the two rectangles under the opposite sides.

This method of computation by the chords, continued in use till about the middle centuries after Christ; when it was changed for that of the sines, which were about that time introduced into trigonometry by the Arabians, who in other respects much improved this science, which they received from the Greeks, introducing among other things the three or four theorems, or axioms, which we use at present as the foundation of our modern trigonometry.

The other great improvements that have been made in this branch, are due to the Europeans. These improvements they have gradually introduced since they received this science from the Arabians. And, although these latter people had long used the Indian, or decimal, scale of arithmetic, it does not appear that they varied from the Greek, or sexagesimal, division of the radius, by which the chords and sines were expressed.

This alteration is said to have been first made by George Purbach, who was so called from his being a native of a place of that name between Austria and Bavaria. He was born in 1423, studied mathematics and astronomy at the university of Vienna, where he was afterwards professor of those sciences, though but for a short time, the learned world quickly suffering a great loss by his immature death, which happened in 1462, at the age of 39 years only. Purbach, besides enriching trigonometry and astronomy with several new tables, theorems, and observations, supposed the radius to be divided into 600,000 equal parts, and computed the sines of the arcs, for every ten minutes, in such equal parts of the radius, by the decimal notation.

This project of Purbach was completed by his disciple, companion, and successor John Muller, or Regiomontanus, who was so called from the place of his nativity, the little town of Mons Regius, or Koningsberg in Franconia, where he was born in the year 1436. Regiomontanus not only extended the sines to every minute, the radius being 600,000, as designed by Purbach, but afterwards disliking that scheme, as evidently imperfect, he computed them likewise to the radius of 1,000,000, for every minute of the quadrant. He also introduced the tangents into trigonometry, the canon of which he called *fecundus*, because of the many and great advantages arising from them. Besides these he enriched trigonometry with many theorems and precepts. Through the benefit of all these improvements, except for the use of logarithms, the trigonometry of Regiomontanus is but little inferior to that of our own time. His treatise on both plane and spherical trigonometry is in 5 books; it was written about the year 1464, and printed in folio at Nuremberg in 1533. And in the fifth book are various problems concerning rectilinear triangles, some of which are resolved by means of algebra: a proof that this science was not wholly unknown in Europe before the treatise of Lucas de Burgo. Regiomontanus died in 1476 at the age of 40 years only, being then at Rome, whither he had been invited by the pope, to assist in the reformation of the calendar, and was suspected to have been poisoned there by the sons of George Trebizonde, in revenge for the death of their father, which was said to have been caused by the grief he felt on account of the criticisms made by Regiomontanus on his translation of Ptolemy's *Almagest*.

Soon after this, several other mathematicians contributed to the improvement of trigonometry, by extending and enlarging the tables, though few of their works have been printed; and particularly John Werner of Nuremberg, who was born in 1468 and died in 1528, and who is said to have written five books on triangles.

About the year 1500, Nicholas Copernicus, the famous modern restorer of the true solar system, wrote a brief treatise on trigonometry both plane and spherical, with the description and construction of the canon of chords, or their halves;

halves, nearly in the manner of Ptolemy; to which is subjoined a canon of sines, with their differences, for every ten minutes of the quadrant, to the radius 100,000. This tract is inserted in the first book of his *Revolutiones Orbium Cælestium*, first printed in folio at Nuremburg 1543. It is remarkable that he does not call these lines *sines*, but *semisses subtensarum*, namely of the double arcs.—Copernicus was born at Thorn in 1473, and died in 1543.

In 1553 was published the *Canon Fæcundus*, or table of tangents, of Erasmus Reinhold, professor of mathematics in the academy of Wurtemberg. He was born at Salfeldt in Upper Saxony, in the year 1511, and died in 1553.

To Franciscus Maurolycus, abbot of Messina in Sicily, we owe the introduction of the *Tabula Benefica*, or canon of secants, which came out about the same time, or little before. But Lansbergius erroneously ascribes this to Rheticus. And the tangents and secants are both ascribed to Reinhold, by Briggs, in his *Mathematica ab antiquis minus cognita* (pa. 30. Appendix to Ward's Lives of the Professors of Gresham College).

Francis Vieta was born in 1540 at Fontenai, or Fontenai-le-Comte, in Lower Poitou, a province of France. He was master of requests at Paris, where he died in 1603, being the 63d of his age. Among other branches of learning in which he excelled, he was one of the most respectable mathematicians of the 16th century, or indeed of any age. His writings abound with marks of great originality, and the finest genius, as well as intense application. Among them are several pieces relating to trigonometry, which may be found in the collection of his works published at Leyden in 1646, by Francis Schooten, besides another large and separate volume in folio, published in the author's life-time at Paris in 1579, containing trigonometrical tables with their construction and use; very elegantly printed by the king's mathematical printer, with beautiful types and rules, the differences of the sines, tangents and secants, and some other parts, being printed with red ink for the better distinction; but inaccurately executed, as he himself testifies in pa. 323 of his other works above-mentioned. The first part of this curious volume is intitled *Canon Mathematicus, seu ad Triangula, cum Appendicibus*, and contains a great variety of tables useful in trigonometry. The first of these is what he more peculiarly calls *Canon Mathematicus, seu ad Triangula*, which contains all the sines, tangents, and secants for every minute of the quadrant, to the radius 100,000 with all their differences; and towards the end of the quadrant the tangents and secants are extended to 8 or 9 places of figures. They are arranged like our tables at present, increasing on the left-hand side to 45 degrees, and then returning upwards by the right-hand side to 90 degrees; so that each number and its complement stand together on the same line. But here the canon of what we now call tangents is denominated *fæcundus*, and that of the secants *fæcundissimus*. For the general idea prevailing in the form of these tables, is not that the lines represented by the numbers are those which are drawn in and about a circle, as sines, tangents and secants, but the three sides of right-angled triangles; this being the way in which those lines had always been considered, and which still continued for some time longer. And therefore he considers the canon as a series of plane right-angled triangles, one side being constantly 100,000; or rather as three series of such triangles, for he makes

makes a distinct series for each of the three varieties, namely, according as the hypotenuse or the base or the perpendicular is represented by the constant number 100,000, which is similar to the radius. Making each side constantly 100,000, the other two sides are computed to every magnitude of the acute angle at the base, from one minute up to 90 degrees or the whole quadrant. Each of the three series therefore consists of two parts, as representing the two variable sides of the triangle. When the hypotenuse is made the constant number 100,000, the two variable sides of the triangle are the perpendicular and base, or our sine and cosine; when the base is 100,000, the perpendicular and hypotenuse are the variable parts, forming the *canon fecundus et fecundissimus*, or our tangent and secant; and when the perpendicular is made the constant 100,000, the series contains the variable base and hypotenuse, or also *canon fecundus et fecundissimus*, or our cotangent and cosecant. Of course therefore the table consists of six columns, 2 for each of the three series, besides the two columns on the right and left for minutes, from 0 to 60 in each degree.

The second of these tables is similar to the first, but all in rational numbers, consisting, like it, of 3 series of 2 columns each, the radius, or constant side of the triangle, in each series, being 100,000, as before, and the other two sides accurately expressed in integers and rational vulgar fractions. So that we have here the canon of accurate sines, tangents and secants, or a series of about 4300 rational right-angled triangles. But then the several corresponding arcs of the quadrant, or angles of those triangles, are not expressed. Instead of them are inserted, in the first column next the margin, a series of numbers decreasing from the beginning to the end of the quadrant, which are called *numeri primi baseos*. It is from these numbers that Vieta constructs the sides of the three series of right-angled triangles, one side in each series being the constant number 100,000, as before. The theorems by which these series of rational triangles are computed from the *numeri primi baseos*, or marginal numbers, are inserted all in one page at the end of this second table, and in the modern notation they may be briefly expressed thus. Let p be the primary or marginal number on any line, and r the constant radius or number 100,000; then if r denote the hypotenuse of the right-angled triangle, the perpendicular and base, or the sine and cosine, will be respectively

$$\frac{pr}{\frac{1}{4}p^2 + 1} \text{ and } r - \frac{2r}{\frac{1}{4}p^2 + 1}, \text{ which last we may reduce to } \frac{\frac{1}{4}p^2 - 1}{\frac{1}{4}p^2 + 1}r);$$

when r denotes the base of the right-angled triangle, the perpendicular and hypotenuse, or the tangent and secant, are expressed by

$$\frac{pr}{\frac{1}{4}p^2 - 1} \text{ and } r + \frac{2r}{\frac{1}{4}p^2 - 1}, \text{ which last we may reduce to } \frac{\frac{1}{4}p^2 + 1}{\frac{1}{4}p^2 - 1}r);$$

and when r denotes the perpendicular of the right-angled triangle, the base and hypotenuse, or the cotangent and cosecant, are then expressed by

$$\frac{1}{4}pr - \frac{r}{p} \text{ (or } \frac{\frac{1}{4}p^2 - 1}{p}r) \text{ and } \frac{1}{4}pr + \frac{r}{p} \text{ (or } \frac{\frac{1}{4}p^2 + 1}{p}r).$$

So that Vieta's general values will be as we have here collected them together in the following expressions immediately under the words sine, cosine, &c; and

just below Vieta's forms I have here placed the others to which they reduce and are equivalent, which are more contracted, but not so well adapted to the expeditious computation as Vieta's forms.

Sine	Cofine	Tangent	Secant	Cotangent	Cofecant
$\frac{pr}{\frac{1}{4}p^2+1}$	$r - \frac{2r}{\frac{1}{4}p^2+1}$	$\frac{pr}{\frac{1}{4}p^2-1}$	$r + \frac{2r}{\frac{1}{4}p^2-1}$	$\frac{1}{4}pr - \frac{r}{p}$	$\frac{1}{4}pr + \frac{r}{p}$
$\frac{p}{\frac{1}{4}p^2+1} r$	$\frac{\frac{1}{4}p^2-1}{\frac{1}{4}p^2+1}$	$\frac{p}{\frac{1}{4}p^2-1} r$	$\frac{\frac{1}{4}p^2+1}{\frac{1}{4}p^2-1} r$	$\frac{\frac{1}{4}p^2-1}{p} r$	$\frac{\frac{1}{4}p^2+1}{p} r$

All these expressions it is evident are rational; and by assuming p of different values, from the first theorems Vieta computed the corresponding sides of the triangles, and so expressed them all in integers and rational fractions.

To the foregoing principal tables are subjoined several other smaller tables, or short specimens of large ones: as, a table of the sines, tangents and secants for every single degree of the quadrant, with the corresponding lengths of the arcs, the radius being 100,000,000; another table of the sines, tangents and secants, for each degree also, expressed in sexagesimal parts of the radius as far as the 3d order of parts; also two other tables for the multiplication and reduction of sexagesimal quantities.

The second part of this volume is intituled *Universalium Inspectionum ad Canonem Mathematicum Liber singularis*. It contains the construction of the tables, a compendious treatise on plane and spherical trigonometry, with the application of them to a great variety of curious subjects in geometry and mensuration, treated in a very learned manner; as also many curious observations concerning the quadrature of the circle, the duplication of the cube, &c. Computations are here given of the ratio of the diameter of a circle to the circumference, and of the length of the sine of 1 minute, both to many places of figures; by which he found that the sine of 1 minute is between 2,908,881,959

2,908,882,056; also that, the diameter of a circle being 100,000,000,00, the perimeter of the inscribed and circumscribed polygon of 393216 sides, will be as follows,

perim. of the inscrib. polygon 314,159,265,35

perim. of the circum. polygon 314,159,265,37

and that therefore the circumference of the circle lies between those two numbers.

Although no author's name appears to the volume I have been describing, there can be no doubt of its being the performance of Vieta; for, besides bearing evident marks of his masterly hand, it is mentioned by himself in several parts of his other works collected by Schooten, and in the preface to those works by Elzevir, the printer of them; as also in M. Montucla's *Histoire des Mathematiques*; which are the only notices I have ever seen or heard of concerning this book, the copies of which are so rare, that I never saw one besides that which is in my own possession, nor ever met with any other person at all acquainted with such a book.

In the other works of Vieta, published at Leyden in 1646 by Schooten, as mentioned above, there are several other pieces relating to trigonometry; some of

of which, on account of their originality and importance, are very deserving of particular notice in this place. And first, the very excellent theorems, here first of all given by our author, relating to angular sections, the geometrical demonstrations of which are supplied by that ingenious geometrician Alexander Anderson, a native of Aberdeen. We find here theorems for the chords (and consequently sines) of the sums and differences of arcs; and for the chords of arcs that are in arithmetical progression, namely, that the first or least chord is to the 2d, as any one after the first is to the sum of the two next less and greater, for example as the 2d to the sum of the 1st and 3d, and as the 3d to the sum of the 2d and 4th, and as the 4th to the sum of the 3d and 5th, &c; so that the 1st and 2d being given, all the rest are found from them by one subtraction and one proportion for each, in which the 1st and 2d terms are constantly the same: next are given theorems for the chords of any multiples of a given arc or angle, as also the chords of their supplements to a semicircle, which are similar to the sines and cosines of the multiples of given angles; and the conclusions from them are expressed in this manner: 1st, that if c be the chord of the supplement of a given arc a , to the radius 1, then the chords of the supplements of the multiple arcs, will be as in the annexed table:

where the author observes that the signs are alternately + and -; that the vertical columns of numeral co-efficients to the terms of the chords, are the several orders of figurate numbers, which he calls triangular, pyramidal, triangulo-triangular, triangulo-pyramidal, &c. *generated in the ordinary way by continual additions; not indeed from unity, AS IN THE GENERATION OF*

Arcs	Chords of the Supplements.
1a	c
2a	$c^2 - 2$
3a	$c^3 - 3c$
4a	$c^4 - 4c^2 + 2$
5a	$c^5 - 5c^3 + 5c$
6a	$c^6 - 6c^4 + 9c^2 - 2$
7a	$c^7 - 7c^5 + 14c^3 - 7c$ &c.
&c.	

POWERS, but beginning with the number 2; and that the powers observe always the same progression: secondly, that if the chord of an arc a be called 1, and d the chord of the double arc $2a$, then the chords of the series of multiple arcs will be as in this table: where the author re-

marks as before on the law of the powers, signs, and co-efficients, these being the orders of figurate numbers, raised from unity by continual additions, *after the manner of the genesis of powers*, which generation in that way he speaks of as a thing generally known, but without giving any hint how the co-efficients of the terms of any power may be found from one another only, and independent of those of any other power, as it was afterwards, and

Arcs	Chords.
1a	1
2a	d
3a	$d^2 - 1$
4a	$d^3 - 2d$
5a	$d^4 - 3d^2 + 1$
6a	$d^5 - 4d^3 + 3d$
7a	$d^6 - 5d^4 + 6d^2 - 1$
8a	$d^7 - 6d^5 + 10d^3 - 4d$
&c.	&c.

first of all, I believe, done by Henry Briggs, about the year 1600: and 3dly, that if C be the chord of any arc a , to the radius 1, then the series of the chords

and supplemental chords of the multiple arcs, will be thus; where the values are alternately chords and chords of the supplements of the arcs on the same line, and the law of the powers and co-efficients as before, but every alternate couplet of lines having their signs changed.

Arcs	Chords and Chords of Sup.
1a	Chord = + C
2a	Sup. ch. = - C ² + 2
3a	Chord = - C ³ + 3C
4a	Sup. ch. = + C ⁴ + 4C ² + 2
5a	Chord = + C ⁵ - 5C ³ + 5C
6a	Sup. ch. = - C ⁶ + 6C ⁴ - 9C ² + 2
7a	Chord = - C ⁷ + 7C ⁵ - 14C ³ + 7C
&c.	&c.

Another curious theorem is added to the above, for finding the sum of all these chords drawn in a semicircle, from one end of the diameter to every point in the circumference, those points dividing the circumference into any number of equal parts; namely, as the least chord is to the diameter, so is the sum of the said least chord and diameter and greatest chord, to double the sum of all the chords including the diameter as one of them.

As the above theorems are chiefly adapted for the chords of multiple angles, a few problems and remarks are then added (whether by Vieta or Anderson does not clearly appear, but I think by the latter) concerning the application of them, to the section of angles into submultiples, and thence to the computation of the chords or sines, or a canon of triangles. The general precept for the angular sections is this; select one of the above equations adapted to the proper number of the section, in which will be concerned the powers of the unknown or required quantity, as high as the index of the section; and from this equation find that quantity by the known methods for the resolution of equations. Examples are given of three different sections, namely, for 3, 5, and 7 equal parts, the forms for which are respectively these

$$\begin{aligned} 3 C - C^3 & \dots\dots\dots = g \\ 5 C - 5C^3 + C^5 & \dots\dots\dots = g \\ 7 C - 14C^3 + C^5 - C^7 & = g \end{aligned}$$

where g is the chord of the given arc or angle, and C the required chord of the 3d, 5th, or 7th part of it. And it is shewn geometrically that the first of these equations has two real positive roots, the second 3, and the last 4; also from the same principles the relations of these roots are pointed out.

The method then annexed for constructing the canon of sines from the foregoing theorems, is thus: By dividing the radius in extreme-and-mean ratio, is obtained the sine of 18 degrees; this quinquisection gives the sine of 3° 36': Again by trisection the arc of 60°, there is obtained the sine of 20°; this again trisection gives that of 6° 40'; and this bisected gives that of 3° 20': Then, by the theorem for the difference of two arcs, there will be found the sine of 16', the difference between 3° 36' and 3° 20': Lastly, by four successive bisections, will at length be found the sines of 8', 4', 2' and 1'. This last being found, the sines of its multiples, and again of the multiples of these multiples, &c, throughout the quadrant, are to be taken by the proper theorems before laid down.

And the same subject is still farther pursued and explained in the tract containing the answer given by Vieta to the problem proposed to the whole world by Adrianus Romanus.

In the same collection of Vieta's works, from page 400 to 432, is given a complete

plete treatise on practical trigonometry, containing rules for resolving all the cases of plane and spherical triangles, by the *Canon Mathematicus*, or table of sines, tangents and secants.

The next authors whose labours in this way have been printed, are Rheticus, Otho, and Pitiscus: to all of whom we owe very great improvements in trigonometry.

George Joachim Rheticus, professor of mathematics in the university of Wurtemberg, and sometime pupil to Copernicus, died in 1576, in the 60th *Wittenberg* year of his age. He conceived and executed the great design of computing the triangular canon for every ten seconds of the quadrant, to the radius (1,000,000,000,000,000) consisting of 1 followed by 15 cyphers. The series of sines which Rheticus computed to this radius, for every ten seconds, and for every single second in the first and last degree of the quadrant, was published in folio at Francfort 1613 by Pitiscus, who himself added a few of the first sines computed to the radius 10,000,000,000,000,000,000,000.

But the large work, or whole trigonometrical canon, computed by Rheticus, was published in 1596 by Valentine Otho, mathematician to the Electoral Prince Palatine. This vast work contains all the three series for the whole canon of right-angled triangles (being similar to the sines, tangents and secants, by which names I shall call them) with all the differences of the numbers, to the radius 10,000,000,000.

Prefixed to these tables are several books on their construction and use in plane and spherical trigonometry, &c. Of these, the first three are by Rheticus himself; namely, book the 1st containing the demonstrations of 9 lemmas concerning the properties of certain lines drawn in and about circles: the 2d book contains 10 propositions relative to the sines and cosines of arcs, together with those of their sums and differences, their halves and doubles, &c. The 3d book teaches, in 13 propositions, the construction of the canon to the radius 1,000,000,000,000,000. By some of the common properties of geometry having determined the sines of a few principal arcs, as 30° , 36° , &c, in the first proposition by continual bisections, he finds the sines of various other arcs, down to 45 minutes. Then in the second proposition, by the theorems for the sums and differences of arcs, he finds all the sines and cosines, up to 90 degrees, in a series of arcs differing by $1^\circ 30'$. And, in the third proposition, by the continual addition of $45'$, he obtains all the sines and cosines in the series whose common difference is $45'$. In the 4th proposition, beginning with $45'$, and continually bisecting, he finds the sines and cosines of the series of half arcs till he arrives at the arc of 14^{viii} 19^{ix} , the sine of which is found to be 1, and its cosine 999,999,999,999,999. In the fifth proposition are computed the sine and cosine of 30 seconds or half a minute. In the 6th and 7th propositions are computed the sines and cosines for every minute, from $1'$ to $45'$, as well as of many larger arcs. The 8th proposition extends the computation for single minutes much farther. In proposition 9 and 10 are computed the tangents and secants for all arcs in the series whose common difference is $45'$; and these are deduced from the sines of the same arcs by one proportion for each. In the remaining three propositions, 11, 12, 13, are computed the tangents and secants for several small angles. And from

from all these primary fines, tangents, and secants, the whole canon is deduced and compleated.

The remaining books in this work are by the editor Otho; namely, a treatise, in one book, on right-angled plane triangles, the cases of which are resolved by the tables; then right-angled spherical trigonometry in four books; next oblique spherical trigonometry in five books; and lastly several other books, containing various spherical problems.

Next after the above are placed the tables themselves, containing, for every 10 seconds, the fines, tangents and secants, with all the differences annexed to each, in a smaller character. The numbers, however, are not called fines, tangents, and secants, but, like Vieta's before described, they are considered as representing the sides of right-angled triangles, and titled accordingly. They are also in like manner divided into three series, namely, according as the radius, or constant side of the triangle, is made the hypotenuse, or the greater leg, or the lesser leg of the triangle. When the hypotenuse is made the constant radius 10,000,000,000, the two columns of this case or series are called the perpendicular and base, which are our sine and cosine; when the greater leg is the constant radius, the two columns of this series are titled hypotenuse and perpendicular, which are our secant and tangent; and when the lesser leg is constant, the two columns in this case are called hypotenuse and base; which are our cosecant and cotangent. After this large canon is printed another smaller table, which is said to be the two columns of the third series, or cosecants and cotangents, with their differences, but to 3 places of figures less, or to the radius 10,000,000. But I cannot discover the reason for adding this lesser table, even if it were correct, which is very far from being the case, the numbers being uniformly erroneous, and different from the former through the greatest part of the table.

Towards the close of the 16th century many persons wrote on the subject of trigonometry, and the construction of the triangular canon. But, their writings being seldom printed till many years afterwards, it is not easy to assign their order in respect of time. I shall therefore mention but a few of the principal authors, and that without pretending to any great precision on the score of chronological precedence.

In 1591 Philip Lansbergius first published his *Geometria Triangulorum* in four books, with the canon of fines, tangents, and secants; a brief but very elegant work; the whole being clearly explained: and it is perhaps the first set of tables titled with those words. The fines, tangents and secants of the arcs to 45 degrees, with those of their complements, are each placed in adjacent columns, in a very commodious manner, continued forwards and downwards to 45 degrees, and then returning backwards and upwards to 90 degrees: the radius is 10,000,000, and a specimen of the first page of the table is as follows.

o	Sines		Tangents		Secants		
	o	10,000,000	o	Infinitum.	10,000,000	Infinitum.	
1	2,909	9,999,999	2,909	34,377,466,738	10,000,000	34,377,468,193	59
2	5,818	9,999,998	5,818	17,188,731,915	10,000,002	17,188,734,824	58
3	8,727	9,999,996	8,727	11,459,152,994	10,000,004	11,459,157,357	57
4	11,636	9,999,993	11,636	8,594,363,048	10,000,007	8,594,368,866	56
5	14,544	9,999,989	14,544	6,875,488,693	10,000,011	6,875,495,966	55
&c							&c
							89

*W! Die Tafel Landstangs
ist richtig aus Funks
Geometria naturae
abgeleitet!*

Of this work, the first book treats of the magnitude and relations of such lines as are considered in and about the circle, as the chords, sines, tangents, and secants. In the second book is delivered the construction of the trigonometrical canon, by means of the properties laid down in the first book: After which follows the canon itself. And in the third and fourth books is shewn the application of the table, in the resolution of plane and spherical triangles.—Lansberg, who was born in Zealand 1561, was many years a minister of the gospel, and died at Middleburg in 1632.

The trigonometry of Bartholomew Pitiscus was first published at Francfort in the year 1599. This is a very compleat work; containing, besides the triangular canon, with its construction and use in resolving triangles, the application of trigonometry to problems of surveying, altimetry, architecture, geography, dialling, and astronomy. The construction of the canon is very clearly described: And in the third edition of the book in the year 1612, he boasts to have added, in this part, arithmetical rules for finding the chords of the 3d, 5th, and other uneven parts of an arc, from the chord of that arc being given; saying that it had been heretofore thought impossible to give such rules: But, after all, those boasted methods are only the application of the double rule of False-Position to the then known rules for finding the chords of multiple arcs; namely, making the supposition of some number for the required chord of a submultiple of any given arc, then from this assumed number computing what will be the chord of its multiple arc, and which is to be compared with that of the given arc; then the same operation is performed with another supposition; and so on as in the double rule of position. The canon contains the sine, tangent, and secant for every minute of the quadrant, in some parts to 7 places of figures, in others to 8; as also the differences for every 10 seconds. The sines, tangents, and secants are also given for every 10 seconds in the first and last degree of the quadrant, for every 2 seconds in the first and last 10 minutes, and for every single second in the first and last minute. In this table the sines, tangents and secants are continued downwards on the left-hand pages as far as to 45 degrees, and then returned upwards on the right-hand pages, so that the complements are always on the same line in the opposite or facing pages.

The mathematical works of Christopher Clavius (a German jesuit, who was born at Bamberg in 1537) in five large folio volumes, were printed at Moguntia, or Mentz, in 1612, the year in which the author died, at the age of 75. In the first volume we find a very ample and circumstantial treatise on trigonometry, with

with Regiomontanus's canon of sines for every minute, as also canons of tangents and secants, each in a separate table, to the radius 10000000, and in a form continued forwards all the way up to 90 degrees. The explanation of the construction of the tables is very compleat, and is chiefly extracted from Ptolemy, Purbach, and Regiomontanus. The sines have the differences set down for each second, that is, the quotients arising from the differences of the sines divided by 60. About the year 1600 Ludolph Van Collen or à Ceulen, a respectable Dutch Mathematician, wrote his book *de circulo & adscriptis*, in which he treats fully and ably of the properties of lines drawn in and about the circle, and especially of chords or subtenses, with the construction of the canon of sines. The geometrical properties from which these lines are computed, are the same as those used by former writers: but his mode of computing and expressing them, is different from theirs; for they actually extracted all the roots, &c, at every step or single operation, in decimal numbers; but he retained the radical expressions to the last, making them however always as simple as possible: thus, for instance, he determines the sides of the polygons of 4, 8, 16, 32, &c, sides inscribed in the circle whose radius is 1, to be as in the table annexed:

where the point before any figure ($\sqrt{\cdot 2}$) signifies the root of all that follows it; so the last line is in our notation the same as

$\sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2}}}}$. And as the perfect management of such surds was then not generally known, he added a very neat tract on that subject, to facilitate the computations. These, together with other dissertations on similar geometrical matters,

No. of sides	Length of each side.
4	$\sqrt{2}$
8	$\sqrt{\cdot 2} - \sqrt{2}$
16	$\sqrt{\cdot 2} - \sqrt{\cdot 2} + \sqrt{2}$
32	$\sqrt{\cdot 2} - \sqrt{\cdot 2} + \sqrt{\cdot 2} - \sqrt{2}$
&c	&c.

were translated from the Dutch language into Latin by Willebrord Snell, and published at (Lugd. Batav.) Leyden in 1619. It was in this work that Ludolph determined the ratio of the diameter to the circumference of the circle to 36 figures, shewing that, if the diameter be 1, the circumference will be

greater than 3.14159,26535,89793,23846,26433,83279,50288,

but less than 3.14159,26535,89793,23846,26433,83279,50289;

which ratio was by his order, in imitation of Archimedes, engraven on his tombstone, as is witnessed by the said Snell, pa. 54, 55, *Cyclometricus*, published at Leyden two years after, in which he treats the same subject in a similar manner, recomputing and verifying Ludolph's numbers. And in the same book he also gives a variety of geometrical approximations, or mechanical solutions, to determine very nearly the lengths of arcs, and the areas of sectors and segments of circles.

Besides the *Cyclometricus*, and another geometrical work (*Apollonius Batavus*) published in 1608, the same Snellius wrote also four others intitled *doctrinae triangulorum canonicae*, in which are contained the canon of secants, and in which the construction of sines, tangents and secants, together with the dimension or calculation of triangles, both plane and spherical, are briefly and clearly treated. After the author's death this work was published in 8vo, at Leyden 1627, by

Martinus Hortensius, who added to it a tract on surveying and spherical problems. Willebrord Snell was born in 1591 at Royen, and died in 1627, being only 35 years of age. He was professor of mathematics in the university of Leyden, as was also his father Rodolph Snell.

Also in 1627, Francis van Schooten published at Amsterdam, in a small neat form, tables of sines, tangents and secants for every minute of the quadrant, to 7 places of figures, the radius being 10,000,000; together with their use in the trigonometry of plane triangles. These tables have a great character for their accuracy, being declared by the author to be without one single error. This, however, must not be understood of the last figure of the numbers, which I find to be very often erroneous, sometimes in excess and sometimes in defect, by not being always set down to the nearest unit. Schooten died in 1659, while the second volume of his second edition of Descartes' geometry was in the press. He was also author of several other valuable works in geometry and other branches of the mathematics.

The foregoing are the principal writers on the tables of sines, tangents and secants, before the invention of logarithms, which happened about this time, namely, soon after the year 1600. Tables of the natural numbers were now all completed, and the methods of computing them nearly perfected: And therefore, before entering on the discovery and construction of logarithms, I shall stop here awhile to give a summary of the manner in which the said natural sines, tangents and secants were actually computed, after having been gradually improved from Hipparchus, Menelaus, and Ptolemy, who used only the chords, down to the beginning of the 17th century, when sines, tangents, secants and versed sines were in use, and when the method hitherto employed had received its utmost improvement.

In this explanation I shall here first enumerate the theorems by which the calculations were made, and then describe the application of them to the computation itself.

Theorem 1. The square of the diameter of a circle is equal to the sum of the squares of the chord of an arc and of the chord of its supplement to a semicircle.

2. The rectangle under the two diagonals of any quadrilateral figure inscribed in a circle, is equal to the sum of the two rectangles under the opposite sides.

3. The sum of the squares of the sine and cosine (hitherto called the sine of the complement), is equal to the square of the radius.

4. The difference between the sines of two arcs that are equally distant from 60 degrees, or $\frac{1}{6}$ of the whole circumference, the one as much greater as the other is less, is equal to the sine of half the difference of those arcs, or of the difference between either arc and the said arc of 60 degrees.

5. The sum of the cosine and versed sine, is equal to the radius.

6. The sum of the squares of the sine and versed sine, is equal to the square of the chord, or to the square of double the sine of half the arc.

7. The sine is a mean proportional between half the radius and the versed sine of double the arc.

8. A mean proportional between the versed sine and half the radius, is equal to the sine of half the arc.

9. As radius is to the sine, so is twice the cosine to the sine of twice the arc.
 10. As the chord of an arc is to the sum of the chords of the single and double arc, so is the difference of those chords to the chord of thrice the arc.
 11. As the chord of an arc is to the sum of the chords of twice and thrice the arc, so is the difference of those chords to the chord of five times the arcs.
 12. And in general, as the chord of an arc is to the sum of the chords of n times and $n + 1$ times the arc, so is the difference of those chords to the chord of $2n + 1$ times the arc.
 13. The sine of the sum of two arcs, is equal to the sum of the products of the sine of each multiplied by the cosine of the other and divided by the radius.
 14. The sine of the difference of two arcs, is equal to the difference of the said two products divided by the radius.
 15. The cosine of the sum of two arcs, is equal to the difference between the products of their sines and of their cosines divided by the radius.
 16. The cosine of the difference of two arcs, is equal to the sum of the said products divided by the radius.
 17. A small arc is equal to its chord or sine, nearly.
 18. As cosine is to sine, so is radius to tangent.
 19. The radius is a mean proportional between the tangent and cotangent.
 20. Half the difference between the tangent and cotangent of an arc, is equal to the tangent of the difference between the arc and its complement. Or, the sum arising from the addition of double the tangent of an arc with the tangent of half its complement, is equal to the tangent of the sum of that arc and the said half complement.
 21. The square of the secant of an arc, is equal to the sum of the squares of the radius and tangent.
 22. The radius is a mean proportional between the secant and cosine. Or, as cosine is to radius, so is radius to secant.
 23. The radius is a mean proportional between the sine and cosecant.
 24. The secant of an arc is equal to the sum of its tangent and the tangent of half its complement. Or, the secant of the difference between an arc and its complement, is equal to the tangent of the said difference added to the tangent of the lesser arc.
 25. The secant of an arc is equal to the difference between the tangent of that arc and the tangent of the arc added to half its complement. Or, the secant of the difference between an arc and its complement, is equal to the difference between the tangent of the said difference and the tangent of the greater arc.
- From some of these 25 theorems, extracted from the writers before mentioned, and a few propositions of Euclid's Elements, they compiled the whole table of sines, tangents, and secants, nearly in the following manner.
- By the elements were computed the sides of a few of the regular figures inscribed in a circle, which were the chords of such parts of the whole circumference as are expressed by the number of sides, and therefore the halves of those chords the sines of the halves of the arcs. So, if the radius be 10,000,000, the sides of the following figures will give the annexed chords and sines.

The

The figure	Arcs sub- tended	Its chord, or side	Half arc	Its fines or $\frac{1}{2}$ -chord
Triangle	120°	17,320,508	60°	8,660,254
Square	90	14,142,136	45	7,071,068
Pentagon	72	11,755,705	36	5,877,853
Hexagon	60	10,000,000	30	5,000,000
Decagon	36	6,180,340	18	3,090,170
Quindecagon	24	4,158,234	12	2,079,117

Of some, or all of these, the fines of the halves were continually taken, by the-orem the 6th, 7th, or 8th, and of their complements, by the same theorems; and so on alternately of the halves and complements, till we arrive at an arc which is nearly equal to its fine. Thus, beginning with the above arc of 12 degrees, and its fine, we obtain the halves as follows:

The halves	Their fines	The comp.	Sines	The halves	Sines.
6°	1,045,285	of these		33°	5,446,390
3	523,360	48°	7,431,448	16 30	2,840,153
1 30	261,769	69	9,335,804	8 15	1,439,426
45	130,896	79 30	9,832,549	27 45	4,656,145
The comp.		84 45	9,958,049	Comps.	
of these		46 30	7,253,744	57	8,386,706
84	9,945,218	68 15	9,288,095	73 30	9,588,197
87	9,986,295	45 45	7,163,019	81 45	9,896,514
88 30	9,996,573	The halves		62 15	8,849,876
89 15	9,999,143	of these		Halves	
The halves		24	4,067,366	28 30	4,771,588
of these		34 30	5,664,062	14 15	2,461,533
42	6,691,306	17 15	2,965,416	36 45	5,983,246
21	3,583,679	39 45	6,394,390	Comps.	
10 30	1,822,355	23 15	3,947,439	61 30	8,788,171
5 15	915,016	The comp.		75 40	9,692,309
43 30	6,883,545	66	9,135,455	53 15	8,012,538
21 45	3,705,574	55 30	8,241,262	Half	
44 15	6,977,905	72 45	9,550,199	30 45	5,112,931
		50 15	7,688,418	Comp.	
		66 45	9,187,912	59 15	8,594,064

The fines of small arcs are then deduced in this manner. From the fine of 45' above determined, are found the halves, which will be thus:

45'	0''		130,896
22	30		65,449,4
11	15		32,724,8
		d 2	

Now

Now these last two fines being evidently in the same ratio as their arcs, the fines of all the less single minutes will be found by single proportion. So the 45th part of the fine of $45'$, gives 2909 for the fine of $1'$; which may be doubled, tripled, &c, for the fines of $2'$, $3'$, &c, up to $45'$.

Then, from all the foregoing primary fines, by the theorems for halving, doubling, or tripling, and by those for the sums and differences, the rest of the fines are deduced, to compleat the quadrant.

But having thus determined the fines and cofines of the first 30° of the quadrant, that is the fines of the first and last 30° , those of the intermediate 30° are, by theor. 4, found by one single subtraction for each fine.

The fines of the whole quadrant being thus compleated, the tangents are found by theor. 18, 19, 20, namely for one half of the quadrant by the 18th and 19th, and the other half, by one single addition or subtraction for each, by the 20th theorem.

And lastly, by theor. 24 and 25, the secants are deduced from the tangents by addition and subtraction only.

Among the various means used for constructing the canon of fines, tangents and secants, the writers above enumerated seem not to have been possessed of the method of differences, so profitably used since, and first of all, I believe, by Briggs, in computing his trigonometrical canon and his logarithms, as we shall see hereafter when we come to describe those works. They took, however, the successive differences of the numbers after they were computed, to verify or prove the truth of them; and, if found erroneous, by any irregularity in the last differences, from thence they had a method of correcting the original numbers themselves. At least this method is used by Pitiscus, *trig. lib. 2*, where the differences are extended to the third order.—In pa. 44 of the same book also is described, for the first time that I know of, the common notation of decimal fractions as now used. And this same notation was afterwards described and used by baron Neper in *positio 4* and *5* of his posthumous work on the construction of logarithms, published by his son in the year 1619. But the decimal fractions themselves may be considered as having been introduced by Regiomontanus, by his decimal division of the radius &c. of the circle; and from that time gradually brought into use; but continued long to be denoted after the manner of vulgar fractions, by a line drawn between the numerator and denominator, which last, however, was soon omitted, and only the numerator set down with the line below it; thus it was first $31 \frac{35}{1000}$, then $31 \frac{35}{1000}$; afterwards, omitting the line, it became 31^{35} , and lastly 31_3 , or 31.35 or $31'35$: As may be traced in the works of Vieta, and others since his time, gradually into the present century.

Having often heard it remarked that the word *sine*, or in Latin and French *sinus*, is of doubtful origin; and as the various accounts which I have seen of its derivation, are very different from one another, it may not be amiss here to employ a few lines on this matter. Some authors say this is an Arabic word; others that it is the single latin word *sinus*; and in Montucla's *Histoire des Mathematiques*, it is conjectured to be an abbreviation of two Latin words. The conjecture is thus expressed by the ingenious and learned author of that excellent history, at pa. xxxiii among the additions and corrections of the first volume: "A l'occasion

l'occasion des sinus dont on parle dans cette page, comme d'une invention des Arabes, voici une étymologie de ce nom, tout-à-fait heureuse et vraisemblable. Je la dois à M. Godin, de l'Académie Royale des Sciences, Directeur de l'Ecole de Marine de Cadix. Les sinus sont, comme l'on sçait, des moitiés de cordes ; et les cordes en Latin se nomment *inscriptæ*. Les sinus sont donc *semiffes inscriptarum*, ce que probablement on écrivit ainsi pour abrégé, S. Ins. Delà ensuite s'est fait par abus le mot de sinus." Now, ingenious as this conjecture is, there appears to be little or no probability for the truth of it. For, in the first place, it is not in the least supported by quotations from any of the more early books to shew that it ever was the practice to write or print the words thus S. Ins. upon which the conjecture is founded. Again, it is said the chords are called in Latin *inscriptæ* ; and it is true that they sometimes are so ; but I think they are more frequently called *subtensæ*, and the fines *semiffes subtensarum* of the double arcs, which will not abbreviate into the word *sinus*. But it may be said, what reason have we to suppose this word to be either a Latin word, or the abbreviation of any Latin words whatever ? that it seems but proper to seek for the etymology of words in the language of the inventors of the things. For which reason it is, that we find the two other words, *tangens* and *secans*, are Latin, as they were invented and used by authors who wrote in that language. But the fines are acknowledged to have been invented and introduced by the Arabians, and thence by analogy it would seem probable that this is a word of *their* language, and from them adopted, together with the use of it, by the Europeans. And indeed Lansbergius, in the 2d pa. of his trigonometry above-mentioned, expressly says that it is Arabic : His words are, *Vox sinus Arabica est, et proinde barbara ; sed, cum longo usu approbata sit, & commodior non suppetat, nequaquam repudianda est : faciles enim in verbis nos esse oportet, cum de rebus convenit.* And Vieta says something to the same purport in pa. 9 of his *Universalium Inspectionum ad Canonem Mathematicum Liber* : His words are, *Breve sinus vocabulum, cum sit artis, Saracenis præsertim quàm familiare, non est ab artificibus explodendum, ad laterum semiffium inscriptorum denotationem, &c.*

Guarinus also is of the same opinion : in his *Euclides Adæutus, &c. tract. xx. pa. 307.* he says, *SINUS vero est nomen Arabicum usurpatum in hanc significationem à mathematicis ;* although he was aware that a Latin origin was ascribed to it by Vitalis, for he immediately adds, *Licet Vitalis in suo Lexico Mathematico ex eo velit sinum appellatum, quòd claudat curvitatē arcus.*

Long before I either saw or heard of any conjecture or observation concerning the etymology of the word *sinus*, I remember that I imagined it to be taken from the same Latin word, signifying breast or bosom, and that our sine was so called allegorically. I had observed that several of the terms in trigonometry were derived from a bow to shoot with, and its appendages ; as *arcus* the bow, *chorda* the string, and *sagitta* the arrow, by which name the versed sine which represents it was sometimes called ; also that the *tangens* was so called from its office, being a line touching the circle, and the *secans* from its cutting the same ; I therefore imagined that the *sinus* was so called, either from its resemblance to the breast or bosom, or from its being a line drawn within the bosom (*sinus*) of the arc, or from its being that part of the string (*chorda*) of a bow (*arcus*) which is drawn near the breast (*sinus*).

(*sinus*) in the act of shooting. And perhaps Vitalis's definition above quoted has some allusion to the same similitude.

Also Vieta seems to allude to the same thing in calling *sinus* an allegorical word, in pa. 417 of his works as published by Schooten, where, with his usual judgment and precision, he treats of the propriety of the terms used in trigonometry for certain lines drawn in and about the circle, of which, as it very well deserves, I shall here extract the principal part, to shew the opinion and arguments of so great a man on those names. "Arabes autem semisses inscriptas duplo, numeris præsertim æstimatas, vocaverunt allegoricè SINUS, atque idè ipsam semidiametrum, quæ maxima est semissium inscriptarum, SINUM TOTUM. Et de iis suâ methodo canones exaraverunt qui circumferuntur, supputante præsertim Regiomontano benè justè & accuratè, in iis etiam particulis qualium semidiameter adsumitur 10,000,000,

"Ex canonibus deinde sinuum derivaverunt recentiores canonem semissium circumscriptarum, quem dixere *Fœcundum*; & canonem eductarum è centro, quem dixere *Fœcundissimum* et *Beneficum*, hypotenusis addictum. Atque adè semisses circumscriptas, numeris præsertim æstimatas, vocaverunt *Fœcundos*, Sinus numerósve videlicet; quanquam nihil vetat *Fœcundi* nomen substantivè accipi. Hypotenusas autem *Beneficas*, vel etiam simpliciter Hypotenusas: quoniam hypotenusâ in primâ serie sinûs totius nomen retinet. Itaque ne novitate verborum res adumbretur, & alioqui sua artificibus, eo nomine debita, præripiatur gloria, præposita in Canone Mathematico canonicis numeris inscriptio candidè admonet primam seriem esse Canonem Sinuum. In secundâ verò, partem canonis fœcundi, partem canonis fœcundissimi, contineri. In tertiâ, reliquam.

"Sanè præter inscriptas & circumscriptas, circulum etiâ adficiunt aliæ lineæ rectæ, velut Incidentes, Tangentes, & Secantes. Verùm illæ voces substantivæ sunt, non peripheriarum relativæ. Ac secare quidem circulum linea recta tunc intelligitur, cum in duobus punctis secat. Itaque non loquuntur benè geometricè, qui eductas è centro ad metas circumscriptarum vocant secantes improprie, cum secantes & tangentes ad certos angulos vel peripherias referunt. Immò verò artem confundunt, cum his vocibus necesse habeat uti geometra abs relatione.

"Quare si quibus arrideat Arabum metaphora; quæ quidem aut omninò retinenda videtur, aut omninò explodenda; ut semisses inscriptas, Arabes vocant sinus; sic semisses circumscriptæ, vocentur Prosinus Amsinúsve; et eductæ è centro, Transsinuosæ. Sin allegoria displiceat, geometrica sanè inscriptarum et circumscriptarum nomina retineantur. Et cum eductæ è centro ad metas circumscriptarum, non habeant hætenus nomen certum neque elegans, vocentur sanè *prosemidiametri*, quasi protensæ semidiametri, se habentes ad suas circumscriptas, sicut semidiametri ad inscriptas."

Against the Arabic origin, however, of this word (*sinus*) may be urged its being varied according to the fourth declension of Latin nouns, like *manus*; and that if it were an Arabic word latinized, it would have been ranked under either the first, second, or third declension, as is usual in such adopted words.

So that, upon the whole, it will perhaps rather seem probable, that the term

sinus

sinus is the Latin word answering to the name by which the Saracens called that line, and not their word itself. And this conjecture seems to be rendered still more probable by some expressions in pa. 4 and 5 of Otho's preface to Rheticus's Canon, where it is not only said that the Saracens called the half-chord of double the arc *sinus*, but also that they called the part of the radius lying between the sine and the arc *sinus versus, vel sagitta*, which are evidently Latin words, and seem to be intended for the Latin translations of the names by which the Arabians called these lines, or the numbers expressing the lengths of them.

O F L O G A R I T H M S.

THE trigonometrical canon of natural sines, tangents and secants, being now brought to a considerable degree of perfection; the great length and accuracy of the numbers, together with the increasing delicacy and number of astronomical problems and spherical triangles, to the resolution of which the canon was applied, urged many persons, conversant in those matters, to endeavour to discover some means of diminishing the great labour and time, requisite for so many multiplications and divisions, in such large numbers as the tables then consisted of. And their chief aim was, to reduce the multiplications and divisions to additions and subtractions, as much as possible.

For this purpose, Nicolas Raymer Ursus Dithmarsus invented an ingenious method, which serves for one case in the sines, namely, when radius is the first term in the proportion, and the sines of two arcs are the second and third terms; for he shewed that the fourth term or sine, would be found by only taking half the sum or difference of the sines of two other arcs, which should be the sum and difference of the less of the two former given arcs and the complement of the greater. This is no more in effect than the following well-known theorem in trigonometry: As half the radius is to the sine of one arc, so is the sine of another arc to the cosine of the difference *minus* the cosine of the sum of the said arcs. The author published this ingenious device in 1588, in his *Fundamentum Astronomie*. And three or four years afterwards it was greatly improved by Clavius, who adapted it to all proportions in the resolution of spherical triangles, both for sines, tangents, secants, versed sines, &c; and that, whether radius be in the proportion or not. All which he explains very fully in *lem. 53 lib. 1.* of his treatise on the *Astrolabe*. This method, although ingenious, depends not on any abstract property of numbers, but only on the relations of certain lines drawn in and about the circle; and it was therefore rather limited, and sometimes attended with trouble in the application.

After perhaps various other contrivances, incessant endeavours at length produced the happy invention of logarithms, which are of direct and universal application to all numbers abstractedly considered, being derived from a property inherent in themselves. This property may be considered, either as the relation between a geometrical series of terms and a corresponding arithmetical one, or as the relation between ratios and the measures of ratios, which comes to much the same thing, they having been conceived in one of these ways by some of the writers on this subject, and in the other by the rest of them, as well as in both ways at different times by the same writer. A summary idea of this property, and of the probable reflections made on it by the first writers on logarithms, may be to the following effect.

The learned calculators, about the close of the 16th, and beginning of the 17th

century, finding the operations of multiplication and division by very long numbers of 7 or 8 places of figures, which they had frequently occasion to perform in solving problems relating to geography and astronomy, to be exceedingly troublesome, set themselves to consider whether it was not possible to find some method of lessening this labour, by substituting other easier operations in their stead. In pursuit of this object they reflected that, since in every multiplication by a whole number, the ratio, or proportion, of the product to the multiplicand, is the same as the ratio of the multiplier to unity, it will follow that the ratio of the product to unity (which, according to Euclid's definition of compound ratios, is compounded of the ratios of the said product to the multiplicand and of the multiplicand to unity) must be equal to the sum of the two ratios of the multiplier to unity, and of the multiplicand to unity. Consequently, if they could find a set of artificial numbers that should be the representatives of, or should be proportional to, the ratios of all sorts of numbers to unity, the addition of the two artificial numbers that should represent the ratios of any multiplier and multiplicand to unity, would answer to the multiplication of the said multiplicand by the said multiplier; or the sum arising from the addition of the said representative numbers, would be the representative number of the ratio of the product to unity; and consequently the natural number to which it should be found, in the table of the said artificial or representative numbers, that the said sum belonged, would be the product of the said multiplicand and multiplier. Having settled this principle, as the foundation of their wished-for method of abridging the labour of calculations, they resolved to compose a table of such artificial numbers, or numbers that should be representatives of, or proportional to, the ratios of all the common or natural numbers to unity.

The first observation that naturally occurred to them, in the pursuit of this scheme, was that, whatever artificial numbers should be chosen to represent the ratios of other whole numbers to unity, the ratio of equality, or of unity to unity, must be represented by 0; because *that* ratio has properly no magnitude, since, when it is added to or subtracted from any other ratio, it neither increases nor diminishes it.

The second observation that occurred to them was, that any number whatever might be chosen at pleasure for the representative of the ratio of any given natural number to unity: but that, when once such choice was made, all the other representative numbers would be thereby determined; because they must be greater or less than that first representative number, in the same proportions in which the ratios represented by them, or the ratios of the corresponding natural numbers to unity, were greater or less than the ratio of the said given natural number to unity. Thus, either 1, or 2, or 3, &c, might be chosen for the representative of the ratio of 10 to 1. But if 1 be chosen for it, the representatives of the ratios of 100 to 1 and 1000 to 1, which are double and triple of the ratio of 10 to 1, must be 2 and 3, and cannot be any other numbers; and, if 2 be chosen for it, the representatives of the ratios of 100 to 1 and 1000 to 1 will be 4 and 6, and cannot be any other numbers; and, if 3 be chosen for it, the representatives of the ratios of 100 to 1 and 1000 to 1 will be 6 and 9, and cannot be any other numbers; and so on.

The third observation that occurred to them was, that, as these artificial numbers were representatives of, or proportional to, the ratios of the natural numbers to unity, they must be expressions of the numbers of some smaller equal ratios that are contained in the said ratios. Thus, if 1 be taken for the representative of the ratio of 10 to 1, then 3, which is the representative of the ratio of 1000 to 1, will express the number of ratios of 10 to 1 that are contained in the ratio of 1000 to 1. And if, instead of 1, we make 10,000,000, or ten millions, the representative of the ratio of 10 to 1 (in which case 1 will be the representative of a very small ratio, or *ratiuncula*, which is only the ten-millionth part of the ratio of 10 to 1, or will be the representative of the ratio of the 10,000,000th root of 10, or of the first, or smallest, of 9,999,999 mean proportionals interposed between 1 and 10, to 1), the representative of the ratio of 1000 to 1, which will in this case be 30,000,000, will express the number of those *ratiunculae*, or small ratios of the 10,000,000th root of 10 to 1, which are contained in the said ratio of 1000 to 1: and the like may be shewn of the representative of the ratio of any other number to unity. And therefore they thought these artificial numbers, which thus represent, or are proportional to, the magnitudes of the ratios of the natural numbers to unity, might not improperly be called the LOGARITHMS of those ratios, since they express the numbers of smaller ratios of which they are composed. And then, for the sake of brevity, they called them the *logarithms of the said natural numbers themselves*, which are the antecedents of the said ratios to unity, of which they are in truth the representatives.

The foregoing method of considering this property, leads to much the same conclusions as the other way, in which the relations between a geometrical series of terms, and their exponents, or the terms of an arithmetical series, are contemplated. In this latter way, it readily occurred that the addition of the terms of the arithmetical series corresponded to the multiplication of the terms of the geometrical series; and that the arithmetics would therefore form a set of artificial numbers, which, when arranged in tables with their geometricals, would answer the purposes desired, as has been explained above.

From this property, by assuming four quantities, two of them as two terms in a geometrical series, and the others as the two corresponding terms of the arithmetics, or artificials, or logarithms, it is evident that all the other terms of both the two series may thence be generated: and therefore there may be as many sets or scales of logarithms as we please, since they depend entirely on the arbitrary assumption of the first two arithmetics. And all possible natural numbers may be supposed to coincide with some of the terms of any geometrical progression whatever, the logarithms or arithmetics determining which of the terms in that progression they are.

It was proper, however, that the arithmetical series should be so assumed, as that the term 0 in it might answer to the term 1 in the geometricals; otherwise the sum of the logarithms of any two numbers would be always to be diminished by the logarithm of 1, to give the logarithm of the product of those numbers: for which reason, making 0 the logarithm of 1, and assuming any quantity whatever for the value of the logarithm of any one number, the logarithms of all other numbers were thence to be derived. And hence, like as the multiplication of two numbers is effected by barely adding their logarithms, so division is per-

formed by subtracting the logarithm of the one from that of the other, raising of powers by multiplying the logarithm of the given number by the index of the power, and extraction of roots by dividing the logarithm by the index of the root. It is also evident that, in all scales or systems of logarithms, the logarithm of 0 will be infinite; namely, infinitely negative if the logarithms increase with the natural numbers, but infinitely positive if the contrary: because that, while the geometrical series must decrease through infinite divisions by the ratio of the progression, before the quotient come to 0 or nothing; the logarithms, or arithmeticals, will in like manner undergo the corresponding infinite subtractions or additions of the common equal difference; which equal increase or decrease, thus indefinitely continued, must needs tend to an infinite result.

This, however, was no newly-discovered property of numbers, but what was always well known to all mathematicians, being treated of in the writings of Euclid, as also by Archimedes, who made great use of it in his *Arenarius*, or treatise on the number of the sands, namely, in assigning the rank or place of those terms, of a geometrical series, produced from the multiplication together of any of the foregoing terms, by the addition of the corresponding terms of the arithmetical series, which served as the indices or exponents of the former: and the reason why tables of these numbers were not sooner composed, was, that the accuracy and trouble of trigonometrical computations had not sooner rendered them necessary. It is therefore not to be doubted that, about the close of the sixteenth and beginning of the seventeenth century, many persons had thoughts of such a table of numbers, besides the few who are said to have attempted it.

Longomontanus has, by some, been said to have invented logarithms: but this cannot well be supposed to have been much more than in idea, since he never published any thing of the kind, nor ever laid claim to the invention, though he lived thirty-three years after they were first published by baron Neper, as he died only in 1647, when they had been long known and received all over Europe. Some circumstances of this matter are indeed related by Wood in his *Athenæ Oxonienses*, under the article Briggs, on the authority of Oughtred and Wingate, viz. "That one Dr. Craig, a Scotchman, coming out of Denmark into his own country, called upon John Neper, baron of Marcheston near Edinburgh, and told him, among other discourses, of a new invention in Denmark (by Longomontanus, as 'tis said) to save the tedious multiplication and division in astronomical calculations. Neper being solicitous to know farther of him concerning this matter, he could give no other account of it, than that it was by proportionable numbers; which hint Neper taking, he desired him at his return to call upon him again. Craig, after some weeks had passed, did so; and Neper then shewed him a rude draught of that he called *Canon mirabilis Logarithmorum*. Which draught, with some alterations, he printing in 1614, it came forthwith into the hands of our author Briggs, and into those of Will. Oughtred, from whom the relation of this matter came."

Kepler also says that one Juste Byrge, assistant-astronomer to the Landgrave of Hesse, invented, or projected, logarithms long before Neper did, but that they had never come abroad on account of the great reservedness of their author with regard to his own compositions. Byrge is also said to have computed a table of natural sines for every two seconds of the quadrant.

But whatever may have been said or conjectured concerning any thing that may have been done by others, it is certain that the world is indebted, for the first publication of logarithms, to John Napier or Nepair *, or in Latin Neper, baron of Merchiston, or Markinston, in Scotland, who died the 3d of April 1618, at the age of 67 years. Baron Napier added considerable improvements to trigonometry; and the frequent numerical computations he performed in this branch, gave occasion to his invention of logarithms, in order to save part of the trouble attending those calculations; and for this reason he adapted his tables peculiarly to trigonometrical uses.

This discovery he published in 1614, in his book intituled *Mirifici logarithmorum canonis descriptio*, reserving the construction of the numbers till the sense of the learned concerning his invention should be known: and, excepting the construction, this is a perfect work on this kind of logarithms, containing in effect the logarithms of all numbers, and the logarithmic sines, tangents, and secants for every minute of the quadrant, together with the description and uses of the tables, as also his definition and idea of logarithms.

Napier explains his notion of logarithms by lines described or generated by the motion of points, in this manner: He first conceives a line to be generated by the equable motion of a point which passes over equal portions of it in equal small moments or portions of time: he then considers another line as generated by the unequal motion of a point, in such manner, that, in the aforesaid equal moments or portions of time, there may be described or cut off, from a given line,

* The origin of which name Crawford informs us was from a (less) peerless action of one of his ancestors, viz. Donald, second son of the earl of Lenox, in the time of David the Second. "Some English writers, mistaking the import of the term *baron*, have called this celebrated person lord Napier, a Scotch nobleman. He was not indeed a peer of Scotland; but the Peerage of Scotland informs us, that he was of a very ancient, honourable, and illustrious family; that his ancestors, for many generations, had been possessed of sundry baronies; and, amongst others, of the barony of Merchistoun, which descended to him by the death of his father in 1608. Mr. Briggs, therefore, very properly styles him *Baro Merchistonii*. Now according to Skene, *de verborum significatione*, 'In this realm (of Scotland) he is called an Barrone, quha halds his landes immediatlie in chiefe of the king, and hes power of Pit and Gallows; *Fossa et Furca*; quhilk was first institute and granted be king Malcolme, quha gave power to the Barrones to ave ane Pit, quhairin women condemned for theft suld be drowned, and ane Gallows, whereupon men thieves and trespassowres suld be hanged, conforme to the doome given in the Barron Court thereanent.' That is, in our modern English, "In this kingdom (of Scotland) a man is called a *Baron* who holds his lands immediately in chief of the King, and has power of Pit and Gallows, *Fossa et Furca*; which was first instituted and granted by king *Malcolm*, who gave power to the Barons to have a Pit, wherein women condemned for theft should be drowned, and a Gallows, whereon men-thieves and offenders (guilty of capital offences) should be hanged, conformably to the judgments given in the court of the Baron concerning the said crimes." So that a Scotch baron, though no peer, was nevertheless a very considerable personage, both in dignity and power. *Reid's Essay on Logarithms*.—The name of the illustrious inventor of logarithms, and his family, has been variously written at different times, and on different occasions. In his own Latin works, and in (perhaps) all other books in Latin, it is *Neper*, or *Neperus Baro Merchistonii*: By Briggs, in a letter to Archbishop Usher, he is called *Naper, lord of Markinston*: In Wright's translation of the logarithms, which was revised by the author himself, and published in 1616, he is called *Nepair, baron of Marchiston*; and the same by Crawford and some others. But M'Kenzy and others write it *Napier, baron of Merchiston*; which, being also the orthography now used by the family, I shall adopt in this work. I observe also that the Scotch Compendium of Honour says he was only Sir John Napier; and that his son and heir, Archibald, was the first lord, being raised to that dignity in 1626. Be this however as it may, I shall conform to the common modes of expression, and call him indifferently *baron Napier* or *lord Napier*.

parts which shall be continually in the same proportion with the respective remainders of that line which had before been left : then are the several lengths of the first line the logarithms of the corresponding parts of the latter. Which description of them is similar to this, that the logarithms are a series of quantities or numbers in arithmetical progression, adapted to another series in geometrical progression. The first or whole length of the line, which is diminished in geometrical progression, he makes the radius of a circle, and its logarithm 0 or nothing, representing the beginning of the first or arithmetical line; and the several proportional remainders of the geometrical line, are the natural sines of all the other parts of the quadrant decreasing down to nothing, while the successive increasing values of the arithmetical line, are the corresponding logarithms of those decreasing sines ; so that, while the natural sines decrease from radius to nothing, their logarithms increase from nothing to infinite. Napier made the logarithm of radius to be 0, that he might save the trouble of adding and subtracting it in trigonometrical proportions, in which it so frequently occurred ; and he made the logarithms of the sines, from the entire quadrant down to 0, to increase, that they might be positive, and so in his opinion the easier to manage, the sines being of more frequent use than the tangents and secants, of which the whole of the latter, and half the former, would in his way be of a different affection from the sines : for it is evident that the logarithms of all the secants of the quadrant, and of all the tangents above 45° , or the half-quadrant, would be negative, being the logarithms of numbers greater than the radius whose logarithm is made equal to 0 or nothing.

As to the contents of Napier's table, it consists of the natural sines and their logarithms, for every minute of the quadrant. Like most other tables, the arcs are continued to 45 degrees from top to bottom, on the left-hand side of the pages ; and then returned backwards from bottom to top, on the right-hand side of the pages : so that the arcs and their complements, with the sines, natural and logarithmic, stand on the same line of the page, in six columns ; and in another column, in the middle of the page, are placed the differences between the logarithmic sines and cosines, on the same lines, and in the adjacent columns on the right and left ; thus making in all seven columns in each page. Of these columns, the first and seventh contain the arc and its complement, in degrees and minutes ; the second and sixth, the natural sine and cosine of each arc ; the third and fifth, the logarithmic sine and cosine ; and the fourth, or middle column, the difference between the logarithmic sine and cosine which are in the third and fifth columns.

To elucidate the description, the first page of the table is here inserted.

Gr. min.	o Sinus	Logarithmi	+ - Differentia	Logarithmi	Sinus	
0	0	Infinitum	Infinitum	0	10000000	60
1	2909	81425681	81425680	1	10000000	59
2	5818	74494213	74494211	2	9999998	58
3	8727	70439560	70439560	4	9999996	57
4	11636	67562746	67562739	7	9999993	56
5	14544	65331315	65331304	11	9999989	55
6	17453	63508099	63508083	16	9999984	54
7	20362	61966595	61966573	22	9999980	53
8	23271	60631284	60631256	28	9999974	52
9	26180	59453453	59453418	35	9999967	51
10	29088	58399857	58399814	43	9999959	50
11	31997	57446759	57446707	52	9999950	49
12	34906	56576646	56576584	62	9999940	48
13	37815	55776222	55776149	73	9999928	47
14	40724	55035148	55035064	84	9999917	46
15	43632	54345225	54345129	96	9999905	45
16	46541	53699843	53699734	109	9999892	44
17	49450	53093600	53093577	123	9999878	43
18	52359	52522019	52521881	138	9999863	42
19	55268	51981356	51981202	154	9999847	41
20	58177	51468431	51468361	170	9999831	40
21	61086	50980537	50980450	187	9999813	39
22	63995	50515342	50515137	205	9999795	38
23	66904	50070827	50070603	224	9999776	37
24	69813	49645239	49644995	244	9999756	36
25	72721	49237030	49236765	265	9999736	35
26	75630	48844826	48844539	287	9999714	34
27	78539	48467431	48467122	309	9999692	33
28	81448	48103763	48103431	332	9999668	32
29	84357	47752859	47752503	356	9999644	31
30	87265	47413852	47413471	381	9999619	30 min.

Gr. o

Besides the columns which are actually contained in this table, as above exhibited and described, namely, the natural and logarithmic sines, and the differences of these, the same table is made to serve also for the logarithmic tangents and secants of the whole quadrant, and for the logarithms of common numbers. For the fourth or middle column contains the logarithmic tangents, being equal to the differences between the logarithmic sines and cosines when the logarithm of radius is 0, because cosine : sine :: radius : tangent, that is, in logarithms, tangent = sine - cosine. Also the logarithmic sines made negative become the

the logarithmic cosecants, and the logarithmic cosines made negative are the logarithmic secants; because $\text{fine} : \text{radius} :: \text{radius} : \text{cosecant}$, and $\text{cosine} : \text{radius} :: \text{radius} : \text{secant}$; that is, in logarithms, $\text{cosecant} = 0 - \text{fine} = - \text{fine}$, and $\text{secant} = 0 - \text{cosine} = - \text{cosine}$. And to make it answer the purpose of a table of logarithms of common numbers, the author directs to proceed thus: A number being given, find that number in any table of natural sines, or tangents, or secants, and note the degrees and minutes in its arc; then in his table find the corresponding logarithmic sine, or tangent, or secant, to the same number of degrees and minutes; and it will be the required logarithm of the given number.

After his definitions and description of logarithms, Napier explains his table, and illustrates the precepts with examples, shewing how to take out the logarithms of sines, tangents, secants, and of common numbers; as also how to add and subtract logarithms. He then proceeds to teach the uses of those numbers; and first, in finding any of the terms of three or four proportionals, shewing how to multiply and divide, and to find powers and roots, by logarithms: 2dly, in trigonometry, both plane and spherical, but especially the latter, in which he is very explicit, turning all the theorems for every case into logarithms, computing examples to each in numbers, and then enumerating a set of astronomical problems of the sphere which properly belong to each case. Napier here teaches also some new theorems in spherical trigonometry; particularly that the tangent of half the base : tang. $\frac{1}{2}$ sum legs :: tang $\frac{1}{2}$ dif. legs : tang. $\frac{1}{2}$ the alternate base; and the general theorem for what are called his five circular parts, by which he condenses into one rule, in two parts, the theorems for all the cases of right-angled spherical triangles, which had been separately demonstrated by Pitiscus, Lansbergius, Copernicus, Regiomontanus, and others.

The description and use of Napier's Canon being in the Latin language, they were translated into English by Mr. Edward Wright, an ingenious mathematician, and inventor of the principles of what has commonly, though erroneously, been called Mercator's Sailing. He sent the translation to the author, at Edinburgh, to be revised by him before publication; who having carefully perused it, returned it with his approbation, and a few lines introduced besides into the translation. But, Mr. Wright dying soon after he received it back, it was after his death published, together with the tables, but each number to one figure less, in the year 1616, accompanied with a dedication, by his son Samuel Wright, to the East India Company; as also a preface by Henry Briggs, of whom we shall presently have occasion to speak more at large, on account of the great share he bore in perfecting the logarithms. In this translation Mr. Briggs gave also the description and draught of a scale that had been invented by Mr. Wright, and several other methods of his own, for finding the proportional parts to intermediate numbers, the logarithms having been only printed for such numbers as were the natural sines of each minute. And the note which baron Napier inserted in this English edition, and which was not in the original, was as follows: "But because the addition and subtraction of these former numbers may seem somewhat painful, I intend (if it shall please God) in a second edition to set out such logarithms as shall make those numbers above written to fall upon
" decimal

“ decimal numbers, such as 100000000, 200000000, 300000000, &c. which are
 “ easie to be added or abated to or from any other number.” This note had reference to the alteration of the scale of logarithms in such manner, that 1 should become the logarithm of the ratio of 10 to 1, instead of the number 2.3025851, which Napier had made that logarithm in his table; and which alteration had before been recommended to him by Briggs, as we shall see presently. Napier also inserted a similar remark in his *Rabdologia*, which he printed at Edinburgh in 1617.

The following is the preface to Wright's book, which, as far as where it mentions the change from the Latin into English, is a literal translation of the preface to Napier's original; but what follows that, is added by Napier himself: and I willingly insert it here, as it contains a declaration of the motives which led to this discovery, and as the book itself is very scarce. “ Seeing there is nothing (right well-beloved students in the mathematics) that is so troublesome to mathematicall practise, nor that doth more molest and hinder calculators, than the multiplications, divisions, square and cubical extractions of great numbers, which, besides the tedious expence of time, are for the most part subject to many slippery errors; I began therefore to consider in my minde, by what certaine and ready art I might remove those hindrances. And having thought upon many things to this purpose, I found at length some excellent briefe rules to be treated of (perhaps) hereafter. But amongst all, none more profitable than this, which together with the hard and tedious multiplications, divisions, and extractions of rootes, doth also cast away from the worke it selfe, even the very numbers themselves that are to be multiplied, divided, and resolved into rootes, and putteth other numbers in their place, which performe as much as they can do, onely by addition and subtraction, division by two, or division by three; which secret invention, being (as all other good things are) so much the better as it shall be the more common, I thought good heretofore to set forth in Latine, for the publique use of mathematicians. But now some of our countrymen in this Island, well affected to these studies, and the more publique good, procured a most learned mathematician to translate the same into our vulgar English tongue; who, after he had finished it, sent the cobby of it to me, to bee seene and considered on by myself. I having most willingly and gladly done the same, finde it to bee most exact and precisely conformable to my minde and the originall. Therefore it may please you, who are inclined to these studies, to receive it from me and the Translator, with as much good will as we recommend it unto you. Fare yee well.”

There are also extant copies of Wright's * translation, with the date 1618 in the title:

* Of this ingenious man I shall here insert in a note the following memoirs, as they have been translated from a Latin piece taken out of the annals of Gonville and Caius College in Cambridge, viz. “ This year (1615) died at London Edward Wright of Garveston in Norfolk, formerly a fellow of this college; a man respected by all for the integrity and simplicity of his manners, and also famous for his skill in the mathematical sciences: insomuch that he was deservedly stiled a most excellent mathematician by Richard Hackluyt, the author of an original-treatise of our English navigations. What knowledge he had acquired in the science of mechanics, and how usefully he employed that knowledge to the public as well as private advantage, abundantly appear both from the writings he published, and from the many mechanical operations still extant, which are standing monuments of his

title; but this is not properly a new edition, but only the old work with a new title page adapted to it (the old one being cancelled), together with the addition of sixteen pages of new matter, called "An Appendix to the Logarithms, shewing the practice of the calculation of triangles, and also a new and ready way for the exact finding out of such lines and logarithms as are not precisely to be found in the canons." But we are not told by what author; probably it was by Briggs.

Besides the trouble attending Napier's canon, in finding the proportional parts, when used as a table of the logarithms of common numbers, and which was in part remedied by the fore-mentioned contrivances of Wright and Briggs, it was also accompanied with another inconvenience, which arose from the logarithms being sometimes $+$ or additive, and sometimes $-$ or negative, and which required therefore the knowledge of algebraic addition and subtraction. And this inconvenience was occasioned partly by making the logarithm of radius to be 0, and the sines to decrease; and partly by the compendious manner in which the author had formed the table, making the three columns of sines, cosines, and tangents, to serve also for the other three of cosecants, secants, and cotangents.

But this latter inconvenience was well remedied by John Speidell in his *New Logarithms*, first published in 1619, which contained all the six columns, and in

his great industry and ingenuity. He was the first undertaker of that difficult but useful work, by which a little river is brought from the town of Ware, in a new canal, to supply the city of London with water; but by the tricks of others he was hindered from completing the work he had begun. He was excellent both in contrivance and execution; nor was he inferior to the most ingenious mechanic in the making of instruments, either of brass or any other matter. To his invention is owing whatever advantage Hondius's geographical charts have above others; for it was our Wright that taught Jodocus Hondius the method of constructing them, which was till then unknown: but the ungrateful Hondius concealed the name of the true author, and arrogated the glory of the invention to himself. Of this fraudulent practice the good man could not help complaining, and justly enough, in the preface to his treatise of the Correction of Errors in the Art of Navigation; which he composed with excellent judgment, and after long experience, to the great advancement of naval affairs. For the improvement of this art he was appointed Mathematical Lecturer by the East India Company, and read lectures in the house of that worthy knight Sir Thomas Smith, for which he had a yearly salary of 50 pounds. This office he discharged with great reputation, and much to the satisfaction of his hearers. He published in English a book on the doctrine of the sphere, and another concerning the construction of sun-dials. He also prefixed an ingenious preface to the learned Gilbert's book on the load-stone. By these, and other his writings, he has transmitted his fame to latest posterity. While he was yet a fellow of this college, he could not be concealed in his private study, but was called forth to the public business of the kingdom, by the queen's majesty, about the year 1593. He was ordered to attend the earl of Cumberland in some maritime expeditions. One of these he has given a faithful account of, in the way of a journal or ephemeris, to which he has prefixed an elegant hydrographical chart of his own contrivance. A little before his death he employed himself about an English translation of the Book of Logarithms, then lately found out by the honourable baron Napier, a Scotchman, who had a great affection for him. This posthumous work of his was published soon after, by his only son Samuel Wright, who was also a scholar of this college. He had formed many other useful designs, but was hindered by death from bringing them to perfection. Of him it may be truly said, that he studied more to serve the public than himself; and though he was rich in fame, and in the promises of the great, yet he died poor, to the scandal of an ungrateful age.* Other anecdotes of him, as well as of many other mathematical authors, may be found in the curious history of navigation, by Dr. James Wilson, prefixed to Mr. Robertson's excellent treatise on that subject.

this order—sines, cosines, tangents, cotangents, secants, cosecants: and they were besides made all positive, by being taken the arithmetical complements of Napier's, that is, they were the remainders left by subtracting each of these latter from 10,000,000. And the former inconvenience was more effectually removed, by the said Speidell, in an additional table, given in the sixth impression of the former work, in the year 1624. This was a table of Napier's logarithms for the round or integer numbers 1, 2, 3, 4, 5, &c, to 1000, together with their differences and arithmetical complements; as also the halves of the said logarithms, with their differences and arithmetical complements; which halves consequently were the logarithms of the square roots of the said numbers. These logarithms are however a little varied in their form from Napier's, namely, so as to increase from 1, whose logarithm is 0, instead of decreasing to 1, or radius, whose logarithm Napier made 0 likewise; that is, Speidell's logarithm of any number n , is equal to Napier's logarithm of its reciprocal $\frac{1}{n}$: so that in this last table of Speidell, the logarithm of 1 being 0, the logarithm of 10 is 2,302,584; the logarithm of 100 is twice as much, or 4,605,168; and that of 1000 thrice as much, or 6,907,753.

This table is now commonly called *hyberbolic* logarithms, because the numbers express the areas between the asymptote and curve of the hyperbola, those areas being limited by ordinates parallel to the other asymptote, the ordinates decreasing in geometrical progression. But this is an improper method of denominating them, as such areas may be made to denote any system of logarithms whatever, as we shall shew more at large in the proper place.

In the year 1619, Robert Napier, son of the inventor of logarithms, published a new edition of his late father's *Logarithmorum Canonis Descriptio*, together with the promised *Logarithmorum Canonis Constructio*, and other miscellaneous pieces written by his father and Mr. Briggs.—Also one Bartholomew Vincent, a bookseller at Lugdunum, or Lyons, in France, printed there an exact copy of the same two works in one volume, in the year 1620; which was four years before the logarithms were carried to France by Wingate, who was therefore erroneously said to have first introduced them into that country. But I shall treat more particularly of the contents of this work after I have enumerated the other writers on this sort of logarithms.

In 1618 or 1619, Benjamin Ursinus, mathematician to the Elector of Brandenburg, published, at Cologne, his *Cursus Mathematicus*, in which is contained a copy of Napier's logarithms, with the addition of some tables of proportional parts. And in 1624 he printed, at the same place, his *Trigonometria*, with a table of natural sines and their logarithms, of the Napierian kind and form, to every ten seconds in the quadrant; which he had been at much pains in computing.

In the same year, 1624, logarithms of nearly the same kind were also published at Marpurg, by the famous John Kepler, mathematician to the Emperor Ferdinand the Second, under the title of *Cebilias Logarithmorum ad totidem Numeros Rotundos; præmissâ Demonstratione legitimâ Ortus Logarithmorum, eorumque Usus, &c.* and, the year following, a supplement to the same; being applied to round or integer numbers, and to such natural sines as nearly coincide with them.

them. These are exactly the same sort of logarithms as Napier's, being the same logarithms of the natural sines of arcs beginning from the quadrant, whose sine or radius is 10,000,000, the logarithm of which is made 0; and from thence the sines decreasing by equal differences down to 0, or the beginning of the quadrant, whilst their logarithms increase to infinity: so that the difference between this table and Napier's consists only in this, namely, that in Napier's table the *arc* of the quadrant is divided into equal parts, differing by one minute each, and consequently their sines, to which the logarithms are adapted, are irrational or interminate numbers, and only expressed by approximate decimals; whereas, in Kepler's table, the *radius* is divided into equal parts, which are considered as perfect and terminate sines, having equal differences, and to which terminate sines the logarithms are here adapted. By this means indeed the proportions for intermediate numbers and logarithms are easier made; but then the corresponding arcs are not terminate, but irrational, and only set down to an approximate degree: so that Kepler's table is more convenient as a table of the logarithms of common numbers, and Napier's as the logarithmic sines of the arcs of the quadrant. In both tables the logarithm of the ratio of 10 to 1 is the same quantity, namely, 23,025,852; and as the radius, or greatest sine, is 10,000,000, whose logarithm is made 0, the logarithms of the decuple parts of it will be found by adding 23,025,852 continually, or multiplying this logarithm by 2, 3, 4, &c. and hence the logarithm of 1, the first number, or smallest sine, in the table, is 161,180,959, or 7 times 2302, &c.

Besides the two columns of the natural sines and their logarithms, with the differences of the logarithms, this table of Kepler consists also of three other columns; the first of which contains the nearest arcs belonging to those sines, expressed in degrees, minutes, and seconds; and the other two express what parts of the radius each sine is equal to, namely, the one of them in 24th parts of the radius, and minutes and seconds of them; and the other in 60th parts of the radius, and minutes of them. As a specimen I have here extracted the last page of the table, printed exactly as in the work.

ARCUS Circuli, cum differentiis.	SINUS, feu numeri absoluti.	Partes vice- simæ-quartæ.	LOGARITHMI cum differentiis	Partes sexagena- riæ
19. 34			101.58	
80. 3. 46	98500.00	23. 38. 24	1511.36 +	59. 6
20. 12			101.47	
80. 23. 58	98600.00	23. 39. 50	1409.89 +	59. 10
20. 53			101.37	
80. 44. 51	98700.00	23. 41. 17	1308.52 +	59. 13
21. 42			101.26	
81. 6. 33	98800.00	23. 42. 43	1207.26	59. 17
22. 53			101.17	
81. 29. 26	98900.00	23. 44. 10	1106.09 +	59. 20
24. 6			101.06	
81. 53. 32	99000.00	23. 45. 36	1005.03 +	59. 24
25. 6			100.96	
82. 18. 38	99100.00	23. 47. 2	904.07 +	59. 28
26. 28			100.85	
82. 45. 6	99200.00	23. 48. 29	803.22 +	59. 31
27. 54			100.76	
83. 13. 0	99300.00	23. 49. 55	702.46	59. 35
30. 20			100.65	
83. 43. 20	99400.00	23. 51. 22	601.81	59. 38
32. 40			100.56	
84. 16. 0	99500.00	23. 52. 48	501.25 +	59. 42
36. 30			100.45	
84. 52. 30	99600.00	23. 54. 14	400.80	59. 46
41. 9			100.35	
85. 33. 39	99700.00	23. 55. 41	300.45	59. 49
48. 54			100.25	
86. 22. 33	99800.00	23. 57. 7	200.20	59. 53
1. 3. 42			100.15	
87. 26. 15	99900.00	23. 58. 34	100.05	59. 56
2. 33. 45			100.05	
90. 0. 0	100000.00	24. 0. 0	000000.00.	60. 0

To the table Kepler prefixes a pretty considerable tract, containing the construction of the logarithms, and a demonstration of their properties and structure, in which he considers logarithms, in the true and legitimate way, as the measures of ratios; as shall be shewn more particularly hereafter, in the next part, where I shall treat of the construction of logarithms.

Kepler also introduced the logarithmic calculus into his Rudolphine tables, published in 1627, and inserted in that work several logarithmic tables; as, first, a table similar to that above described, except that the second, or column of sines, or of absolute numbers, is omitted, and instead of it another column is added, shewing what part of the quadrant each arc is equal to, namely the quotient, expressed in integers and sexagesimal parts, arising from the dividing the whole quadrant by each given arc; 2dly, Napier's table of logarithmic sines to every minute of the quadrant; also two other smaller tables adapted for the purposes

purposes of eclipses and the latitudes of the planets. In this work also Kepler gives a summary account of logarithms, with the description and use of those that are contained in these tables: and here it is that he mentions Justus Byrgius, as having had logarithms before Napier published them.

Besides the above, some few others published logarithms of the same sort about this time.—But let us now return to treat of the history of the common or Briggs's logarithms, so called because he first computed them, and first mentioned them, and recommended them to Napier, instead of the first sort by him invented.

Mr. Henry Briggs, not less esteemed for his great probity and other eminent virtues than for his excellent skill in mathematics, was, at the time of the publication of Napier's logarithms, in 1614, professor of geometry in Gresham college in London, having been appointed the first professor after its institution; which appointment he held till January 1620, when he was chosen also the first Savilian professor of geometry at Oxford, where he died January the 26th, 1630, aged about 74 years.

On the publication of Napier's logarithms, Briggs immediately applied himself to the study and improvement of them. In a letter to Mr. (afterwards archbishop) Usher, dated the 10th of March 1615, he writes "that he was wholly taken up and employed about the noble invention of logarithms lately discovered:" and again, "Napier, lord of Markinston, hath set my head and hands at work with his new and admirable logarithms; I hope to see him this summer, if it please God; for I never saw a book which pleased me better, and made me more wonder." Thus we find that Briggs began very early to compute logarithms: but these were not of the same kind with Napier's, in which the logarithm of the ratio of 10 to 1 was 2.302,585,1, &c; for in Briggs's first attempt he made 1 the logarithm of that ratio; and, from the evidence we have, he appears to be the first person who formed the idea of this change in the scale, which he presently and generously communicated both to the public in his lectures, and to lord Napier himself, who afterwards said that he also had thought of the same thing; as appears by the following extract, translated from the preface to Briggs's *Arithmetica Logarithmica*: "Wonder not (says he) that these logarithms are different from those which the excellent baron of Marchiston published in his Admirable Canon. For when I explained the doctrine of them to my auditors at Gresham College in London, I remarked that it would be much more convenient, the logarithm of the sine total or radius being 0 (as in the *Canon Mirificus*), if the logarithm of the 10th part of the said radius, namely of $5^{\circ} 44' 21''$, were 100000 &c. and concerning this I presently wrote to the author; also as soon as the season of the year, and my public teaching, would permit, I went to Edinburgh, where being kindly received by him, I staid a whole month. But when we began to converse about the alteration of them, he said that he had formerly thought of it, and wished it; but that he chose to publish those that were already done, till such time as his leisure and health would permit him to make others more convenient. And as to the nature of the change, he thought it more expedient that 0 should be made the logarithm of 1, and 100000, &c, the logarithm of radius, which
I could

I could not but acknowledge was much better. Therefore rejecting those which I had before prepared, I proceeded at his exhortation to calculate these; and the next summer I went again to Edinburgh, to shew him the principal of them; and should have been glad to do the same the third summer, if it had pleased God to spare him so long."

So that it is plain that Briggs was the inventor of the present scale of logarithms, in which 1 is the logarithm of the ratio of 10 to 1, and 2 that of 100 to 1, &c. and that the share which Napier had in them was only advising Briggs to begin at the lowest number 1, and make the logarithms, or artificial numbers, as Napier had also called them, to *increase* with the natural numbers instead of *decreasing*; which made no alteration in the figures that expressed Briggs's logarithms, but only in their affection or signs, changing them from negative to positive; so that Briggs's first logarithms to the numbers in the second column of the annexed tablet, would have been as in the first column; but, after they were changed, as they are here in the third column; which is a change of no essential difference, as the logarithm of the ratio of 10 to 1, the radix of the natural system of numbers, continues the same, a change in the logarithm of that ratio being the only circumstance that can essentially alter the system of logarithms, the logarithm of 1 being 0. And the reason why Briggs, after that interview, rejected what he had before done, and began anew, was probably because he had adapted his new logarithms to the approximate sines of arcs, instead of the round or integer numbers; and not from their being logarithms of another system, as were those of Napier.

B	Num.	N
n	10^n	$-n$
3	.001	-3
2	.01	-2
1	.1	-1
0	1	0
-1	10	1
-2	100	2
-3	1000	3
$-n$	10^n	n

On Briggs's return from Edinburgh to London the second time, namely in 1617, he printed the first thousand logarithms, to eight places of figures besides the index, under the title of *Logarithmorum Cbiliarum Prima*. But these seem not to have been published till after the death of Napier, which happened on the third of April 1618, as before said; for in the preface to them Briggs says, "why these logarithms differ from those set forth by their most illustrious inventor, of ever respectful memory, in his *Canon Mirificus*, IT IS TO BE HOPED his posthumous work will shortly make appear." And as Napier, after communication had with Briggs on the subject of altering the scale of logarithms, had given notice, both in Wright's translation and in his own *Rabdologia*, printed in 1617, of his intention to alter the scale (though it appears very plainly that he never intended to compute any more), without making any mention of the share which Briggs had in the alteration, this gentleman modestly gave the above hint. But not finding any regard paid to it in the said posthumous work, published by lord Napier's son in 1619, where the alteration is again adverted to, but still without any mention of Briggs; this gentleman thought he could not do less than state the grounds of that alteration himself, as they are above extracted from his work published in 1624.

Thus, upon the whole matter, it seems evident that Mr. Briggs, whether he

he had thought of this improvement in the construction of logarithms, of making 1 the logarithm of the ratio of 10 to 1, before lord Napier, or not, (which is a secret that could be known only to Napier himself), was the first person who communicated the idea of such an improvement to the world; and that he did this in his lectures to his auditors at Gresham College, in the year 1615, very soon after his perusal of Napier's *Canon Mirificus Logarithmorum* in the year 1614. He also mentioned it to Napier, both by letter in the same year, and on his first visit to him in Scotland in the summer of the year 1616; when Napier approved the idea, and said it had already occurred to himself, and that he had determined to adopt it. It would therefore have been more candid in lord Napier to have told the world, in the second edition of this book, that Mr. Briggs had mentioned this improvement to him, and that he had thereby been confirmed in the resolution he had already taken, before Mr. Briggs's communication with him, to adopt it in that his second edition, as being better fitted to the decimal notation of arithmetic which was in general use. Such a declaration would have been but a piece of justice to Mr. Briggs; and the not having made it, cannot but incline us to suspect that lord Napier was desirous that the world should ascribe to him alone the merit of this very useful improvement of the logarithms, as well as that of having originally invented them; though, if the having first communicated an invention to the world be sufficient to entitle a man to the honour of having first invented it, Mr. Briggs had the better title to be called the first inventor of this happy improvement of logarithms.

In 1620, two years after the *Cblias Prima* of Briggs came out, Mr. Edmund Gunter published his *Canon of Triangles*, which contains the artificial or logarithmic sines and tangents, for every minute, to seven places of figures, besides the index, the logarithm of radius being 10.0 &c. These logarithms are of the kind last agreed upon by Napier and Briggs, and they were the first tables of logarithmic sines and tangents that were published of this sort. Gunter also in 1623 reprinted the same in his book *De Sector et Radio*, together with the *Cblias Prima* of his old colleague Mr. Briggs, he being professor of Astronomy at Gresham College when Briggs was professor of Geometry there; Gunter having been elected to that office the 6th of March 1619, and enjoyed it till his death, which happened on the 10th of December 1626, about the forty-fifth year of his age. In 1623 also Gunter applied these logarithms of numbers, sines, and tangents, to straight lines drawn on a ruler; with which proportions in common numbers and trigonometry were resolved by the mere application of a pair of compasses; a method founded on this property, that the logarithms of the terms of equal ratios are equidifferent. This instrument, in the form of a two-foot scale, is now in common use for navigation and other purposes, and is commonly called the Gunter. He also greatly improved the sector for the same uses. Gunter was the first who used the word *co-sine* for the sine of the complement of an arc. He also introduced the use of arithmetical complements into the logarithmical arithmetic, as is witnessed by Briggs, chap. xv. Arith. Log. And it has been said that he started the idea of the logarithmic

logarithmic curve, which was so called because the segments of its axis are the logarithms of the corresponding ordinates.

The logarithmic lines were afterwards drawn in various other ways. In 1627 they were drawn by Wingate, on two separate rulers, sliding against each other, to save the use of compasses in resolving proportions. They were also in 1627 applied to concentric circles, by Oughtred. Then in a spiral form by a Mr. Milburne of Yorkshire, about the year 1650. And lastly, in 1657, on the present sliding rule, by Seth Partridge.

The discoveries relating to logarithms were carried to France by Mr. Edmund Wingate; but not first of all, as he erroneously says in the preface to his book. He published at Paris, in 1624, two small tracts in the French language; and afterwards at London, in 1626, an English edition of the same, with improvements. In the first of these he teaches the use of Gunter's ruler; and in the other that of Briggs's logarithms, and the artificial sines and tangents. Here are contained also tables of those logarithms, sines, and tangents, copied from Gunter. The edition of these logarithms printed at London in 1635, and the former editions also I suppose, have the units figures disposed along the tops of the columns, and the tens down the margins, like our tables at present; but the whole logarithm, which was only to six places of figures, in the angle of meeting: which is the first instance that I have seen of this mode of arrangement.

But proceed we now to the larger structure of logarithms.

Briggs had continued, from the beginning, to labour with great industry at the computation of those logarithms of which he before published a short specimen in small numbers: and in 1624 he produced his *Arithmetica Logarithmica*, a stupendous work for so short a time! containing the logarithms of 30000 natural numbers, to fourteen places of figures besides the index, namely from 1 to 20000, and from 90000 to 100000; together with the differences of the logarithms. Some writers say that there was another *chiliad*, namely from 100000 to 101000; but none of the copies that I have seen have more than the 30000 above mentioned, and they were all regularly terminated in the usual way with the word FINIS. The preface to these logarithms contains, among other things, an account of the alteration made in the scale by Napier and himself, from which we before gave an extract; and an earnest solicitation to others to undertake the computation for the intermediate numbers, offering to give instructions, and paper ready ruled for that purpose, to any persons so inclined to contribute to the completion of so valuable a work. In the introduction he gives also an ample treatise on the construction and uses of these logarithms, which will be particularly described hereafter.—By this invitation, and other means, he had hopes of collecting materials for the logarithms of the intermediate 70000 numbers, whilst he should employ his own labour more immediately upon the canon of logarithmic sines and tangents, and so carry on both works at once; as indeed they were both equally necessary, and he himself was now pretty far advanced in years.

Soon after this Adrian Vlacq, or Flack, of Gouda in Holland, completed the intermediate seventy chiliads, and republished the *Arithmetica Logarithmica*

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at that place, in 1627 and 1628, with those intermediate numbers, making in the whole the logarithms of all numbers to 10000, but only to ten places of figures. To these was added a table of artificial sines, tangents, and secants, to every minute of the quadrant.

Briggs himself lived also to complete a table of logarithmic sines and tangents for the hundredth part of every degree, to fourteen places of figures besides the index; together with a table of natural sines for the same parts to fifteen places, and the tangents and secants for the same to ten places; with the construction of the whole. These tables were printed at *Gouda*, under the care of Adrian Vlacq, and mostly finished off before 1631, though not published till 1633. But his death, which then happened, prevented him from completing the application and uses of them. However, the performing of this office, when dying, he recommended to his friend Henry Gellibrand, who was then Professor of Astronomy in Gresham College, having succeeded Mr. Gunter in that appointment. Gellibrand accordingly added a preface, and the application of the logarithms to plane and spherical trigonometry, &c. and the whole was printed at Gouda, by the same printer, and brought out in the same year, 1633, as the *Trigonometria Artificialis* of Vlacq, who had the care of the press, as abovesaid. This work was called *Trigonometria Britannica*; and besides the arcs in degrees, and centesims of degrees, it has another column containing the minutes and seconds, answering to the several centesims in the first column.

In 1633, as mentioned above, Vlacq printed, at Gouda in Holland, his *Trigonometria Artificialis, sive Magnus Canon Triangulorum Logarithmicus ad Decadas Secundorum Scrupulorum constructus*. This work contains the logarithmic sines and tangents to 10 places of figures, with their differences, for every 10 seconds in the quadrant. To them is also added Briggs's table of the first 20000 logarithms, but carried only to 10 places of figures besides the index, with their differences. The whole is preceded by a description of the tables, and the application of them to plane and spherical trigonometry, chiefly extracted from Briggs's *Trigonometria Britannica*, mentioned above.

Gellibrand published also, in 1635, *An Institution Trigonometricall*, containing the logarithms of the first 10000 numbers, with the natural sines, tangents, and secants, and the logarithmic sines and tangents, for degrees and minutes, all to seven places of figures, besides the index; as also other tables proper for navigation; with the uses of the whole. Gellibrand died the 9th of February 1636, in the 40th year of his age, to the great loss of the mathematical world.

Besides the persons hitherto mentioned, who were mostly computers of logarithms, many others have also published tables of those artificial numbers, more or less complete, and sometimes improved and varied in the manner and form of them. I shall here just advert to a few of the principal.

In 1626, D. Henrion published, at Paris, a treatise concerning Briggs's logarithms of common numbers from 1 to 20000, to eleven places of figures; with the sines and tangents to eight places only.

In 1631 was printed at London, by one George Miller, a book containing Briggs's logarithms, with their differences, to ten places of figures, besides the

index, for all numbers to 100000; as also the logarithmic fines, tangents, and secants for every minute of the quadrant; with the explanation and uses in English.

The same year, 1631, Richard Norwood published his *Trigonometrie*; in which we find Briggs's logarithms for all numbers to 10000, and for the fines, tangents, and secants to every minute, both to seven places, besides the index.—In the conclusion of the trigonometry, he complains of the unfair practices of printing Vlacq's book in 1627 or 1628, and the book mentioned in the last article. His words are, "Now whereas I have here, and in sundry places in this book, cited Mr. Briggs his *Arithmetica Logarithmica* (lest I may seem to abuse the reader), you are to understand not the book put forth about a month since in English, as a translation of his, and with the same title; being nothing like his, nor worthy his name; but the book which himself put forth with this title in Latin, being printed at London, anno 1624. And here I have just occasion to blame the ill dealing of these men, both in the matter before-mentioned, and in printing a second edition of his *Arithmetica Logarithmica* in Latin, whilst he lived, against his mind and liking; and brought them over to sell when the first were unfold; so frustrating those additions which Mr. Briggs intended in his second edition, and moreover leaving out some things that were in the first edition of special moment. A practice of very ill consequence, and tending to the great disparagement of such as take pains in this kind."

Francis Bonaventure Cavalierius published at Bologna, in 1632, his *Directorium Generale Uranometricum*, in which are tables of Briggs's logarithms of fines, tangents, secants, and versed fines, each to eight places, for every second of the first five minutes, for every five seconds from five to ten minutes, for every ten seconds from ten to twenty minutes, for every twenty seconds from twenty to thirty minutes, for every thirty seconds from 30' to 1° 30', and for every minute in the rest of the quadrant; which is the first table of logarithmic versed fines that I know of. In this book are contained also the logarithms of the first ten chiliads of natural numbers, namely from 1 to 10000, disposed in this manner, all the twenties at top, and from 1 to 19 on the side, the logarithm of the sum being in the square of meeting. In this work also I think Cavalierius first gave the method of finding the area or spherical surface contained by various arcs described on the surface of a sphere.

Also in the *Trigonometria* of the same author, printed in 1643, besides the logarithms of numbers from 1 to 1000, to eight places, with their differences, we find both natural and logarithmic fines, tangents, and secants, the former to seven and the latter to eight places; namely, to every 10" of the first 30 minutes, to every 30" from 30' to 1°; and the same for their complements, or backwards through the last degree of the quadrant; the intermediate 88° being to every minute only.

Mr. Nathaniel Roe, "Pastor of Benacre in Suffolke," also reduced the logarithmic tables to a contracted form, in his *Tabulæ Logarithmicæ*, printed at London in 1633. Here we have Briggs's logarithms of numbers from 1 to 100000, to eight places; the fifties placed at top, and from 1 to 50 on the

side; also the first four figures of the logarithms at top, and the other four down the columns. They contain also the logarithmic sines and tangents to every 100th part of the several successive degrees, to ten places.

Ludovicus Frobenius published at Hamburgh, in 1634, his *Clavis Universa Trigonometriae*, containing tables of Briggs's logarithms of numbers from 1 to 2000; and of sines, tangents, and secants, for every minute; both to seven places.

But the table of logarithms of common numbers was reduced to its most convenient form by John Newton, in his *Trigonometria Britannica*, printed at London in 1658, having availed himself of both the improvements of Wingate and Roe, namely, uniting Wingate's disposition of the natural numbers with Roe's contracted arrangement of the logarithms; the numbers being all disposed as in our best tables at present, namely, the units along the top of the page, and the tens down the left-hand side; also the first three figures of each logarithm in the first column, and the remaining five figures in the other columns, the logarithms being to eight places. This work contains also the logarithmic sines and tangents, to eight figures besides the index, for every 100th part of a degree, with their differences, and for 1000th parts in the first three degrees.—In the preface to this work, Newton takes occasion, as Wingate and Norwood had done before, as well as Briggs himself, to censure the unfair practices of some other publishers of logarithms. He says, “In the second part of this institution, thou art presented with Mr. Gellibrand's *Trigonometrie*, faithfully translated from the Latin copy, that which the author himself published under the title of *Trigonometria Britannica*, and not that which Vlacq the Dutchman styles *Trigonometria Artificialis*, from whose corrupt and imperfect copy that seems to be translated, which is amongst us generally known by the name of *Gellibrand's Trigonometry*; but those who either knew him, or have perused his writings, can testify that he was no admirer of the old sexagenary way of working; nay, that he did preferre the decimal way before it, as he hath abundantly testified in all the examples of this his *Trigonometry*, which differs from that other which Vlacq hath published, and that which hath hitherto borne his name in English, as in the form, so likewise in the matter of it; for in the two last-mentioned editions there is something left out in the second chapter of plain triangles, the third chapter wholly omitted, and a part of the third in the spherical, but in this edition nothing; something we have added to both, by way of explanation and demonstration.”

In 1670, John Caramuel published his *Mathesis Nova*, in which are contained 1000 logarithms both of Napier's and Briggs's form, as also 1000 of what he calls the Perfect Logarithms, namely, the same as those which Briggs first thought of; which differ from the last only in this, that the one increases while the other decreases; the radix, or logarithm of the ratio of 10 to 1, being the same in both.

The books of logarithms have since become very numerous; but the logarithms are mostly of that sort invented by Briggs, and which are now in common use. Of these the most noted for their accuracy or usefulness, besides the works above mentioned, are Vlacq's small volume of tables, particularly

that edition printed at Lyons in 1670; also tables printed at the same place in 1760; but most especially the tables of Sherwin and Gardiner. Of these, Sherwin's *Mathematical Tables* in 8vo. form the most complete collection of any; containing, besides the logarithms of all numbers to 101,000, the sines, tangents, secants, and versed sines, both natural and logarithmic, to every minute of the quadrant. The first edition was in 1706; but the third edition, in 1742, which was revised by Gardiner, is esteemed the most correct of any: as to the last or fifth edition, in 1771, it is so erroneously printed, that no dependence can be placed in it, and it is the most inaccurate book of tables I ever knew; I have a list of several thousand errors which I have corrected in it.

Gardiner also printed at London, in 1742, a quarto volume of "Tables of Logarithms, for all numbers from 1 to 102,100, and for the sines and tangents to every ten seconds of each degree in the quadrant; as also, for the sines of the first 72 minutes to every single second: with other useful and necessary tables;" namely, a table of Logistical Logarithms, and three smaller tables to be used for finding the logarithms of numbers to twenty places of figures. Of these tables of Gardiner, only a small number was printed, and that by subscription; and they are now in the highest estimation of any logarithms for their accuracy and usefulness.

An edition of Gardiner's collection was also elegantly printed at Avignon in France, in 1770, with some additions, namely, the sines and tangents for every single second in the first four degrees; and a small table of hyperbolic logarithms, copied from a treatise on Fluxions by the late ingenious Mr. Thomas Simpson: but this is not quite so correct as Gardiner's own edition. The tables in all these books are to seven places of figures.

"The logarithmic canon serves to find readily the logarithm of any assigned number; and we are told by Dr. Wallis, in the second volume of his *Mathematical Works*, that an antilogarithmic canon, or one to find as readily the number corresponding to every logarithm, was begun he thinks by Mr. Harriot the algebraist (who died in 1621), and completed by Mr. Walter Warner, the editor of Harriot's works, before 1640; which ingenious performance it seems was lost, for want of encouragement to publish it."

"A small specimen of such numbers was published in the *Philosophical Transactions*, for the year 1714, by Mr. Long of Oxford; but it was not till 1742 that a complete antilogarithmic canon was published, by Mr. James Dodson, wherein he has computed the numbers corresponding to every logarithm from 1 to 100,000, to 11 places of figures."

Since the preceeding account was written, and whilst it was in the press, there has been printed at Paris, "*Tables Portatives de Logarithmes, publiées à Londres par Gardiner*," &c. This work is most beautifully printed in a neat portable 8vo. volume, and contains all the tables in Gardiner's 4to. volume, with some additions and improvements; but with what degree of accuracy remains yet to be determined. And on this, as well as several other occasions, it is but justice to remark the extraordinary spirit and elegance with which the learned men and the artisans of the French nation undertake and execute works of merit.

THE

THE CONSTRUCTION OF LOGARITHMS, &c.

HAVING described the several sorts of logarithms, their rise and invention, their nature and properties, and given some account of the principal early cultivators of them, with the chief collections that have been published of such tables; I proceed now to deliver a more particular account of the ideas and methods employed by each author, and the peculiar modes of construction which they made use of.

And first of the great inventor himself, lord Napier.

Napier's Construction of Logarithms.

The inventor of logarithms did not adapt them to the series of natural numbers 1, 2, 3, 4, 5, &c, as it was not his principal idea to extend them to all arithmetical operations whatever; but he confined his labours to that circumstance which first suggested the necessity of the invention, and adapted his logarithms to the approximate numbers expressing the natural sines of every minute in the quadrant, as they had been set down by former writers on trigonometry.

The same restricted idea was pursued through his method of constructing the logarithms. As the lines of the sines of all arcs are parts of the radius or sine of the quadrant, therefore called the *sinus totus* or whole sine, he conceived the line of the radius to be described, or run over, by a point moving along it in such manner, that in equal portions of time it generated, or cut off, parts in a decreasing geometrical progression, leaving the several remainders, or sines, in geometrical progression also; whilst another point, in an indefinite line, described equal parts of it in the same equal portions of time; so that the respective sums of these, or the whole line generated, were always the arithmeticals or logarithms of those sines. Thus, *az* is the given radius upon which all the sines are to be taken, and *A* &c the indefinite line containing the logarithms; these lines being each generated by the motion of a point, beginning at *A*, *a*. Now at the end of the 1st, 2d, 3d, &c, moments, or equal small portions of time, let the moving points be found at the places marked 1, 2, 3, &c; then *za*, *z1*, *z2*, *z3*, &c, will be the series of natural sines, and *Ao* (or *o*), *A1*, *A2*, *A3*, &c, will be their logarithms; supposing the point which generates *az* to move every where with a velocity decreasing in proportion to its distance from *z*, namely, its velocity in the points *o*, 1, 2, 3, &c, to be respectively as the distances *zo*, *z1*, *z2*, *z3*, &c; whilst the velocity of the point generating the logarithmic line *A* &c remains constantly the same as at first in the point *A* or *o*.

Sines.	Log.
<i>a</i> or <i>o</i>	<i>A</i> or <i>o</i>
-1	-1
-2	-2
-3	-3
-4	-4
-5	-5
-6	-6
-7	-7
&c	&c

Hitherto

Hitherto the author had not fully limited his system or scale of logarithms, having only supposed one condition or limitation, namely, that the logarithm of the radius az should be 0. But two independent conditions, no matter what they are, were necessary to limit the scale, or system, of logarithms. It did not occur to him, that it was proper to form the other limit by affixing some particular logarithm to an assigned number, or part of the radius: but as another condition was necessary, he assumed *this* for it, namely, that the two generating points should begin to move at a, A , with equal velocities; or that the increments a_1, A_1 , described in the first moments, should be equal; as he thought this circumstance would be attended with some little ease in the computation: and this is the reason that, in his table, the natural sines and their logarithms, at the complete quadrant, have equal differences; and this is also the reason why his scale of logarithms happens accidentally to agree with what have since been called the hyperbolic logarithms, which have numerical differences equal to those of their natural numbers at the beginning; except only that these latter increase with the natural numbers, and his on the contrary decrease; the logarithms of the ratio of 10 to 1 being the same in both, namely, 2.302,585,09.

And here by the way it may be observed, that Napier's manner of conceiving the generation of the lines of the natural numbers and their logarithms, by the motion of points, is very similar to the manner in which Newton afterwards considered the generation of magnitudes in his doctrine of fluxions; and it is also remarkable that, in Art. 2. of the *Habitudines Logarithmorum & suorum naturalium numerorum invicem*, in the appendix to the *Constructio Logarithmorum*, Napier speaks of the velocities of the increments or decrements of the logarithms, in the same way as Newton does of his fluxions, namely, where he shews that those velocities, or fluxions, are inversely as the sines, or natural numbers of the logarithms; which is a necessary consequence of the nature of the generation of those lines, as described above; with this alteration, however, that now the radius az must be considered as generated by an equable motion of the point, and the indefinite line A &c by a motion increasing in the same ratio as the other before decreased; which is a supposition that Napier must have had in view when he stated that relation of the fluxions.

Having thus limited his system, Napier proceeds, in the posthumous work of 1619, to explain his construction of the logarithmic canon: and this he effects in various ways; but chiefly by generating, in a very easy manner, a series of proportional numbers and their arithmeticals, or logarithms; and then finding, by proportion, the logarithms to the natural sines, from those of the nearest numbers among the original proportionals.

After describing the necessary cautions he made use of to preserve a sufficient degree of accuracy, in so long and complex a process of calculation; such as annexing several cyphers, as decimals separated by a point, to his primitive numbers, and rejecting the decimals thence resulting after the operations were completed, setting the numbers down to the nearest unit in the last figure; and teaching the arithmetical processes of adding, subtracting, multiplying, and dividing the limits between which certain unknown numbers must lie, so

as

as to obtain the limits between which the results must also fall; I say, after describing such particulars, in order to clear and smooth the way, he enters on the great field of calculation itself. Beginning at radius 10,000,000, he first constructs several descending geometrical series, but of such a nature that they are all quickly formed by an easy continual subtraction, and a division by 2, or by 10, or 100, &c, which is done by only removing the decimal point so many places towards the left hand, as there are cyphers in the divisor. He constructs three tables of such series: The first of these consists of 100 numbers, in the proportion of radius to radius minus 1, or of 10,000,000 to 9,999,999; all which are found by only subtracting from each its 10,000,000th part, which part is also found by only removing each figure 7 places lower: the last of these 100 proportionals is found to be 9,999,900.000,495,0.

The 2d table contains 50 numbers, which are in the continual proportion of the first to the last in the first table, namely, of 10,000,000.000,000,0 to 9,999,900.000,495,0, or nearly the proportion of 100,000 to 99,999; these therefore are found by only removing the figures of each number 5 places lower, and subtracting them from the same number: the last of these he finds to be 9,995,001.222,927. And a specimen of these two tables is here annexed.

No.	First Table.	2d Table.
1	10,000,000.000,000,0	10,000,000.000,000
2	9,999,999.000,000,0	9,999,900.000,000
3	9,999,998.000,000,1	9,999,800.001,000
4	9,999,997.000,000,3	9,999,700.003,000
&c	&c till the 100th	&c to the 50th term
50	term, which will be	9,995,001.222,927
100	9,999,900.000,495,0	

The 3d table consists of 69 columns, and each column of twenty-one numbers or terms; which terms, in every column, are in the continual proportion of 10,000 to 9,995, that is, nearly as the first is to the last in the 2d table; and as 10,000 exceeds 9,995 by the 2000th part, the terms in every column will be constructed by dividing each upper number by 2, removing the figures of the quotient 3 places lower, and then subtracting them; and in this way it is proper to construct only the first column of 21 numbers, the last of which will be 9,900,473.578,0: but the 1st, 2d, 3d, &c, numbers in all the columns, are in the continual proportion of 100 to 99, or nearly the proportion of the first to the last in the first column; and therefore these will be found by removing the figures of each preceeding number 2 places lower, and subtracting them, for the like number in the next column. A specimen of this 3d table is as here below.

The 3d Table.					
Terms	1st Column.	2d Column.	3d Column.	&c. till the	69 Column.
1	10,000,000.000,0	9,900,000.000,0	9,801,000.000,0	&c for the	5,048,858.890,0
2	9,995,000.000,0	9,895,050.000,0	9,796,099.500,0	4th, 5th, 6th,	5,046,334.460,5
3	9,990,002.500,0	9,890,102.475,0	9,791,201.450,3	7th, &c, col.	5,043,811.293,2
4	9,985,007.498,7	9,885,157.423,7	9,786,305.849,5	till the last	5,041,289.387,9
5	9,980,014.995,0	9,880,214.845,1	9,781,412.696,7	or	5,038,768.743,5
&c	&c till	&c	&c		&c
21	9,900,473.578,0	9,801,468.842,3	9,703,454.153,9		4,998,609.403,4

Thus

Thus he had, in this 3d table, interposed between the radius and its half, 68 numbers in the continual proportion of 100 to 99; and interposed between every two of these 20 numbers in the proportion of 10,000 to 9,995: and again, in the 2d table, between 10,000,000 and 9,995,000, the two first of the 3d table, he had 50 numbers in the proportion of 100,000 to 99,999: And lastly, in the 1st table, between 10,000,000 and 9,999,900, or the 2 first of the 2d table, 100 numbers in the proportion of 10,000,000 to 9,999,999. That is, in all about 1600 proportionals; all found in the most simple manner by little more than easy subtractions; which proportionals nearly coincide with all the natural sines from 90° down to 30° .

To obtain the logarithms of all those proportionals, he demonstrates several properties and relations of the numbers and logarithms, and illustrates the manner of applying them. The principal of these properties are as follows: 1st, that the logarithm of any sine is greater than the difference between that sine and the radius, but less than the said difference when increased in the proportion of the sine to radius*; and 2dly, that the difference between the logarithms of two sines, is less than the difference of the sines increased in the proportion of the less sine to radius, but greater than the said difference of the sines increased in the proportion of the greater sine to radius†.

Hence, by the 1st theorem, the logarithm of 10,000,000, the radius or first term in the first table, being 0, the logarithm of 9,999,999, the 2d term, will be between 1 and 1.000,000,1, and will therefore be equal to 1.000,000,05 very nearly: and this will be also the common difference of all the terms or proportionals in the first table; and therefore by the continual addition of this logarithm, there will be obtained the logarithms of all these 100 proportionals: consequently 100 times the said first logarithm, or the last of the above sums, will give 100.000,005, for the logarithm of 9,999,900.000,495,0, the last of the said 100 proportionals.

Then, by the 2d theorem, it easily appears that .000,495,0 is the difference between the logarithms of 9,999,900.000,495,0 and 9,999,900, the last term of the first table and the 2d term of the second table; this then being added to

* By this first theorem, r being radius, the logarithm of the sine s , is between $r-s$ and $\frac{r-s}{s}r$; and therefore, when s differs but little from r , the logarithm of s will be nearly equal to $\frac{r+s \times r-s}{2s}$, the arithmetical mean between the limits $r-s$ and $\frac{r-s}{s}r$; but still nearer to $\frac{r-s}{s}\sqrt{\frac{r}{s}}$ or $\frac{r-s}{s}\sqrt{rs}$, the geometrical mean between the said limits.

† By this second theorem, the difference between the logarithms of the two sines S and s , lying between the limits $\frac{S-s}{s}r$ and $\frac{S-s}{S}r$, will, when those sines differ but little, be nearly equal to $\frac{S^2-s^2}{2Ss}r$ or $\frac{S+s \times S-s}{2Ss}r$, their arithmetical mean; or nearly $= \frac{S-s}{\sqrt{Ss}}r$, the geometrical mean; or nearly $= \frac{S-s}{S+s}2r$, by substituting, in the last denominator, $\frac{1}{2} \cdot \overline{S+s}$ for $\sqrt{Ss}$, to which it is nearly equal.

the

the last logarithm, gives 100.0005000 for the logarithm of the said 2d term, as also the common difference of the logarithms of all the proportionals in the 2d table; and therefore by continually adding it, there will be generated the logarithms of all these proportionals in the 2d table; the last of which is 5000.025, answering to 9995001.222927^{*} the last term of that table.

Again, by the 2d theorem, the difference between the logarithms of this last proportional of the 2d table, and the 2d term in the first column of the 3d table, is found to be 1.223,538,7; which being added to the last logarithm, gives 5001.248,538,7 for the logarithm of 9,995,000, the said 2d term of the 3d table, as also the common difference of the logarithms of all the proportionals in the first column of that table; and this therefore, being continually added, gives all the logarithms of that first column, the last of which is 100,024.970,77, the logarithm of 9,900,473.5780, the last term of the said column.

Finally, by the 2d theorem again, the difference between the logarithms of this last number and 9,900,000, the first term in the 2d column, is 478.3502; which being added to the last logarithm, gives 100,503.3210 for the logarithm of the said first term in the 2d column, as well as the common difference of the logarithms of all the numbers on the same line in every line of the table, namely, of all the 1st terms, of all the 2d, of all the 3d, of all the 4th, &c. terms in all the columns; and which therefore, being continually added to the logarithms in the first column, will give the corresponding logarithms in all the other columns.

And thus is compleated what the author calls the radical table, in which he retains only one decimal place in the logarithms (or *artificials*, as he always calls them in his tract on the construction), and four in the naturals. A specimen of the table is as here follows:

Radical Table.						
Terms	1st Column.		2d Column.		69th Column.	
	Naturals.	Artificials.	Naturals.	Artificials.	Naturals.	Artificials.
1	10,000,000.0000	0	9,900,000.0000	100,503.3	5,048,858.8900	6,834,225.8
2	9,995,000.0000	5001.2	9,895,050.0000	105,504.6	5,046,333.4605	6,839,227.1
3	9,990,002.5000	10,002.5	9,890,102.4750	110,505.8	5,043,811.2932	6,844,228.3
4	9,985,007.4987	15,003.7	9,885,157.4237	115,507.1	5,041,289.3879	6,849,229.6
5	9,980,014.9950	20,005.0	9,880,214.8451	120,508.3	5,038,768.7435	6,854,230.8
&c	&c till	&c	&c	&c	&c	&c
21	9,900,473.5780	100,025.0	9,801,468.8423	200,528.2	4,998,609.4034	6,934,250.8

Having thus, in the most easy manner, compleated the radical table, by little more than mere addition and subtraction, both for the natural numbers and logarithms; the logarithmic fines were easily deduced from it by means of the 2d theorem, namely, taking the sum and difference of each tabular fine and the nearest number in the radical table, annex 7 ciphers to the difference, divide the result by the sum, and half the quotient will be the difference between the logarithms of the said numbers, namely, between the tabular fine and radical number; consequently, adding or subtracting this difference, to or from the given logarithm of the radical number, there will be obtained the logarithmic fine required. And thus the logarithms of all the fines from radius to the half of it, or from 90° to 30°, were perfected.

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Next,

* Line 2. of page 49. Hist. Journal des Savants. 1835. p. 363.

Next, for determining the sines of the remaining 30 degrees, he delivers two methods. In the first of these he proceeds in this manner: Observing that the logarithm of the ratio of 2 to 1, or of half the radius, is 6,931,469.22, of 4 to one is the double of this, of 8 to 1 is triple of it, &c; that of 10 to 1 is 23,025,842.34, of 20 to 1 is the sum of the logarithms of 2 and 10, and so on by composition for the logarithms of the ratios between 1 and 40, 80, 100, 200, &c. to 10,000,000; he multiplies any given sine, for an arc less than 30 degrees, by some of these numbers, till he finds the product nearly equal to one of the tabular numbers; then by means of this and the second theorem, the logarithm of this product is found; to which adding the logarithm that answers to the multiple abovementioned, the sum is the logarithm sought. But the other method is still much easier, and is derived from this property, which he demonstrates, namely, as half radius is to the sine of half an arc, so is the cosine of the said half arc to the sine of the whole arc; or as $\frac{1}{2}$ radius : sine of an arc :: cosine of the arc : sine of double the arc; hence the logarithmic sine of an arc is found, by adding together the logarithms of half radius and of the sine of the double arc, and then subtracting the logarithmic cosine from the sum.

And thus the remainder of the sines, from 30° down to 0, are easily obtained. But in this latter way, the logarithmic sines for full one half of the quadrant, or from 0 to 45 degrees, he observes, may be derived; the other half having already been made by the general method of the radical table, by one easy division and addition or subtraction for each.

I have dwelt the longer on this work of the inventor of logarithms, because I have not seen in any author an account of his method of constructing his table, although it is perfectly different from any other method used by the later computers, and indeed almost peculiar to his species of logarithms. The whole of this work manifests great ingenuity in the designer, as well as much accuracy. But notwithstanding the caution he took to obtain his logarithms true to the nearest unit in the last figure set down in the tables, by extending the numbers in the computations to several decimals, and other means; he had been disappointed of that end, either by the inaccuracy of his assistant computers or transcribers, or through some other cause; as the logarithms in the table are commonly very inaccurate. It is remarkable too that in this tract on the construction of the logarithms, Lord Napier never calls them logarithms, but every where *artificials*, as opposed in idea to the natural numbers: and this notion of natural and artificial numbers I take to have been his first idea of this matter, and that he altered the word *artificials* to *logarithms* in his first book, on the description of them, when he printed it, in the year 1614, and that he would also have altered the word every where in this posthumous work if he had lived to print it: for in the two or three pages of appendix, annexed to the work by his son from Napier's papers, he again always calls them logarithms. This appendix relates to the change of the logarithms to that scale in which 1 is the logarithm of the ratio of 10 to 1, the logarithm of 1, with or without ciphers, being 0; and it appears to have been written after Briggs had communicated to him his idea of that change.

Napier

Napier here in this appendix also briefly describes some methods by which this new species of logarithms may be constructed. Having supposed 0 to be the logarithm of 1, and 1 with any number of ciphers, as 10,000,000,000, the logarithm of 10; he directs to divide this logarithm of 10, and the successive quotients, ten times by 5, by which divisions there will be obtained these other ten logarithms, namely 2,000,000,000, 400,000,000, 80,000,000, 16,000,000, 3,200,000, 640,000, 128,000, 25,600, 5,120, 1024: then this last logarithm, and its quotients, being divided ten times by 2, will give these other ten logarithms 512, 256, 128, 64, 32, 16, 8, 4, 2, 1. And the numbers answering to these twenty logarithms, we are directed to find in this manner; namely, extract the 5th root of 10 (with ciphers), then the 5th root of that root; and so on for ten continual extractions of the 5th root; so shall these ten roots be the natural numbers belonging to the first ten logarithms, above found in continually dividing by 5: Next, out of the last 5th root we are to extract the square root, then the square root of this last root, and so on for ten successive extractions of the square root; so shall these last ten roots be the natural numbers corresponding to the logarithms or quotients arising from the last ten divisions by the number 2. And from these twenty logarithms, 1, 2, 4, 8, 16, &c. and their natural numbers, the author observes that other logarithms and their numbers may be formed, namely, by adding the logarithms and multiplying their correspondent numbers. It is evident that this process would generate rather an antilogarithmic canon, such as Dodson's, than the table of Briggs; and that the method would also be very laborious, since, besides the very troublesome original extractions of the 5th roots, all the numbers would be very large, by the multiplication of which the successive secondary natural numbers are to be found.

Our author next mentions another method of deriving a few of the primitive numbers and their logarithms, namely, by taking continually geometrical means, first between 10 and 1, then between 10 and this mean, and again between 10 and the last mean, and so on; and taking the arithmetical means between their corresponding logarithms. He then lays down various relations between numbers and their logarithms, such as that the products and quotients of numbers answer to the sums and differences of their logarithms; and that the powers and roots of numbers answer to the products and quotients of the logarithms by the index of the power or root, &c; as also that, of any two numbers, whose logarithms are given, if each number be raised to the power denoted by the logarithm of the other, the two results will be equal. He then delivers another method of making the logarithms to a few of the prime integer numbers, which is well adapted for constructing the common table of logarithms. This method easily follows from what has been said above, and it depends on this property, "that the logarithm of any number in this scale, is 1 less than the number of places or figures contained in that power of the given number whose exponent is 10,000,000,000, or the logarithm of 10, at least as to integer numbers," for they really differ by a fraction, as is shewn by Mr. Briggs in his illustrations of these properties, printed at the end of this appendix to the con-

struction of logarithms. I shall here set down one more of these relations, as the manner in which it is expressed is exactly similar to that of fluxions and fluents, and it is this: Of any two numbers, as the greater is to the less, so is the velocity of the increment or decrement of the logarithms at the less, to the velocity of the increment or decrement of the logarithms at the greater: that is, in our modern notation, as $X : Y :: \dot{y}$ to $\dot{x}$, where $\dot{x}$ and $\dot{y}$ are the fluxions of the logarithms of X and Y .

Kepler's Construction of Logarithms.

The logarithms of Briggs and Kepler were both printed the same year, 1624; but as the latter are of the same sort as Napier's, I shall here give the author's construction of them, before we proceed to that of Briggs's.

We have already (p. 34 & seq.) described the nature and form of Kepler's logarithms, shewing that they are of the same kind as Napier's, but only a little varied in the form of the table. It may also be added that, in general, the ideas which these two masters had on this subject, were of the same nature, only it was more fully and methodically laid down by Kepler, who expanded, and delivered in the form of a regular science, the hints that were given by the illustrious inventor. The foundation and nature of their methods of construction are also the same, but only a little varied in their modes of applying them. Kepler here first of any treats of logarithms in the true and genuine way of the measures of ratios, or proportions *, as he calls them, and that in a very full and scientific manner: and this method of his was afterwards followed and abridged by Mercator, Halley, Cotes, and others, as we shall see in the proper places. Kepler first erects a regular and purely mathematical system of proportions, and the measures of proportions, treated at considerable length in a number of propositions, which are fully and chastely demonstrated by genuine mathematical reasoning, and illustrated by numerical examples. This part contains and demonstrates both the nature and the principles of the structure of logarithms. And in the second part he applies those principles in the actual construction of his table, which contains only 1000 numbers and their logarithms, in the form which we before described: and in this part he indicates the various contrivances made use of in deducing the logarithms of proportions one from another, after a few of the leading ones had been first formed by the general and more remote principles. He uses the name *logarithms*, given them by the inventor, being the most proper, as expressing the very nature and essence of those artificial numbers, and containing, as it were, a definition in the very name of them; but without taking any notice of the inventor, or of the origin of those useful numbers.

* Kepler almost always uses the term *proportion* instead of *ratio*, which I also shall do in my account of his work, as well as conform in expressions and notations to his other peculiarities. It may also be here remarked, that I observe the same practice in describing the works of other authors, the better to convey the idea of their several methods and style. And this may serve to account for some seeming inequalities in the language of this history.

As this tract is very curious and important in itself, and is besides very rare and little known, instead of a particular description only, I shall here give a brief translation of both the parts, omitting only the demonstrations of the propositions, and some rather long illustrations of them.

The book is dedicated to Philip, landgrave of Hesse, but is without either preface or introduction, and commences immediately with the subject of the first part, which is intituled *The Demonstration of the Structure of Logarithms*; and the contents of it are as follows :

Postulate 1. That all proportions equal among themselves, by whatever variety of couplets of terms they may be denoted, are measured or expressed by the same quantity.

Axiom 1. If there be any number of quantities of the same kind, the proportion of the extremes is understood to be composed of all the proportions of every adjacent couplet of terms, from the first to the last.

1 Proposition. The mean proportional between two terms divides the proportion of those terms into two equal proportions.

Axiom 2. Of any number of quantities, regularly increasing the means, divide the proportion of the extremes into one proportion more than the number of the means.

Postulate 2. That the proportion between any two terms is divisible into any number of parts, until those parts become less than any proposed quantity.

An example of this section is then inserted in a small table, in dividing the proportion which is between 10 and 7 into 1,073,741,824 equal parts, by as many mean proportionals wanting one, namely, by taking the mean proportional between 10 and 7, then the mean between 10 and this mean, and the mean between 10 and the last, and so on for 30 means, or 30 extractions of the square-root, the last, or 30th, of which roots is 9.999,999,996,678,205,690,0 and the 30th power of 2, which is 1,073,741,824, shews into how many parts the proportion between 10 and 7, or between 1000 &c and 700 &c, is divided by 1,073,741,824 means, each of which parts is equal to the proportion between 10.00 &c, and the 30th mean 9.99 &c, that is, the proportion between 10.00 &c and 9.99 &c, is the 1,073,741,824th part of the proportion between 10 and 7. Then by assuming the small difference 0.000,000,003,321,794,310,0, for the measure of the very small element of the proportion of 10 to 7, or for the measure of the proportion of 10.00 &c to 9.99 &c, or for the logarithm of this last term, and multiplying it by 1,073,741,824 the number of parts, the product will give 3.566,749,481,372,221,440,0 for the logarithm of the less term 7 or 700 &c.

Postulate 3. That the extremely small quantity or element of a proportion, may be measured or denoted by any quantity whatever; as for instance, by the difference of the terms of that element.

2 Proposition. Of three continued proportionals, the difference of the two first has to the difference of the two latter, the same proportion which the first term has to the 2d, or the 2d to the 3d.

3 Prop. Of any continued proportionals, the greatest terms have the greatest difference, and the least terms the least.

4 Prop. In any continued proportionals, if the difference of the greatest terms be made the measure of the proportion between *them*, the difference of any other couplet will be less than the true measure of *their* proportion.

5 Prop. In continued proportionals, if the difference of the greatest terms be made the measure of their proportion, then the measure of the proportion of the greatest to any other term will be greater than *their* difference.

6 Prop.

6 *Prop.* In continued proportionals, if the difference of the greatest term and any one of the less, taken not immediately next to it, be made the measure of their proportion; then the measure of the proportion which is between the greatest and any other term greater than the one before taken, will be less than the difference of those terms; but the measure of the proportion which is between the greatest term and any one less than that first taken, will be greater than their difference.

7 *Prop.* Of any quantities placed according to the order of their magnitudes, if any two successive proportions be equal, the three successive terms which constitute them, will be continued proportionals.

8 *Prop.* Of any quantities placed in the order of their magnitudes, if the intermediates lying between any two terms, be not among the mean proportionals which can be interposed between the said two terms, then such intermediates do not divide the proportion of those two terms into commensurable proportions.

Besides the demonstrations, as usual, several definitions are here given; as of commensurable proportions, &c.

9 *Prop.* When two expressible lengths are not to one another as two figurate numbers of the same species, such as two squares, or two cubes; there cannot fall between them other expressible lengths, which shall be mean proportionals, and as many in number as that species requires, namely, one in the squares, two in the cubes, three in the biquadrates, &c.

10 *Prop.* Of any expressible quantities, following in the order of their magnitudes, if the two extremes be not in the proportion of two square numbers, or two cubes, or two other powers of the same kind; none of the intermediates divide the proportion into commensurables.

11 *Prop.* All the proportions, taken in order, which are between expressible terms that are in arithmetical proportion, are incommensurable to one another. As between 8, 13, 18.

12 *Prop.* Of any quantities placed in the order of magnitude, if the difference of the greatest terms be made the measure of their proportion, then the difference between any two others will be less than the measure of *their* proportion; and if the difference of the two least terms be made the measure of their proportion, then the differences of the rest will be greater than the measure of the proportion between their terms.

Corol. If the measure of the proportion between the greatest exceed their difference, then the proportion of this measure to the said difference, will be less than that of a following measure to the difference of its terms. Because proportionals have the same ratio.

13 *Prop.* If three quantities follow one another in the order of magnitude; the proportion of the two last will be contained in the proportion of the extremes, a less number of times than the difference of the two least is contained in the difference of the extremes: And on the contrary, the proportion of the two greatest will be contained in the proportion of the extremes, oftener than the difference of the former is contained in that of the latter.

Corol. Hence if the difference of the two greater be equal to the difference of

of the two less terms, the proportion between the two greater will be less than the proportion between the two less.

14 *Prop.* Of three equidifferent quantities taken in order, the proportion between the extremes is more than double the proportion between the two greater terms.

Corol. Hence it follows, that half the proportion of the extremes, is greater than the proportion of the two greatest terms, but less than the proportion of the two least.

15 *Prop.* If two quantities constitute a proportion, and each quantity be lessened by half the greater; the remainders will constitute a proportion greater than double the former.

16 *Prop.* The aliquot parts of incommensurable proportions, are incommensurable to each other.

17 *Prop.* If one thousand numbers follow one another in the natural order, beginning at 1000, and differing all by unity, viz. 1000, 999, 998, 997, &c; and the proportion between the two greatest 1000, 999, by continual bisection, be cut into parts that are smaller than the excess of the proportion between the next two 999, 998, over the said proportion between the two greatest 1000, 999; and then for the measure of that small element of the proportion between 1000, 999, there be taken the difference of 1000 and that mean proportional which is the other term of the element. Again, if the proportion between 1000, 998, be likewise cut into double the number of parts which the former proportion between 1000, 999, was cut into; and then for the measure of the small element in this division, be taken the difference of its terms, of which the greater is 1000. And, in the same manner, if the proportion of 1000 to the following numbers, as 997, &c, by continual bisection, be cut into particles of such magnitude as may be between $\frac{3}{2}$ and $\frac{3}{4}$ of the element arising from the section of the first proportion between 1000 and 999; the measure of each element will be given from the difference of its terms. Then, this being done, the measure of any one of the 1000 proportions, will be composed of as many measures of its element, as there are of those elements in the said divided proportion. And all these measures, for all the proportions, will be sufficiently exact for the nicest calculations.

All these sections and measures of proportions are performed in the manner of that described at postulate 2, and the operation is abundantly explained by numerical calculations.

18 *Prop.* The proportion of any number to the first term 1000 being known; there will also be known the proportion of the rest of the numbers in the same continued proportion, to the said first term.

So from the known proportion between 1000 and 900,

there is also known the proportion of 1000 to 810, and to 729;

And from 1000 to 800, also 1000 to 640, and to 512;

And from 1000 to 700, also 1000 to 490, and to 343;

And from 1000 to 600, also 1000 to 360, and to 216;

And from 1000 to 500, also 1000 to 250, and to 125.

Corol. Hence arises the precept for squaring, cubing, &c; as also for extracting the square root, cube root, &c, out of the first figures of numbers.

For it will be, as the greatest number of the chiliad as a denominator, is to the number proposed as a numerator, so is this to the square of the fraction, and so is this to the cube.

19 *Prop.* The proportion of a number to the first, or 1000, being known; if there be two other numbers in the same proportion to each other, then the proportion of one of these to 1000 being known, there will also be known the proportion of the other to the same 1000.

Corol. 1. Hence from the 15 proportions mentioned in prop. 18, will be known 120 others below 1000, to the same 1000.

For so many are the proportions, equal to some one or other of the said 15, that are among the other integer numbers which are less than 1000.

Corol. 2. Hence arises the method of treating the Rule-of-Three, when 1000 is one of the given terms.

For this is effected by adding to, or subtracting from, each other, the measures of the two proportions of 1000 to each of the other two given numbers, according as 1000 is, or is not, the first term in the Rule-of-Three.

20 *Prop.* When four numbers are proportional, the first to the second as the third to the fourth, and the proportions of 1000 to each of the three former are known, there will also be known the proportion of 1000 to the fourth number.

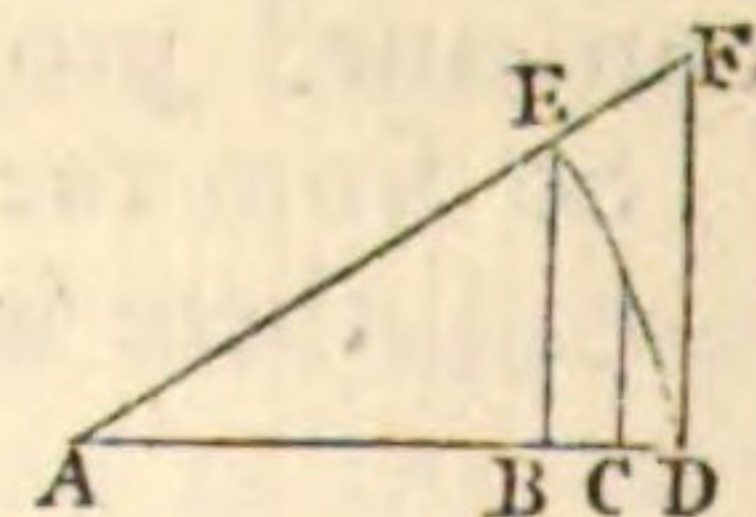
Corol. 1. By this means other chiliads are added to the former.

Corol. 2. Hence arises the method of performing the Rule-of-Three, when 1000 is not one of the terms. Namely, from the sum of the measures of the proportions of 1000 to the second and third, take that of 1000 to the first, and the remainder is the measure of the proportion of 1000 to the fourth term.

Definition. The measure of the proportion between 1000 and any less number, as before described, and expressed by a number, is set opposite to that less number in the chiliad, and is called its LOGARITHM, that is, the number ($\alpha\lambda\theta\mu\acute{o}\varsigma$) indicating the proportion ($\lambda\omicron\gamma\omicron\nu$) which 1000 bears to that number, to which the logarithm is annexed.

21 *Prop.* If the first or greatest number be made the radius of a circle, or *sinus totus*; every less number, considered as the cosine of some arc, has a logarithm greater than the versed sine of that arc, but less than the difference between the radius and secant of the arc; except only in the term next after the radius, or greatest term, the logarithm of which by the hypothesis is made equal to the versed sine.

That is, if CD be made the logarithm of AC, or the measure of the proportion of AC to AD; then the measure of the proportion of AB to AD, that is the logarithm of AB, will be greater than BD, but less than EF. And this is the same as Napier's first rule in page 45.



22 *Prop.* The same things being supposed; the sum of the versed sine and excess of the secant over the radius, is greater than double the logarithm of the cosine of an arc.

Corol.

Corol. The log. cosine is less than the arithmetic mean between the versed sine and the excess of the secant.

Precept 1. Any sine being found in the canon of sines, and its defect below radius to the excess of the secant above radius; then shall the logarithm of the sine be less than half that sum, but greater than the said defect or co-versed sine.

Let there be the fine 99,970.1490 of an arc :
 Its defect below radius is 29.8510 the co-vers. and less than logarithm fine ;
 Add the excess of the secant 29.8599

Sum 59.7109
its half or 29.8555 greater than the logarithm.

Therefore the logarithm is between $\begin{cases} 29.8510 \text{ and} \\ 29.8555. \end{cases}$

Precept 2. The logarithm of the sine being found, you will also find nearly the logarithm of the round or integer number which is next less than your sine with a fraction, by adding that fractional excess to the logarithm of the said sine.

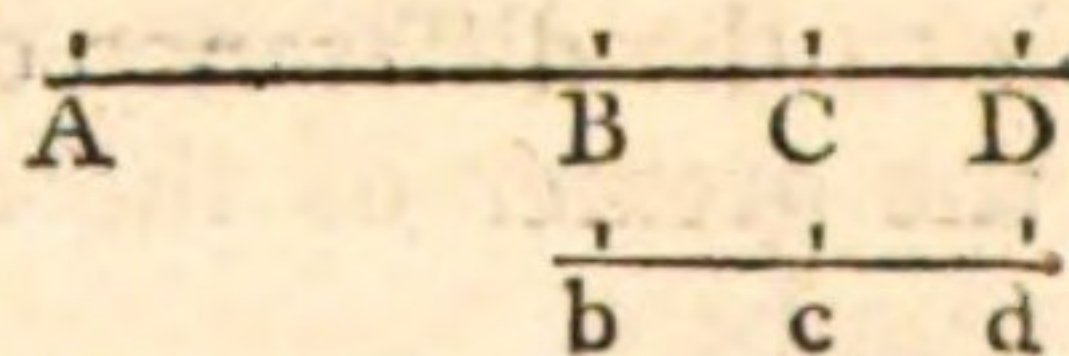
Thus the logarithm of the fine 99,970.149 is found to be about 29.854; if now the logarithm of the round number 99,970.000 be required, add 149 the fractional part of the fine to its logarithm, observing the point, thus

29.854	}	is the logarithm of the round number 99,970.000 nearly.
149		
30.003		

the sum

23 *Prop.* Of three equidifferent quantities, the measure of the proportion between the two greater terms, with the measure of the proportion between the two lesser terms, will constitute a proportion, which will be greater than the proportion of the two greater terms, but less than the proportion of the two lesser terms.

Thus if AB, AC, AD be three quantities having the equal differences BC, CD; and if the measure of the proportion of AD, AC be cd, and that of AC, AB be bc; then the proportion of cd to cb will be greater than the proportion of AC to AD, but less than the proportion of AB to AC.



24 *Prop.* The said proportion between the two measures, is less than half the proportion between the extreme terms. That is, the proportion between bc , cd , is less than half the proportion between AB , AD .

Corol. Since therefore the arithmetical mean divides the proportion into unequal parts, of which the one is greater, and the other less, than half the whole; if it be enquired what proportion is between these proportions, the answer is, that it is a little less than the said half.

An example of finding nearly the limits, greater and less, to the measure of any proposed proportion.

It being known that the measure of the proportion between 1000 and 900 is 10,536.05, required the measure of the proportion 900 to 800, where the terms 1000, 900, 800, have equal differences.

Therefore as 9 is to 10 so is 10,536.05 to 11,706.72, which is less than 11,778.30 the measure of the proportion 9 to 8. Again, as the mean proportional between 8 and 10 (which is 8.944,271,9) is to 10 so is 10,536.05 to 11,779.66, which is greater than the measure of the proportion between 9 and 8.

Axiom. Every number denotes an expressible quantity.

25. *Prop.* If the 1000 numbers, differing by 1, follow one another in the natural order; and there be taken any two adjacent numbers, as the terms of some proportion; the measure of this proportion will be to the measure of the proportion between the two greatest terms of the chiliad, in a proportion greater than that which the greatest term 1000 bears to the greater of the two terms first taken, but less than the proportion of 1000 to the less of the said two selected terms.

So of the first 1000 numbers taking any two successive terms, as 501 and 500, the logarithm of the former being 69114.92, and of the latter 69314.72, the difference of which is 199.80. Wherefore by the definition, the measure of the proportion between 501 and 500 is 199.80. In like manner, because the logarithm of the greatest term 1000 is 0, and of the next 999 is 100.05, the difference of these logarithms, and the measure of the proportion between 1000 and 999, is 100.05. Couple now the greatest term 1000 with each of the selected terms 501 and 500; couple also the measure 199.80 with the measure 100.05; so shall the proportion between 199.80 and 100.05 be greater than the proportion between 1000 and 501, but less than the proportion between 1000 and 500.

Corol. 1. Any number below the first 1000 being proposed, as also its logarithm; the differences of any logarithms antecedent to that proposed, towards the beginning of the chiliad, are to the first logarithm (viz. that which is assigned to 999), in a greater proportion than 1000 to the number proposed; but of those which follow towards the last logarithm, they are to the same in a less proportion.

Corol. 2. By this means the places of the chiliad may easily be filled up, which have not yet had logarithms adapted to them by the former propositions.

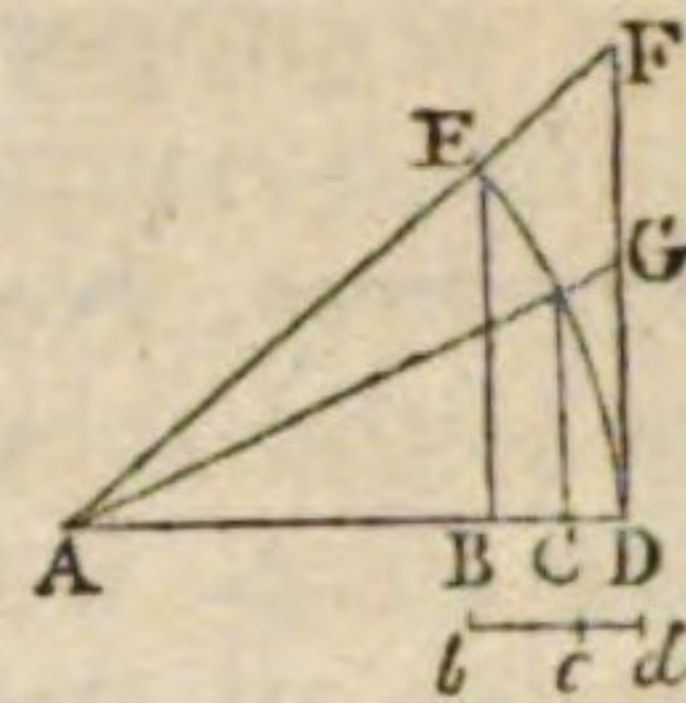
26 *Prop.* The difference of two logarithms, adapted to two adjacent numbers, is to the difference of these numbers, in a proportion greater than 1000 bears to the greater of those numbers, but less than that of 1000 to the less of the two numbers.

This 26 prop. is the same as Napier's second rule at page 47.

27 *Prop.* Having given two adjacent numbers of the 1000 natural numbers, with their logarithmic indices, or the measures of the proportions which those absolute, or round, numbers constitute with 1000 the greatest; the increments or differences of these logarithms will be to the logarithm of the small element of the proportions, as the secants of the arcs whose cosines are the two absolute numbers, are to the greatest number, or the radius of the circle: so that, however, of the said two secants, the less will have to the radius a less proportion, than the proposed difference has to the first of all, but the greater will have a greater proportion, and so also will the mean proportional between the said secants have a greater proportion.

Thus

Thus if BC , CD be equal, also bd the logarithm of AB , and cd the logarithm of AC ; then the proportion of bc to cd will be greater than the proportion of AG to AD , but less than that of AF to AD , and also less than that of the mean proportional between AF and AG to AD .



Corol. 1. The same obtains also when the two terms differ, not only by the unit of the small element, but by another unit which may be ten-fold, a hundred-fold, or a thousand-fold of that.

Corol. 2. Hence the differences will be obtained sufficiently exact, especially when the absolute numbers are pretty large, by taking the arithmetical mean between two small secants, or (if you will be at the labour) by taking the geometrical mean between two larger secants, and then by continually adding the differences, the logarithms will be produced.

Corol. 3. Precept. Divide the radius by each term of the assigned proportion, and the arithmetical mean (or, still nearer, the geometrical mean) between the quotients will be the required increment, which, being added to the logarithm of the greater term, will give the logarithm of the less term.

Example.

Let there be given the logarithm of 700, viz. 35,667.4948, to find the logarithm to 699.

Here radius divided by 700 gives 142.8571 &c.

and divided by 699 gives 143.0672 &c.

the arithmetic mean is 142.962

which added to 35,667.4948

gives the logarithm to 699 35,810.4568

Corol. 4. Precept for the logarithms of sines.

The increment between the logarithms of two sines, is thus found: find the geometrical mean between the cosecants, and divide it by the difference of the sines; the quotient will be the difference of the logarithms.

Example.

∞ 1' sine 2909 cosec.	34,377,468.2
0 2' sine 5818 cosec.	17,188,731.9
dif. 2909 geom. mean 24,280,000.0 nearly.	

The quotient 8000.0 exceeds the required increment of the logarithms, because the secants are here so large.

Appendix. Nearly in the same manner it may be shewn, that the second differences are in the duplicate proportion of the first, and the third in the duplicate of the second. Thus, for instance, in the beginning of the logarithms the first difference is 100.00000, viz. equal to the difference of the numbers 100,000.00000 and 99,900.00000; the second, or difference of the differences, 10000; the third 20. Again, after arriving at the number 5000.00000, the logarithms have for a difference 200.00000, which is to the first difference, as the number 100,000.00000 to 50,000.00000; but the second difference is 40,000, in which 10,000 is contained four times; and the third 328, in which 20 is contained sixteen times. But since in treating of new matters we labour under the

want of proper words, therefore, lest we should become too obscure, the demonstration is omitted untried.

28 *Prop.* No number expresses exactly the measure of the proportion, between two of the 1000 numbers, constituted by the foregoing method.

29 *Prop.* If the measures of all proportions be expressed by numbers or logarithms; all proportions will not have assigned to them their due portion of measure, to the utmost accuracy.

30 *Prop.* If to the number 1000, the greatest of the chiliad, be referred others that are greater than it, and the logarithm of 1000 be made 0, the logarithms belonging to those greater numbers will be negative.

This concludes the first, or scientific, part of the work, the principles of which Kepler applies, in the second part, to the actual construction of the first 1000 logarithms, which is pretty minutely described. This part is intitled *A very compendious method of constructing the Chiliad of Logarithms*; and it is not improperly so called, the method being very concise and easy. The fundamental principles are briefly these: That at the beginning of the logarithms, their increments, or differences, are equal to those of the natural numbers: that the natural numbers may be considered as the decreasing cosines of increasing arcs: and that the secants of those arcs at the beginning have the same differences as the cosines, and therefore the same differences as the logarithms. Then, since the secants are the reciprocals of the cosines, by these principles and the third corol. to the twenty-seventh proposition, he establishes the following method of constituting the 100 first or smallest logarithms to the 100 largest numbers, 1000, 999, 998, 997, &c. to 900. viz. Divide the radius 1000, increased with seven ciphers, by each of these numbers separately, disposing the quotients in a table, and they will be the secants of those arcs which have the divisors for their cosines; continuing the division to the 8th figure, as it is in that place only that the arithmetical and geometrical means differ. Then by adding successively the arithmetical means between every two successive secants, the sums will be the series of logarithms. Or by adding continually every two secants, the successive sums will be the series of the double logarithms.

Besides these 100 logarithms, thus constructed, he constitutes two others by continual bisection, or extractions of the square root, after the manner described in the second postulate. And first he finds the logarithm which measures the proportion between 100,000.00 and 97,656.25, which latter term is the third proportional to 1024 and 1000, each with two ciphers; and this is effected by means of twenty-four continual extractions of the square root, determining the greatest term of each of twenty-four classes of mean proportionals; then the difference between the greatest of these means and the first or whole number 1000, with ciphers, being as often doubled, there arises 2371.6526 for the logarithm sought, which made negative is the logarithm of 1024. Secondly, the like process is repeated for the proportion between the numbers 1000 and 500, from which arises 69,314.7193 for the logarithm of 500; which he also calls the logarithm of duplication, being the measure of the proportion of 2 to 1.

Then from the foregoing he derives all the other logarithms in the chiliad, beginning with those of the prime numbers 1, 2, 3, 5, 7, &c, in the first 100.

And first, since 1024, 512, 256, 128, 64, 32, 16, 8, 4, 2, 1, are all in the continued proportion of 1000 to 500, therefore the proportion of 1024 to 1 is decuple of the proportion of 1000 to 500, and consequently the logarithm of 1 would be decuple of the logarithm of 500, if 0 were taken as the logarithm of 1024; but since the logarithm of 1024 is applied negatively, the logarithm of 1 must be diminished by as much: diminishing therefore 10 times the logarithm of 500, which is 693,147.1928, by 2371.6526, the remainder 690,775.5422 is the logarithm of 1, or of 100.00 what is set down in the table.

And because 1, 10, 100, 1000, are continued proportionals, therefore the proportion of 1000 to 1 is triple of the proportion of 1000 to 100, and consequently $\frac{1}{3}$ of the logarithm of 1 is to be put for the logarithm of 100, viz. 23,025.85141, and this is also the logarithm of decuplication, or of the proportion of 10 to 1, And hence multiplying this logarithm of 100 successively by 2, 3, 4, 5, 6, and 7, there arise the logarithms to the numbers in the decuple proportion, as in the little table here annexed.

Numbers.	Logarithms.
100	230,258.5141
10	460,517.0282
1	690,775.5422
.1	921,034.0563
.01	1,151,292.5703
.001	1,381,551.0844
.0001	1,611,809.5985

Also if the logarithm of duplication, or of the proportion of 2 to 1, be taken from the logarithm of 1, there will remain the logarithm of 2; and from the logarithm of 2 taking the logarithm of 10, there remains the logarithm of the proportion of 5 to 1; which taken from the logarithm of 1, there remains the logarithm of 5. See the little table here annexed.

Log. of 1	690,775.5422
of 2 to 1	69,314.7193
log. of 2	621,460.8229
log. of 10	460,517.0281
of 5 to 1	160,943.7948
log. of 5	529,831.7474

For the logarithms of other prime numbers he has recourse to those of some of the first or greatest century of numbers, before found, viz. of 999, 998, 997, &c. And first, taking 960, whose logarithm is 4082.2001; then by adding to this logarithm the logarithm of duplication, there will arise the several logarithms of all these numbers, which are in duplicate proportion continued from 960, namely 480, 240, 120, 60, 30, 15. Hence the logarithm of 30 taken from the logarithm of 10, leaves the logarithm of the proportion of 3 to 1; which taken from the logarithm of 1, leaves the logarithm of 3, viz. 580,914.3106. And the double of this diminished by the logarithm of 1, gives 471,053.0790 for the logarithm of 9.

Next, from the logarithm of 990, or $9 \times 10 \times 11$, which is 1005.0331, he finds the logarithm of 11, namely, subtract the sum of the logarithms of 9 and 10 from the sum of the logarithm of 990 and double the logarithm of 1, there remains 450,986.0106 the logarithm of 11.

Again, from the logarithm of 980, or $2 \times 10 \times 7 \times 7$, which is 2020.2711, he finds 496,184.5228 for the logarithm of 7.

And from 5129.3303 the logarithm of 950 or $5 \times 10 \times 19$, he finds 396,331.6392 for the logarithm of 19.

In like manner the logarithm

- to 998 or $4 \times 13 \times 19$, gives the logarithm of 13;
- to 969 or $3 \times 17 \times 19$, gives the logarithm of 17;
- to 986 or $2 \times 17 \times 29$, gives the logarithm of 29;
- to 966 or $6 \times 7 \times 23$, gives the logarithm of 23;
- to 930 or $3 \times 10 \times 31$, gives the logarithm of 31.

And

And so on for all the primes below 100, and for many of the primes in the other centuries up to 900. After which he directs to find the logarithms of all numbers composed of these, by the proper addition and subtraction of their logarithms, namely, in finding the logarithm of the product of two numbers, from the sum of the logarithms of the two factors take the logarithm of 1, the remainder is the logarithm of the product. In this way he shews that the logarithms of all numbers under 500 may be derived, except those of the following 36 numbers, namely, 127, 149, 167, 173, 179, 211, 223, 251, 257, 263, 269, 271, 277, 281, 283, 293, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397, 401, 409, 419, 421, 431, 433, 439, 443, 449. Also, besides the composite numbers between 500 and 900, made up of the products of some numbers whose logarithms have been before determined, there will be 59 primes not composed of them; which with the 36 above mentioned, make 95 numbers in all not composed of the products of any before them, and the logarithms of which he directs to be derived in this manner; namely, by considering the differences of the logarithms of the numbers interspersed among them; then by that method by which were constituted the differences of the logarithms of the smallest 100 numbers in a continued series, we are to proceed here in the discontinued series, that is, by prop. 27, corol. 3, and especially by the appendix to it, if it be rightly used, from whence those differences will be very easily supplied.

This closes the second part, or the actual construction of the logarithms; after which follows the table itself, which has been before described, p. 35. Before I dismiss Kepler's work, however, it may not be improper in this place to take notice of an erroneous property laid down by him in the appendix to the 27th prop. just now referred to; both because it is an error in principle, tending to vitiate the practice, and because it serves to shew that Kepler was unacquainted with the true nature of the orders of differences of the logarithms, notwithstanding what he says above with respect to the construction of them by means of their several orders of differences, and that consequently he has no legal claim to any share in the discovery of the differential method, known at that time to Briggs, and (it would seem) to him alone, it being published in his logarithms in the same year 1624, as Kepler's book, together with the true nature of the logarithmic orders of differences, as we shall presently see in the following account of his works. Now this error of Kepler's, here alluded to, is in that expression where he says the third differences are in the *duplicate* ratio of the second differences, like as the second differences are in the duplicate ratio of the first; or, in other words, that the third differences are as the *squares* of the second differences, as well as the second differences as the squares of the first; or that the third differences are as the *fourth powers* of the first differences. Whereas in truth the third differences are only as the *cubes* of the first differences. Kepler seems to have been led into this error by a mistake in his numbers, viz. when he says in that appendix, that *the third difference is 328, in which 20 is contained 16 times*; for when the numbers are accurately computed, the third difference comes out only 161, in which therefore 20 is contained only 8 times, which is the cube of 2, the number of times the one first difference contains the other. It would hence seem that Kepler had hastily drawn the above erroneous principle from
this

this one numerical example, or little more, false as it is : for had he made the trial in many instances, although erroneously computed, they could not easily have been so uniformly so, as to afford the same false conclusion. And therefore from hence, and what he says at the conclusion of that appendix, it may be inferred that he either never attempted the demonstration of the property in question, or else that he found himself embarrassed with it, and unable to accomplish it, and therefore dispatched it in the ambiguous manner in which it appears.

But it may easily be shewn, not only that the third differences of the logarithms at different places, are as the cubes of the first differences ; but, in general, that the numbers in any one and the same order of differences, at different places, are as that power of the numbers in the first differences, whose index is the same as that of the order ; or that the second, third, fourth, &c. differences, will be as the second, third, fourth, &c. powers of the first differences. For the several orders of differences, when the absolute numbers differ by indefinitely small parts, are as the several orders of fluxions of the logarithms ; but if x be any number, then $\frac{m \dot{x}}{x}$ is the fluxion of the logarithm of x , to the modulus m , and the second fluxion, or the fluxion of this fluxion, is $-\frac{m \dot{x}^2}{x^2}$ since x is constant ; and the third, fourth, &c. fluxions, are $\frac{2 m \dot{x}^3}{x^3}$, $-\frac{2.3 m \dot{x}^4}{x^4}$, &c ; that is, the first, second, third, fourth, fifth, sixth, &c. orders of fluxions, are equal to the modulus m multiplied into each of these terms, $\frac{\dot{x}}{x}$, $-\frac{1 \dot{x}^2}{x^2}$, $\frac{1.2 \dot{x}^3}{x^3}$, $-\frac{1.2.3 \dot{x}^4}{x^4}$, $\frac{1.2.3.4 \dot{x}^5}{x^5}$, $-\frac{1.2.3.4.5 \dot{x}^6}{x^6}$ &c, where it is evident that the fluxion of any order, is as that power of the first fluxion, whose index is the same as the number of the order. And these quantities would actually be the several terms of the differences themselves, if the differences of the numbers were indefinitely small. But they vary the more from them, as the differences of the absolute numbers differ from $\dot{x}$, or as the said constant numerical difference 1, approaches towards the value of x the number itself. However, upon the whole, the several orders vary proportionably, so as still sensibly to preserve the same analogy, namely, that two n th differences are in proportion as the n th powers of their respective first differences.

Of Briggs's Construction of his Logarithms.

Nearly according to the methods described in page 47, Mr. Briggs constructed the logarithms of the prime numbers, as appears from his relation of this business in the *Arithmetica Logarithmica*, printed in 1624, where he details, in an ample manner, the whole construction and use of his logarithms. The work is divided into thirty-two chapters or sections. In the first of these, logarithms in a general sense are defined, and some properties of them illustrated. In the second chapter he remarks, that it is most convenient to make 0 the logarithm of 1 ; and on that supposition he exemplifies these following properties, namely, that

that the logarithms of all numbers are either the indices of powers, or proportional to them; that the sum of the logarithms of two or more factors, is the logarithm of their product; and that the difference of the logarithm of two numbers, is the logarithm of their quotient. In the third section he states the other assumption which is necessary to limit his system of logarithms, namely, making 1 the logarithm of 10, as that which produces the most convenient form of logarithms; he hence also takes occasion to shew that the powers of 10, namely 100, 1000, &c, are the only numbers which can have rational logarithms. The fourth section treats of the characteristic; by which name he distinguishes the integral, or first part of a logarithm towards the left-hand, which expresses one less than the number of integer places or figures in the number belonging to that logarithm, or how far the first figure of this number is removed from the place of units; namely, that 0 is the characteristic of the logarithms of all numbers from 1 to 10; and 1 the characteristic of all those from 10 to 100; and 2 that of those from 100 to 1000; and so on.

He begins the fifth chapter with remarking, that his logarithms may chiefly be constructed by the two methods which were mentioned by Napier, as above related, and for the sake of which he here premises several *lemmata*, concerning the powers of numbers and their indices, and how many places or figures are in the products of numbers, observing that the product of two numbers will consist of as many figures as there are in both factors, unless perhaps the product of the first figures in each factor be expressed by one figure only, which often happens, and then commonly there will be one figure in the product less than in the two factors; as also that, of any two of the terms, in a series of geometricals, the results will be equal by raising each term to the power denoted by the index of the other; or any number raised to the power denoted by the logarithm of the other, will be equal to this latter number raised to the power denoted by the logarithm of the former; and consequently if the one number be 10, whose logarithm is 1 with any number of ciphers, then any number raised to the power whose index is 1000 &c, or the logarithm of 10, will be equal to 10 raised to the power whose index is the logarithm of that number; that is, the logarithm of any number in this scale, where 1 is the logarithm of 10, is the index of that power of 10 which is equal to the given number. But the index of any integral power of 10, is one less than the number of places in that power, consequently the logarithm of any other number, which is no integral power of 10, is not quite one less than the number of places in that power of the given number whose index is 1000, &c, or the logarithm of 10.

Find therefore the 10th, or 100th, or 1000th, &c, power of any number, as suppose 2, with the number of figures in such power; then shall that number of figures always exceed the logarithm of 2, although the excess will be constantly less than 1.

An

An example of this process is here given in the margin; where the first column contains several of the powers of 2, the 2d their corresponding indices, and the 3d contains the number of places in the powers in the first column; and of these numbers in the third column, such as are on the lines of those indices that consist of 1 with ciphers, are continual approximations to the logarithm of 2, being always too great by less than 1 in the last figure, that logarithm being. 301,029,995,663,98 &c.

And here, since the exact powers of 2 are not required, but only the number of figures they consist of, as shewn by the third column, only a few of the first figures of the powers in the first column are retained, those being sufficient to determine the number of places in them; and the multiplications in raising these powers are performed in a contracted way, so as to have the fifth, or last, figure in them true to the nearest unit. Indeed these multiplications might be performed in the same manner, retaining only the first three figures, and those to the nearest unit in the third place; which would make this a very easy way indeed of finding the logarithms of a few prime numbers.

It may also be remarked, that those several powers, whose indices are 1 with ciphers, are raised by thrice squaring from the former powers, and multiplying the first by the third of these squares: making also the corresponding doublings and additions of their indices: thus, the square of 2 is 4, the square of 4 is 16, the square of 16 is 256, and 256 multiplied by 4 is 1024; in like manner, the double of 1 is 2, the double of 2 is 4, the double of 4 is 8, and 8 added to 2 makes 10. And the

same for all the following powers and indices. The numbers in the third column, which shew how many places are in the corresponding powers in the first column, are produced in the very same way as those in the second column, namely, by three duplications and one addition; only observing to subtract 1

Powers of 2	Indices.	No. of places, or logs.
2	1	1
4	2	1
16	4	2
256	8	3
1024	10	4 log. of 2
10486	20	7 log. of 4
10995	40	13 log of 16
12089	80	25 log. of 256
12676	100	31 log. of 2
16069	200	61 log. of 4
25823	400	121 log. 16
66680	800	241 log. 256
10715	1000	302 log. 2
11481	2000	603 log. 4
13182	4000	1205 log 16
17377	8000	2409 log. 256
19950	10,000	3011 log. 2
39803	20,000	6021 log. 4
15843	40,000	12042 log. 16
25099	80,000	24083 l. 256
99900	100,000	30103 log. 2
99801	200,000	60206 l. 4
99601	400,000	120412 $\frac{8}{2}$
99204	800,000	240824 $\frac{7}{2}$
99006	1,000,000	301030
98023	2,000,000	602060
96085	4,000,000	1204120
92323	8,000,000	2408240
90498	10,000,000	3010300
81899	20,000,000	6020600
67075	40,000,000	12041200
44990	80,000,000	24082400
36846	100,000,000	30103000
13577	200,000,000	60206000
18433	400,000,000	120411999
33977	800,000,000	240823997
46129	1000,000,000	301029996

when the product of the first figures is expressed by one figure, or when the first figures exceed those of the number or power next above them. It may farther be observed that, like as the first number in each quaternion, or space of four lines or numbers, in the third column, approximates to the logarithm of 2, the first number in the first quaternion of the first column; so the second, third, and fourth terms of each quaternion in the third column, approximate to the logarithm of 4, 16, and 256, the second, third, and fourth numbers in the first quaternion of the first column. And moreover, by cutting off one, two, three, &c, figures, as the index or integral part, from the said logarithms of 2, 4, 16, and 256, the first, second, third and fourth numbers in the first quaternion of the first column, the remaining figures will be the decimal part of the logarithms of the corresponding first, second, third, and fourth numbers in the following second, third, fourth, &c, quaternions: the reason of which is, that any number of any quaternion in the first column, is the tenth power of the corresponding term in the next preceding quaternion. So that the third column contains the logarithms of all the numbers in the first column: a property which, if Dr. Newton had been aware of, he could not well have committed such gross mistakes as are found in a table of his similar to that above given, in which most of the numbers in the latter quaternions are totally erroneous; and his confused and imperfect account of this method, would induce one to believe that he did not well understand it.

In the sixth chapter our illustrious author begins to treat of the other general method of finding the logarithms of prime numbers, which he thinks is an easier way than the former, at least when the logarithm is required to a great many places of figures. This method consists in taking a great number of continued geometrical means between 1 and the given number whose logarithm is required; that is, first extracting the square root of the given number, then the root of the first root, the root of the second root, the root of the third root, and so on till the last root shall exceed 1 by a very small decimal, greater or less according to the intended number of places to be in the logarithm sought: then finding the logarithm of this small number, by methods described below, he doubles it as often as he made extractions of the square root, or, which is the same thing, he multiplies it by such power of 2 as is denoted by the said number of extractions, and the result is the required logarithm of the given number; as is evident from the nature of logarithms. The rule to know how far to continue this extraction of roots is, that the number of decimal places in the last root be double the number of true places required to be found in the logarithm, and that the first half of them be ciphers; the integer being 1: The reason of which is, that then the significant figures in the decimal, after the ciphers, are directly proportional to those in the corresponding logarithms; such figures in the natural number being the half of those in the next preceeding number, like as the logarithm of the last number is the half of the preceeding logarithm. Therefore, any one such small number, with its logarithm, being once found, by the continual extractions of square roots out of a given number, as 10, and corresponding bisections of its given logarithm 1; the logarithm for any other such small number, derived by like continual extractions from another given number, whose
logarithm

Logarithm is sought, will be found by one single proportion: which logarithm is then to be doubled according to the number of extractions, or multiplied at once by the like power of 2, for the logarithm of the number proposed. To find the first small number and its logarithm, our author begins with the number 10, and its logarithm 1, and extracts continually the root of the last number, and bisects its logarithm, as here registered in the annexed table, but to far more places of figures, till he arrives at the 53d and 54th roots, with their annexed logarithms, as here below:

	10, given no.	1, its log.
1	3.162,277 &c.	0.5
2	1.778,279	0.25
3	1.333,521	0.125
4	1.154,781	0.062,5
5	1.074,607	0.031,25
	&c.	&c.

	Numbers.	Logarithms.
53	1.000,000,000,000,000,255,638,298,640,064,70	0.000,000,000,000,000,111,022,302,462,515,654,04
54	1.000,000,000,000,000,127,819,149,320,032,35	0.000,000,000,000,000,055,511,151,231,257,827,02

where the decimals in the natural numbers are to each other in the ratio of the logarithms, namely, in the ratio of 2 to 1: and therefore any other such small number being found, by continual extraction or otherwise, it will then be as 0.000,000,000,000,000,127,819,149,320,032,35 &c, is to 0.000,000,000,000,000,055,511,151,231,257,827,02 &c, so is that other small decimal, to the corresponding significant figures of its logarithm. But as every repetition of this proportion requires both a very long multiplication and a very long division, he reduces this constant ratio to another equivalent ratio whose antecedent is 1, by which all the divisions are saved: thus,

as 0.000,000,000,000,000,127,819,149,320,032,35 &c, is to 0.000,000,000,000,000,055,511,151,231,257,827,02, &c, so is 1.000,000,000,000,000,000, to 0.434,294,481,903,251,804, &c, that is, the logarithm of 1.000,000,000,000,000,000,1, is 0.000,000,000,000,000,043,429,448,190,325,180,4; and therefore this last number being multiplied by any such small decimal, found as above by continual extraction, the product will be the corresponding logarithm of such last root.

But as the extraction of so many roots is a very troublesome operation, our author devises some ingenious contrivances to abridge that labour. And first, in the 7th chapter, by the following device, to have fewer and easier extractions to perform: namely, raising the powers from any given prime number, whose logarithm is sought, till a power of it be found such that its first figure on the left hand is 1, and the next to it either one or more ciphers; then, having divided this power by 1 with as many ciphers as it has figures after the first, or supposing all after the first to be decimals, the continual roots from this power are extracted till the decimal become sufficiently small, as when the first fifteen places are ciphers; and then by multiplying the decimal by 43429 &c, we have the logarithm of this last root: which logarithm multiplied by the like power of the number 2, gives the logarithm of the first number from which the extraction was begun: to this logarithm prefixing a 1, or 2, or 3, &c, according as this number was found by dividing the power of the given prime number by 10, or 100, or 1000, &c; and lastly, dividing the result by the index of that power, the quotient will be the required logarithm of the given prime number. Thus, to

find the logarithm of 2 : it is first raised to the 10th power, as in the margin, before the first figures come to be 10 ; then, dividing by 1000, or cutting off for decimals, all the figures after the first or 1, the root is continually extracted from the quotient 1,024, till the 47th extraction, which gives 1 000,000,000,000,000,168,516,057,053,949,77; the decimal part of which multiplied by 43429, &c, gives 0.000,000,000,000,000,073,185,593,690,623,936,8 for its logarithm ; and this being continually doubled for 47 times, will give the logarithms of all the roots up to the first number : or being at once multiplied by the 47th power of 2, viz. 140,737,488,355,328, which is raised as in the annexed table, it gives 0.010,299,956,639,811,952,652,774,4 for the logarithm of the number 1,024, true to 17 or 18 decimals : to this prefix 3, so shall 3.010,2 &c, be the logarithm of 1024 : and lastly, because 2 is the tenth root of 1024, divide by 10, so shall 0.301,029,995,663,981,195,2 be the logarithm required to the given number 2.

2	1
4	2
8	3
16	4
32	5
64	6
128	7
256	8
512	9
1024	10

The logarithms of 1, 2, and 10 being now known ; it is remarked that the logarithm of 5 becomes known ; for since $10 \div 2$ is $= 5$, therefore $\log. 10 - \log. 2 = \log. 5$, which is 0.698,970,004,336,018,805,8 ; and that from the multiplications and divisions of these three 2, 5, 10, with the corresponding additions and subtractions of their logarithms, a multitude of other numbers and their logarithms are produced ; so from the powers of 2 are obtained 4, 8, 16, 32, 64, &c ; from the powers of 5 these 25, 125, 625, 3125, &c ; also the powers of 5 by those of 10 give 250, 1250, 6250, &c ; and the powers of 2 by those of 10 give 20, 200, 2000, &c ; 40, 400, 80, 800, &c ; likewise by division are obtained $2\frac{1}{2}$, $1\frac{1}{4}$, $12\frac{1}{2}$, $6\frac{1}{4}$, $1\frac{3}{5}$, $3\frac{1}{5}$, $6\frac{2}{5}$, &c.

2	1
4	2
8	3
16	4
32	5
64	6
128	7
256	8
512	9
1024	10
1,048,576	20
1,073,741,824	30
1,099,511,627,776	40
140,737,488,355,328	47

He then observes that the logarithm of 3, the next prime number, will be best derived from that of 6, in this manner : 6 raised to the 9th power becomes 10,077,696, which divided by 10,000,000, gives 1.007,769,6, and the root from this continually extracted till the 46th, is 1.000,000,000,000,000,109,985,934,588,155,718,66 ; the decimal part of which multiplied by 43429 &c, gives 0.000,000,000,000,000,047,766,284,478,608,030,4 for its logarithm ; and this 46 times doubled, or multiplied by the 46th power of 2, gives 0.003,361,253,452,792,69 for the logarithm of 1.007,769,6 ; to which adding 7, the logarithm of the divisor 10,000,000, and dividing by 9, the index of the power of 6, there results 0.778,151,250,383,643,63 for the logarithm of 6 ; from which subtracting the logarithm of 2, there remains 0.477,121,254,719,662,44 for the logarithm of 3.

In the eighth chapter our ingenious author describes an original and easy method of constructing, by means of differences, the continual mean proportionals which were before found by the extraction of roots. And this, with the other methods of generating logarithms by differences, in this book as well as in our author's *Trigonometria Britannica*, are, I believe, the first instances that are to be

be found of making such use of differences, and shew him to have been the inventor of what may be called the *Differential Method*. He seems to have discovered this method in the following manner: Having observed that these continual means between 1 and any number proposed, found by the continual extraction of the square roots, approach always nearer and nearer to the halves of each preceeding root, as is visible when they are placed together under each other; and indeed it is found that as many of the significant figures of each decimal part, as there are ciphers between them and the integer 1, agree with the half of those above them; I say, having observed this evident approximation, he subtracted each of these decimal parts, which he called A or the first differences, from half the next preceeding one, and by comparing together the remainders or second differences, called B, he found that the succeeding were always nearly equal to $\frac{1}{4}$ of the next preceeding ones; then taking the difference between each second difference and $\frac{1}{4}$ of the preceeding one, he found that these third differences, called C, were nearly in the continual ratio of 8 to 1; again taking the difference between each C and $\frac{1}{8}$ of the next preceeding, he found that these fourth differences, called D, were nearly in the continual ratio of 16 to 1; and so on, the 5th (E), 6th (F), &c, differences, being nearly in the continual ratio of 32 to 1, of 64 to 1, &c: these plain observations being made, they

1	1,00776,96	
2	1,00387,72833,36962,45663,84655,1	A
3	1,00193,67661,36946,6167,87022,9	$\frac{1}{2}A$
4	1,00096,79146,39099,01728,89072,0	B
5	1,00048,38402,68846,62985,40253,5	$\frac{1}{4}B$
6	1,00024,18908,78824,68563,80872,7	C
7	24,19201,34423,31492,74626,7	$\frac{1}{2}C$
8	292,55598,62928,93754,0	B
9	1,00012,09381,26397,13459,43919,4	$\frac{1}{4}B$
10	12,09454,39412,34281,90436,3	C
11	73,13015,20822,26516,9	$\frac{1}{2}C$
12	73,13899,65732,23438,5	D
13	884,44909,76921,5	$\frac{1}{4}D$
14	1,00006,04672,35055,30968,01600,5	A
15	6,04690,63198,56729,71959,7	$\frac{1}{2}A$
16	18,28143,25761,70359,2	B
17	18,28253,80205,61629,2	$\frac{1}{4}B$
18	110,54443,91270,0	C
19	110,55613,72115,2	$\frac{1}{2}C$
20	1169,80843,2	D
21	1,00003,02331,60505,65775,90479,4	A
22	3,02336,17527,65484,00800,2	$\frac{1}{2}A$
23	4,57021,99708,04320,8	B
24	4,57035,81440,42589,8	$\frac{1}{4}B$
25	13,81732,38269,0	C
26	13,81805,48908,7	$\frac{1}{2}C$
27	73,10639,7	D
28	73,11302,8	$\frac{1}{4}D$
29	663,1	E
30	1,00001,51164,65999,05672,95048,8	A
31	1,51165,80252,82887,98239,7	$\frac{1}{2}A$
32	1,14253,77215,03190,9	B
33	Hitherto the 1,14255,49927,01080,2	$\frac{1}{4}B$
34	smaller differences 1,72711,97889,3	C
35	are found by sub- 1,72716,54783,6	$\frac{1}{2}C$
36	tracting the larger from 4,56894,3	D
37	the parts of the like pre- 4,56915,0	$\frac{1}{4}D$
38	ceding ones. 20,7	E
39	20,7	$\frac{1}{2}E$
40	Here the greater differences 65	$\frac{1}{4}E$
41	remain after subtracting 28555,89	$\frac{1}{2}E$
42	the smaller from the parts 28555,24	D
43	of the difference of 21588,99736,16	$\frac{1}{4}D$
44	the next preceeding 21588,71180,92	C
45	number. 28563,34303,75797,72	$\frac{1}{2}C$
46	28563,22715,04616,80	B
47	75582,32999,52836,47523,40	$\frac{1}{4}B$
48	1,00000,75582,03436,30121,42907,60	A
49	2	$\frac{1}{2}A$
50	1784,70	$\frac{1}{4}A$
51	1784,68	D
52	2698,58897,62	$\frac{1}{2}D$
53	2698,57112,94	C
54	7140,80678,76154,20	$\frac{1}{4}C$
55	7140,77980,19041,26	B
56	37791,02218,15060,71453,80	$\frac{1}{2}B$
57	1,00000,37790,95077,37080,52412,54	A

very

very naturally and clearly suggested to him the notion and method of constructing all the remaining numbers from the differences of a few of the first, found by extracting the roots in the usual way. This will evidently appear from the annexed specimen of a few of the first numbers in the last example for finding the logarithm of 6; where after the 9th number the rest are supposed to be constructed from the preceeding differences of each, as here shewn in the 10th and 11th. And it is evident that, in proceeding, the trouble will become always less and less, the differences gradually vanishing, till at last only the first differences remain. And that generally each less difference is shorter than the next greater, by as many places as there are ciphers at the beginning of the decimal in the number to be generated from the differences.

He then concludes this chapter with an ingenious, but not obvious, method of finding the differences B, C, D, E, &c. belonging to any number, as suppose the 9th from that number itself, independent of any of the preceeding 8th, 7th, 6th, 5th, &c; and it is this: Raise the decimal A to the 2d, 3d, 4th, 5th, &c powers; then will the 2d (B), 3d (C), 4th (D), &c differences, be as here below, viz.

$$\begin{aligned}
 B &= \frac{1}{2}A^2, \\
 C &= \frac{1}{2}A^3 + \frac{1}{8}A^4, \\
 D &= \frac{7}{8}A^4 + \frac{7}{8}A^5 + \frac{7}{16}A^6 + \frac{1}{8}A^7 + \frac{1}{64}A^8, \\
 E &= 2\frac{5}{8}A^5 + 7A^6 + 10\frac{1}{8}A^7 + 12\frac{6}{16}A^8 + 11\frac{1}{64}A^9 + 7\frac{1}{128}A^{10}, \\
 F &= 13\frac{9}{16}A^6 + 81\frac{3}{8}A^7 + 296\frac{8}{16}A^8 + 834\frac{4}{16}A^9 + 1953\frac{2}{16}A^{10} \&c. \\
 G &= 122\frac{1}{16}A^7 + 1510\frac{6}{16}A^8 + 11475\frac{7}{16}A^9 + 68372\frac{7}{16}A^{10} \&c. \\
 H &= 1937\frac{9}{16}A^8 + 47151\frac{9}{16}A^9 + 706845\frac{1}{16}A^{10} \&c. \\
 I &= 54902\frac{8}{16}A^9 + 2558465\frac{2}{16}A^{10} \&c. \\
 K &= 2805527A^{10} \&c. \\
 \&c.
 \end{aligned}$$

Thus in the 9th number of the foregoing example, omitting the ciphers at the beginning of the decimals, we have

$$\begin{aligned}
 A &= 1,51164,65999,05672,95048,8 \\
 A^2 &= 2,28507,54430,06381,6726 \\
 A^3 &= 3,45422,65239,48546,2 \\
 A^4 &= 5,22156,97802,288 \\
 A^5 &= 7,89316,8205 \\
 A^6 &= 11,93168,1 \\
 \&c.
 \end{aligned}$$

Consequently

$$\frac{1}{2}A^2 = 1,14253,77215,03190,8363 = B$$

$$\frac{1}{2}A^3 = 1,72711,32619,74273$$

$$\frac{1}{8}A^4 = 65269,62225$$

$$\frac{1}{2}A^3 + \frac{1}{8}A^4 = 1,72711,97889,36498 = C$$

$$\frac{7}{8}A^4 = 4,56887,35577$$

$$\frac{7}{8}A^5 = 6,90652$$

$$\frac{7}{16}A^6 = 5$$

$$\frac{7}{8}A^4 + \frac{7}{8}A^5 + \frac{7}{16}A^6 = 4,56894,26234 = D$$

$$2\frac{5}{8}A^5 = 20,71957$$

$$7A^6 = 83$$

$$2\frac{5}{8}A^5 + 7A^6 = 20,72040 = E$$

which agree with the like differences in the foregoing specimen.

In the ninth chapter, after observing that from the logarithms of 1, 2, 3, 5, and 10, before found, are to be determined, by addition and subtraction, the logarithms of all other numbers which can be produced from these by multiplication and division; for finding the logarithms of other prime numbers, instead of that in the seventh chapter, our author then shews another ingenious method of obtaining numbers beginning with 1 and ciphers, and such as to bear a certain relation to some prime number by means of which its logarithm may be found. The method is this: Find three products having the common difference 1, and such that two of them are produced from factors having given logarithms, and the third produced from the prime number, whose logarithm is required, either multiplied by itself, or by some other number whose logarithm is given; then the greatest and least of these three products being multiplied together, and the mean by itself, there arise two other products also differing by 1, of which the greater divided by the less, gives for a quotient 1 with a small decimal, having several ciphers at the beginning. Then the logarithm of this quotient being found as before, from thence will be deduced the required logarithm of the given prime number. Thus, if it be proposed to find the logarithm of the prime number 7; here $6 \times 8 = 48$, $7 \times 7 = 49$, and $5 \times 10 = 50$ will be the three products, of which the logarithms of 48 and 50, the 1st and 3d, will be given from those of their factors 6, 8, 5, 10; also $48 \times 50 = 2400$, and $49 \times 49 = 2401$ are the two new products, and $2401 \div 2400 = 1,00041\frac{2}{3}$ their quotient: then the least of 44 means between 1 and this quotient is 1,00000,00000,00000,02367,98249,04333,6405, which multiplied by 43429 &c, produces 0.00000,00000,00000,01028,40172,88387,29715 for its logarithm; which being 44 times doubled, or multiplied by 17592186044416, produces 0,00018,09183,45421,30 for the logarithm of the quotient $1,00041\frac{2}{3}$; which being added to the logarithm of the divisor 2400, gives the logarithm of the dividend 2401; then the half of this logarithm is the logarithm of 49 the root of 2401, and the half of this again gives 0,84509,80400,14256,82 for the logarithm of 7 which is the root of 49.—The author adds another example to illustrate this method; and then sets down the requisite factors, products, and quotients for finding the logarithms of all other prime numbers up to 100.

The 10th chapter is employed in teaching how to find the logarithms of fractions, namely by subtracting the logarithm of the denominator from that of the numerator, then the logarithm of the fraction is the remainder; which therefore is either abundant or defective, that is positive or negative, as the fraction is greater or less than 1.

In the 11th chapter we are shewn an ingenious contrivance for very accurately finding intermediate numbers to given logarithms, by the proportional parts. On this occasion it is remarked, that while the absolute numbers increase uniformly, the logarithms increase unequally, with a decreasing increment; for which reason it happens, that either logarithms or numbers corrected by means of the proportional parts, will not be quite accurate, the logarithms so found being always too small, and the absolute numbers so found too great; but yet so however as that they approach much nearer to accuracy towards the end of the table, where the increments or differences become much nearer to equality, than in the former parts

parts of the table. And from this property our author, ever fruitful in happy expedients to obviate natural difficulties, contrives a device to throw the proportional part, to be found from the numbers and logarithms, always near the end of the table, in whatever part they may happen naturally to fall. And it is this: Rejecting the characteristic of any given logarithm, whose number is proposed to be found, take the arithmetical complement of the decimal part, by subtracting it from 1,000 &c, the logarithm of 10; then find in the table the logarithm next less than this arithmetical complement, together with its absolute number; to this tabular logarithm add the logarithm that was given, and the sum will be a logarithm necessarily falling among those near the end of the table; find then its absolute number, corrected by means of the proportional part, which will not be very inaccurate, as falling near the end of the table; this being divided by the absolute number, before found for the logarithm next less than the arithmetical complement, the quotient will be the required number answering to the given logarithm; which will be much more correct than if it had been found from the proportional part of the difference where it naturally happened to fall: and the reason of this operation is evident from the nature of logarithms. But as this divisor, when taken as the number answering to the logarithm next less than the arithmetical complement, may happen to be a large prime number; it is farther remarked, that instead of this number and its logarithm, we may use the next less composite number which has small factors, and *its* logarithm; because the division by those small factors, instead of by the number itself, will be performed by the short and easy way of division in one line. And for the more easy finding proper composite numbers and their factors, our author here subjoins an abacus or list of all such numbers, with their logarithms and component factors, from 1000 to 10000; from which the proper logarithms and factors are immediately obtained by inspection. Thus, for example, to find the root of 10800, or the mean proportional between 1 and 10800: The logarithm of 10800 is 4.033,423,755,486,95, the half of which is 2.016,711,877,743,47, the logarithm of the number sought. The arithmetical complement of the decimal part of this logarithm, to wit, the decimal fraction 0.016,711,877,743,47, to 1, (the logarithm of 10,) is 0.983,288,122,256,53. Now the nearest logarithm to this in the *abacus* is 0.982,271,233,039,57, and its annexed number is 9600, the factors of which are 2, 6, 8, besides the ciphers. To this last logarithm, 0.982,271,233,039,57, if we add the decimal part of the logarithm sought, to wit, 0.016,711,877,743,47, the sum will be 0.998,983,110,783,04, whose absolute number, corrected by the proportional part, is 9.976,612,651,652,1; which being divided continually by 2, 6, 8, the factors of 96, the last quotient is 103.923,048,454,71; which is pretty correct, the true number being $103.923,048,454,133 = \sqrt{10800}$.

We now arrive at the 12th and 13th chapters, in which our ingenious author first of all teaches the rules of the *Differential Method*, in constructing logarithms by interpolation from differences. This is the same method which has since been more largely treated of by later authors, and particularly by the learned Mr. Cotes, in his *Canonotechnia*. How Mr. Briggs came by it, does not well appear, as he only delivers the rules, without laying down the principles or investigation

of

of them. He divides the method into two cases, namely, when the second differences are equal, or nearly equal, and when the differences run out to any length whatever. The former of these is treated in the 12th chapter; and he particularly adapts it to the interpolating 9 equidistant means between two given terms, evidently for this reason, that then the powers of 10 become the principal multipliers or divisors, and so the operations may be performed mentally. The substance of his process is this: Having given two absolute numbers with their logarithms, to find the logarithms of 9 arithmetical means between the given numbers: Between the given logarithms take the 1st difference, as well as between each of them and their next or equidistant greater and less logarithms; and likewise the 2d differences, or the two differences of these three 1st differences; then if these 2d differences be equal, multiply one of them severally by the numbers 45, 35, &c, in the annexed tablet, dividing each product by 1000, that is, cutting off three figures from each; lastly, to $\frac{1}{10}$ of the 1st difference of the given logarithms add severally the first five quotients, and subtract the other five: so shall the ten results be the respective 1st differences to be continually added, to compose the required series of logarithms. Now this amounts to the same thing as what is at this day taught in the like case: we know that if A be any term of an equidistant series of terms, and $a, b, c, \&c$, the first of the 1st, 2d, 3d, &c, order of differences; then the term z , whose distance from A is expressed by x , will be thus, $z = A + x a + x \cdot \frac{x-1}{2} b + x \cdot \frac{x-1}{2}$

1	45	Additive products.
2	35	
3	25	
4	15	
5	5	
6	5	Subtractive products.
7	15	
8	25	
9	35	
10	45	

$\frac{x-2}{3} c + \&c$. And if now, with our author, we make the 2d differences equal, then $c, d, e, \&c$, will all vanish, or be equal to 0, and z will become barely $= A + x a + x \cdot \frac{x-1}{2} b$. Therefore

if we take x successively equal to $\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \&c$, we shall have the annexed series of terms with their differences. Where it is to be observed, that our author had reduced the differences from the 1st to the 2d form, as he thought it easier to multiply by 5 than to divide by 2. Also all the last terms ($x \cdot \frac{x-1}{2} b$) are set down positive, because in the logarithms b is negative.

Series of terms.	The Differences.
A	
$A + \frac{1}{10}a + \frac{2}{200}b$	$\frac{1}{10}a + \frac{2}{200}b = \frac{1}{10}a + \frac{4}{1000}b$
$A + \frac{2}{10}a + \frac{1}{100}b$	$\frac{1}{10}a + \frac{7}{200}b = \frac{1}{10}a + \frac{3}{1000}b$
$A + \frac{3}{10}a + \frac{2}{200}b$	$\frac{1}{10}a + \frac{5}{200}b = \frac{1}{10}a + \frac{2}{1000}b$
$A + \frac{4}{10}a + \frac{3}{200}b$	$\frac{1}{10}a + \frac{3}{200}b = \frac{1}{10}a + \frac{1}{1000}b$
$A + \frac{5}{10}a + \frac{2}{200}b$	$\frac{1}{10}a + \frac{1}{200}b = \frac{1}{10}a + \frac{5}{1000}b$
$A + \frac{6}{10}a + \frac{1}{100}b$	$\frac{1}{10}a - \frac{1}{200}b = \frac{1}{10}a - \frac{5}{1000}b$
$A + \frac{7}{10}a + \frac{2}{200}b$	$\frac{1}{10}a - \frac{3}{200}b = \frac{1}{10}a - \frac{1}{1000}b$
$A + \frac{8}{10}a + \frac{1}{100}b$	$\frac{1}{10}a - \frac{5}{200}b = \frac{1}{10}a - \frac{2}{1000}b$
$A + \frac{9}{10}a + \frac{2}{200}b$	$\frac{1}{10}a - \frac{7}{200}b = \frac{1}{10}a - \frac{3}{1000}b$
$A + a$	$\frac{1}{10}a - \frac{9}{200}b = \frac{1}{10}a - \frac{4}{1000}b$

If the two 2d differences be only nearly equal, take an arithmetical mean between them, and proceed with it the same as above with one of the equal 2d differences. He also shews how to find any one single term, independent of the rest; and concludes the chapter with pointing out a method of finding the proportional part more accurately than before.

In the 13th chapter our author remarks, that the best way of filling up the intermediate

intermediate chiliads of his table, namely, from 20,000 to 90,000, is by quinquisection, or interposing four equidistant means between two given terms; the method of performing which he thus particularly describes. Of the given terms, or logarithms, and two or three others on each side of them, take the 1st, 2d, 3d, &c, differences, till the last differences come out equal, which suppose to be the 5th differences: divide the 1st differences by 5, the 2d by 25, the 3d by 125, the 4th by 625, and the 5th by 3125, and call the respective quotients the 1st, 2d, 3d, 4th, 5th *mean* differences; or, instead of dividing by these powers of 5, multiply by their reciprocals $\frac{1}{5}, \frac{1}{25}, \frac{1}{125}, \frac{1}{625}, \frac{1}{3125}$, that is multiplied by 2, 4, 8, 16, 32, cutting off respectively one, two, three, four, five figures from the end of the products, for the several mean differences: then the 4th and 5th of these mean differences are sufficiently accurate, but the 1st, 2d, and 3d, are to be corrected in this manner: From the mean third differences subtract three times the 5th difference, and the remainders are the *correct* 3d differences; from the mean 2d differences subtract double the 4th differences, and the remainders are the *correct* 2d differences; lastly, from the mean 1st differences take the *correct* 3d differences, and $\frac{1}{5}$ of the 5th difference, and the remainders will be the *correct* first differences. Such are the corrections when the differences extend as far as the 5th. However, in compleating those chiliads in this way, there will be only 3 orders of differences, as neither the 4th nor 5th will enter the calculation, but will vanish through their smallness: therefore the mean 2d and 3d difference will need no correction, and the mean 1st differences will be corrected by barely subtracting the 3d from them. These preparatory numbers being thus found, all the 2d differences of the logarithms required, will be generated by adding continually, from the less to the greater, the constant 3d difference; and the series of 1st differences will be found by adding the several 2d differences; and, lastly, by adding continually these 1st differences to the 1st given logarithm, &c, the required logarithmic terms will be generated.

These easy rules being laid down, Mr. Briggs next teaches how by them the remaining chiliads may best be compleated: namely, having here the logarithms for all numbers up to 20,000, find the logarithms to every 5 beyond this, or of 20,005, 20,010, 20,015, &c, in this manner; to the logarithms of the 5th part of each of those, namely 4001, 4002, 4003, &c. add the constant logarithm of 5, and the sums will be the logarithms of all the terms of the series 20,005, 20,010, 20,015, &c: and these logarithms will have the very same differences as those of the series 4001, 4002, 4003, &c; by means of which therefore interpose 4 equidistant terms by the rules above; and thus the whole canon will be easily compleated.

He here also extends the rules for correcting the mean differences in quinquisection, as far as the 20th difference; he also lays down similar rules for trisection, and speaks of general rules for any other section, but which are omitted as being less easy. So that he appears to have been possessed of all that Cotes afterwards delivered in his *Canonotechnia, sive Constructio Tabularum per Differentias*, drawn from the *Differential Method*, as their general rules exactly agree; Briggs's mean and *correct* differences being by Cotes called round and quadrat differences, because he expresses them by the numbers 1, 2, 3, &c. written respectively in a small circle and square.

Mr.

Mr. Briggs also observes that the same rules equally apply to the construction of equidistant terms of any other kind, such as sines, tangents, secants, the powers of numbers, &c: and farther remarks, that of the sines of three equidifferent arcs, all the remote differences may be found by the rule of proportion, because the sines and their 2d, 4th, 6th, 8th, &c, differences are continued proportionals, as are also the 1st, 3d, 5th, 7th, &c, differences among themselves; and like as the 2d, 4th, 6th, &c, differences are proportional to the sines of the mean arcs, so also are the 1st, 3d, 5th, &c, differences proportional to the cosines of the same arcs. Moreover, with regard to the powers of numbers, he remarks the following curious properties: 1st, that they will each have as many orders of differences as are denoted by the index of the power, the squares having two orders of differences, the cubes three, the 4th powers four, &c: second, that the last differences will be all equal, and each equal to the common difference of the sides or roots raised to the given power, and multiplied by $1 \times 2 \times 3 \times 4$ &c, continued to as many terms as there are units in the index; so if the roots differ by 1, the 2d difference of the squares will be each 1×2 or 2, the third differences of the cubes each $1 \times 2 \times 3$ or 6, the 4th differences of the 4th powers each $1 \times 2 \times 3 \times 4$ or 24, and so on; and if the common difference of the roots be any other number n , then the last differences of the squares, cubes, 4th powers, 5th powers, &c, will be respectively $2n^2$, $6n^3$, $24n^4$, $120n^5$, &c.

Besides what was shewn in the eleventh chapter concerning the taking out the logarithms of large numbers by means of proportional parts, he employs the next or 14th chapter in teaching how, from the first ten chiliads only, and a small table of one page here given, to find the number answering to any logarithm, and the logarithm to any number consisting of fourteen places of figures*.

Having thus fully shewn the construction and chief properties of his logarithms, our ingenious author, in the remaining eighteen chapters, exemplifies their uses in various curious and important subjects: such as the Rule-of-three, or rule of proportion; finding the roots of given numbers; finding any number of mean proportionals between two given terms; with other arithmetical rules: Also various geometrical subjects; as, 1st, Having given the sides of any plane triangle, to find the area, perpendicular, angles, and diameters of the inscribed and circumscribed circles; 2d, In a right-angled triangle, having given any two of these, to find the rest, viz. one leg and the hypotenuse, one leg and the sum or

* It is no more than a large exemplification of this method of Briggs's that has been printed so late as 1771, in a 4to. tract by Mr. Rob. Flower, under the title of *The Radix, a New Way of making Logarithms*; although Briggs's work might not be known to this writer.—Since this was written I have been favoured with the following anecdote concerning Mr. Flower and his work, by the Rev. Dr. Horsley, the learned editor of the works of Sir I. Newton. "This Robert Flower was a very obscure, and probably an illiterate man. He was master of a writing school in the town of Bishop Stortford in Hertfordshire. He communicated his Radix, before he published it, to my late learned friend Math. Raper, Esq. of Thorley Hall. I was at Thorley at the time, upon a visit to my father, who was rector of the parish; and I well remember that Mr. Raper told me, with great surprize, that Flower (who was known to us both by name as the writing-master of the neighbouring market town) had fallen upon Briggs's way of finding all logarithms from the first ten chiliads. And he was so well persuaded that Flower had made the discovery for himself, without any light from Briggs, that with his accustomed munificence he rewarded the man's ingenuity with a present of ten guineas; informing him, I believe, that his work had been done before, and dissuading the publication."

difference of the hypotenuse and the other leg, the two legs, one leg and the area, the area and the sum or difference of the legs, the hypotenuse and sum or difference of the legs, the hypotenuse and area, and the perimeter and area; 3d, Upon a given base to describe a triangle equal and isoperimetrical to another triangle given; 4th, To describe the circumference of a circle so, that the three distances from any point in it to the three angles of a given plane triangle, shall be to one another in a given ratio; 5th, Having given the base, the area, and the ratio of the two sides of a plane triangle, to find the sides; 6th, Given the base, difference of the sides, and area of a triangle, to find the sides; 7th, To find a triangle whose area and perimeter shall be expressed by the same number; 8th, Of four given lines, of which the sum of any three is greater than the fourth, to form a quadrilateral figure about which a circle may be described; 9th, Of the diameter, circumference, and area of a circle, and the surface and solidity of the sphere generated by it, having any one given, to find any of the rest; 10th, Concerning the ellipse, spheroid, and gauging; 11th, To cut a line or a number in extreme and mean ratio; 12th, Given the diameter of a circle, to find the sides and areas of the inscribed and circumscribed regular figures of 3, 4, 5, 6, 8, 10, 12, and sixteen sides; 13th, Concerning the regular figures of 7, 9, 15, 24, and 30 sides; 14th, Of isoperimetrical regular figures; 15th, Of equal regular figures; and 16th, Of the sphere and the 5 regular bodies; which closes this introduction. Such of these problems as can admit of it, are determined by elegant geometrical constructions, and they are all illustrated by accurate arithmetical calculations performed by logarithms; for the exemplification of which they are purposely given.

At the end he remarks, that the chief and most necessary use of logarithms, is in the doctrine of spherical trigonometry, which he here promises to give in a future work, and which was accomplished in his *Trigonometria Britannica*, to the description of which we now proceed.

Of BRIGGS's *Trigonometria Britannica*.

At the close of the account of writings on the natural sines, tangents, and secants, I omitted the description of this work of our learned author, although it is perhaps the greatest of this kind, all things considered, that ever was executed by one person; purposely reserving my account of it to this place, not only as it is connected with the invention and construction of logarithms, but thinking it deserved more peculiar and distinguished notice, on account of the importance and originality of its contents. The division of the quadrant, and the mode of construction, are both new; and the numbers are far more accurate, and are extended to more places, than they had ever been before. The circular arcs had always been divided in a sexagesimal proportion; but here the quadrant is divided into degrees and decimals, as this is a much easier mode of computation than by 60ths; the division being completed only to 100ths of degrees, though his design was to have extended it to 1000ths of degrees. And, besides his own private opinion, he was induced to adopt this method of decimal divisions, partly
at

at the request of other persons, and partly perhaps from the authority of Vieta, p. 29 *Calendarii Gregoriani*. And it is probable that computations by this decimal division would have come into general use, had it not been for the publication of Vlacq's tables, which were extended to every 10 seconds, or 6th parts of minutes. But besides this method by a decimal division of the degrees, of which the whole circle contains 360, or the quadrant 90, in the 14th chapter he remarks that some other persons were inclined rather to adopt a complete decimal division of the whole circle, first into 100 parts, and each of these into 1000 parts; and for *their* sakes he subjoins a small table of the sines of every 40th part of the quadrant, and remarks that from these few the whole may be made out by continual quinquisections; namely, 5 times these 40 make 200; then 5 times these give 1000; thirdly, 5 times these give 5000; and, lastly, 5 times these give 25,000 for the whole quadrant, or 100,000 for the whole circumference.

But to return. Our author's large table consists of natural sines to 15 places, natural tangents and secants each to 10 places, logarithmic sines to 14 places, and logarithmic tangents to 10 places, each besides the characteristic: a most stupendous performance! The table is preceeded by an introduction, divided into two books, the one containing an account of the truly ingenious construction of the table, by the author himself; and the other its uses in trigonometry, &c, by Henry Gellibrand, professor of astronomy in Gresham College, who remarks in the preface that the work was composed by the author about the year 1600; though it was only published by the direction of Gellibrand in 1633, it having been printed at Gouda under the care of Vlacq, and by the printer of his *Trigonometria Artificialis*, which came out the same year.

After briefly mentioning the common methods of dividing the quadrant, and constructing the tables of sines, &c, from the ancients down to his own time, he hastens to the description of his own peculiar and truly ingenious method, which is briefly this: Having first divided the quadrant into a small number of parts, as 72, he finds the sine of one of those parts; then from it the sines of the double, triple, quadruple, &c, up to the quadrant, or 72 parts. He next quinquisects each of these parts, by interposing four equidistant means, by differences; he then quinquisects each of these; and finally each of these again; which completes the division as far as degrees and centesims. The rules for performing all these things, he investigates and illustrates in a very ample manner. In treating of multiple and submultiple arcs, he gives general algebraical expressions, for the sine or chord of any multiple whatever of a given arc, which he deduced from a geometrical figure, by finding the law for the series of successive multiple chords or sines, after the manner of Vieta, who was the first person that I know of, who laid down general rules for the chords of multiples and submultiples of arcs or angles; and the same was afterwards improved by Sir I. Newton, to such form, that radius, and double the cosine of the first given angle, are the first and second terms of all the proportions for finding the sines and cosines of the multiple angles. For assigning the coefficients of the terms in the multiple expressions, our author here delivers the construction of figurate or polygonal numbers, inserts a large table of them, and teaches their several uses; one of which is, that every other number taken in the diagonal lines, furnishes the coefficients of the terms of

the general equation by which the sines and chords of multiple arcs are expressed, which he amply illustrates; and another, that the same diagonal numbers constitute the coefficients of the terms of any power of a binomial; which property was also mentioned by Vieta in his *Angulares Sectiones*, theor. 6, 7: and this is the first mention I have seen made of this law of the coefficients of the powers of a binomial, commonly called Sir I. Newton's binomial theorem, although it is very evident that Sir Isaac was not the first inventor of it; the part of it properly belonging to him seems to be only the extending of it to fractional indexes, which was indeed an immediate effect of the general method of denoting all roots like powers with fractional exponents, the theorem being not at all altered.

Of the binomial theorem. However it appears that our author Briggs was the first who taught the rule for generating the coefficients of the terms, successively one from another, of any power of a binomial, independent of those of any other power. For having shewn, in his *Abacus Παγκρητος* (which he so calls on account of its frequent and excellent use, and of which a small specimen is here annexed), that the numbers

ABACUS ΠΑΓΧΡΗΣΤΟΣ.							
H	G	F	E	D	C	B	A
−⑧	−⑦	+⑥	+⑤	−④	−③	+②	①
1	1	1	1	1	1	1	1
9	8	7	6	5	4	3	2
	36	28	21	15	10	6	3
		84	56	35	20	10	4
			126	70	35	15	5
				126	56	21	6
					84	28	7
						56	8
							9

in the diagonal directions, ascending from right to left, are the coefficients of the powers of a binomial, the indexes being the figures in the first perpendicular column A, which are also the coefficients of the 2d terms of each power (those of the first terms, being 1, are here omitted); and that any one of these diagonal numbers is in proportion to the next higher in the diagonal, as the vertical of the former is to the marginal of the latter, that is, as the uppermost number in the column of the former is to the first or right-hand number in the line of the latter; having shewn these things, I say, he thereby teaches the generation of the coefficients of any power, independently of all other powers, by the very same law or rule which we now use in the binomial theorem. Thus, for the 9th power; 9 being the coefficient of the 2d term, and 1 always that of the first, to find the 3d coefficient we have $2 : 8 :: 9 : 36$; for the 4th term, $3 : 7 :: 36 : 84$; for the 5th term, $4 : 6 :: 84 : 126$; and so on for the rest. That is to say, the coefficients of the terms in any power m , are inversely as the vertical numbers or first line $1, 2, 3, 4, \dots, m$, and directly as the ascending numbers $m, m - 1, m - 2, m - 3, \dots, 1$, in the first column A; and that consequently those coefficients

cients are found by the continual multiplication of these fractions $\frac{m}{1}, \frac{m-1}{2}, \frac{m-2}{3}, \frac{m-3}{4}, \dots, \frac{1}{m}$, which is the very theorem as it stands at this day, and as applied by Newton to roots or fractional exponents, as it had before been used for integral powers. This theorem then being thus plainly taught by Briggs about the year 1600, I am surpris'd how a man of such general reading as Dr. Wallis was, could possibly be ignorant of it, as he plainly appears to be by the 85th chapter of his Algebra, where he fully ascribes the invention to Newton, and adds that he himself had formerly sought after such a rule, but without success: or how Mr. John Bernoulli, not half a century since, could himself first dispute the invention of this theorem with Newton, and then give the discovery of it to M. Pascal, who was not born till long after it had been taught by Briggs. See Bernoulli's *Works*, vol. 4. p. 173. But I do not wonder that Briggs's remark was unknown to Newton, who owed almost every thing to genius, and very little to reading: and I have no doubt that he made the discovery himself, without any light from Briggs; and that he thought it was new for all powers in general, as it was indeed for roots and quantities with fractional and irrational exponents.

When the above table of the sums of figurate numbers is used by our author in determining the coefficients of the terms of the equation, whose root is the chord of any submultiple of an arc, as when the section is expressed by any uneven number, he remarks that the powers of that chord or root will be the 1st, 3d, 5th, 7th, &c, in the alternate uneven columns, A, C, E, G, &c, with their signs + or -, as marked to the powers, continued till the highest power be equal to the index of the section; and that the coefficients of those powers are the sums of two continuous numbers in the same column with the powers, beginning with 1 at the highest power, and gradually descending one line obliquely to the right at each lower power: so for a trisection, the numbers are 1 in C, and $1 + 2 = 3$ in A; and therefore the terms are $- 1 \textcircled{3} + 3 \textcircled{1}$: for a quinquisection, the numbers are 1 in E, $1 + 4 = 5$ in C, $2 + 3 = 5$ in A; so that the terms are $1 \textcircled{5} - 5 \textcircled{3} + 5 \textcircled{1}$: for a septisection, the numbers are 1 in G, $1 + 6 = 7$ in E, $4 + 10 = 14$ in C, and $3 + 4 = 7$ in A; and so the terms are $- 1 \textcircled{7} + 7 \textcircled{5} - 14 \textcircled{3} + 7 \textcircled{1}$: and so on; the sum of all these terms being always equal to the chord of the whole or multiple arc. But when the section is denominated by an even number, the squares of the chords enter the equation instead of the first powers as before, and the dimensions of all the powers are doubled, the coefficients being found as before, and therefore the powers and numbers will be those in the 2d, 4th, 6th, &c, columns: and the uneven sections may also be expressed the same way: hence for a bisection the terms will be $- 1 \textcircled{4} + 4 \textcircled{2}$; for a trisection $1 \textcircled{6} - 6 \textcircled{4} + 9 \textcircled{2}$; for the quadrisection $- 1 \textcircled{8} + 8 \textcircled{6} - 20 \textcircled{4} + 16 \textcircled{2}$; for the quinquisection $1 \textcircled{10} - 10 \textcircled{8} + 35 \textcircled{6} - 50 \textcircled{4} + 25 \textcircled{2}$; and so on.

Our author also subjoins another table, a small specimen of which is here annexed, in which the first column consists of the uneven numbers, 1, 3, 5, &c; the rest being found by addition as before, and the alternate diagonal numbers themselves are the coefficients.

F	E	D	C	B	A
+⑥	+⑤	-④	-③	+②	①
1	1	1	1	1	1
	7	6	5	4	3
		20	14	9	5
			30	16	7
				25	9
					11

The method is quite different from that of Vieta, who gives another table for the like purpose, a small part of which is here annexed, which is formed by adding from the number 2 downwards obliquely towards the right; and the coefficients of the terms stand upon the horizontal line.

VIETA'S Table.					
1st					
2					
3	2c				
4	2				
5	5	3d			
6	9	2			
7	14	7	4th		
8	20	16	2		
9	27	30	9	5th	
10	35	50	25	2	6th

These angular sections were afterwards further discussed by Oughtred and Wallis. And the same theorems of Vieta and Briggs have been since given in a different form, by Messrs. Herman, and the Bernoullis in the *Leipfic Acts*, and the *Memoirs of the Royal Academy of Sciences*. These theorems they expressed by the alternate terms of the power of a binomial, whose exponent is that of the multiple angle or section. And Mr. De Lagny, in the same Memoirs, first shewed that the tangents and secants of multiple angles are also expressed by the terms of a binomial, in the form of a fraction, of which some of those terms form the numerator, and others the denominator. Thus, if r express the radius, s the sine, c the cosine, t the tangent, and f the secant of the angle A ; then the sine, cosine, tangent, and secant of n times the angle are expressed thus, viz.

$$\text{Sin. } nA = \frac{1}{r^{n-1}} \times : \frac{n}{1} c^{n-1} s - \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} c^{n-3} s^3 + \frac{n \cdot n-1 \cdot n-2 \cdot n-3 \cdot n-4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} c^{n-5} s^5 \text{ \&c.}$$

$$\text{Cosine } nA = \frac{1}{r^{n-1}} \times : c^n - \frac{n \cdot n-1}{1 \cdot 2} c^{n-2} s^2 + \frac{n \cdot n-1 \cdot n-2 \cdot n-3}{1 \cdot 2 \cdot 3 \cdot 4} c^{n-4} s^4 \text{ \&c.}$$

$$\text{Tang. } nA = r \times \frac{\frac{n}{s} r^{n-1} t - \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} r^{n-3} t^3 + \frac{n \cdot n-1 \cdot n-2 \cdot n-3 \cdot n-4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} r^{n-5} t^5 \text{ \&c.}}{r^n - \frac{n \cdot n-1}{1 \cdot 2} r^{n-2} t^2 + \frac{n \cdot n-1 \cdot n-2 \cdot n-3}{1 \cdot 2 \cdot 3 \cdot 4} r^{n-4} t^4 \text{ \&c.}}$$

$$\text{Sec. } nA = r \times \frac{f^2 \text{ or } r^2 + t^2}{r^n - \frac{n \cdot n-1}{1 \cdot 2} r^{n-2} t^2 + \frac{n \cdot n-1 \cdot n-2 \cdot n-3}{1 \cdot 2 \cdot 3 \cdot 4} r^{n-4} t^4 \text{ \&c.}}$$

where it is evident that the series in the value of the sine of the multiple angle nA consists of the even terms of the series that is equal to $\overline{c+s}^n$, or the n th power of the binomial $c+s$, and the series in the value of the cosine of the same multiple angle consists of the uneven terms of the same power; also the series in the numerator of the tangent consists of the even terms of the power $\overline{r+t}^n$, and the

the denominator, both of the tangent and secant, consists of the uneven terms of the same power $r + 1^n$. And, if the diameter, chord, and chord of the supplement, be substituted for the radius, sine, and cosine, in the expressions for the sine and cosine of the multiple arc, the result will give the chord and chord of the supplement of n times the arc, or angle, A . These and various other expressions for multiple and submultiple arcs, with other improvements in trigonometry, have also been given by Euler and other eminent writers on the subject.

The before-mentioned M. De Lagny offered a project for substituting, instead of the common logarithms, a binary arithmetic, which he called the *natural logarithms*, and which he and M. Leibnitz seem to have both invented about the same time, independently of each other: but the project came to nothing. Mr. De Lagny also published, in several Memoirs of the Royal Academy, a new method of determining the angles of figures, which he called *Goniometry*. It consists in measuring with a pair of compasses the arc which subtends the angle in question; however this arc is not measured by applying its extent to any pre-constructed scale, but by examining what part it is of half the circumference of the same circle, in this manner: from the proposed angular point as a center, with a sufficiently large radius, a semicircle being described, a part of which is the arc intercepted by the sides of the proposed angle, the extent of this arc is taken with a fine pair of compasses, and applied continually upon the arc of the semicircle, by which he finds how often it is contained in the semicircle, with usually a small arc remaining; in the same manner he measures how often this remaining arc is contained in the first arc, and what remains again is applied continually to the first remainder, and so the 3d remainder to the 2d, the 4th to the 3d, and so on till there be no remainder, or else till it become insensibly small. By this process he obtains a series of quotients, or fractional parts, one of another, which being properly reduced into one, give the ratio of the first arc to the semicircumference, or the ratio of the proposed angle to two right angles or 180 degrees, and consequently the magnitude of that angle in degrees, minutes, &c, if required, and that commonly to a degree of accuracy far exceeding the calculation of the same by means of any tables of sines, tangents, or secants, notwithstanding the apparent paradox in this expression at first sight. Thus, if the 1st arc be 4 times contained in the semicircle, the remainder once contained in the first arc, the next five times in the second, and finally the fourth two times in the third: here the quotients are 4, 1, 5, 2; consequently the fourth or last arc was $\frac{1}{2}$ the 3d, therefore the 3d was $\frac{1}{5\frac{1}{2}}$ or $\frac{2}{11}$ of the 2d, and the 2d was $\frac{1}{1\frac{2}{11}}$ or $\frac{11}{13}$ of the 1st, and the first or arc sought, was $\frac{1}{4\frac{11}{13}}$ or $\frac{13}{63}$ of the semicircle; and consequently it contains $37\frac{1}{7}$ degrees, or $37^{\circ} 8' 34''\frac{2}{7}$.

But to return from this long digression, Mr. Briggs next treats of interpolation by differences, and chiefly of quinquisection, after the manner used in the 13th chapter of his construction of logarithms before described. He here proves that curious property of the sines and their several orders of differences, before mentioned, namely, that, of equidifferent arcs, the sines with the 2d, 4th, 6th, &c, differences, are continued proportionals; as also the cosines of the means be-

tween those arcs, and the 1st, 3d, 5th, &c, differences. And to this treatise on interpolation by differences, he adds a marginal note, complaining that this 13th chapter of his *Arithmetica Logarithmica* had been omitted by Vlacq in his edition of it; as if he were afraid of an intention to deprive him of the honour of the invention of interpolation by successive differences. The note is this: *Modus correctionis à me traditus est in Arithmetica Logarithmica capite 13, in editione Londinensi: Istud autem caput, unà cum sequenti in editione Batavâ, me inconsulto et inscio omissum fuit: nec in omnibus editionis illius author (vir alioqui industrius et non indoctus) meam mentem videtur assequutus. Ideoque, ne quicquam desit cuiquam qui integrum canonem conficere cupiat, quædam maximè necessaria illinc huc transferenda censui.*

A large specimen of quinquisection by differences is then given, and he shews how it is to be applied to the construction of the whole canon of sines, both for 100th and 1000th parts of degrees; namely, for centesms, divide the quadrant first into 72 equal parts, and find their sines by the primary methods; then these quinquisectioned give 360 parts, a second quinquisection gives 1800 parts, and a third gives 9000 parts, or centesms of degrees: but for millesms, divide the quadrant into 144 equal parts; then one quinquisection gives 720, a second gives 3600, a third 18,000, and a fourth gives 90,000 parts or millesms.

He next proceeds to the natural tangents and secants, which are directed to be raised in the same manner, by interpolations from a few primary ones, constructed from the known proportions between sines, tangents, and secants; excepting that half the tangents and secants are to be formed by addition and subtraction only, by means of some such theorems as these, namely, 1st, the secant of an arc is equal to the sum of the tangent of the same arc, and the tangent of half its complement, which will find every other secant; 2d, Double the tangent of an arc, added to the tangent of half its complement, is equal to the tangent of the sum of that arc and the said half complement, by which rule half the tangents will be found; &c.

In the two remaining chapters of this book are treated the construction of the logarithmic sines, tangents, and secants. This is preceded by some remarks on the origin and invention of them. Our author here observes that logarithms may be of various kinds: that others had followed the plan of Baron Napier, the first inventor; among whom Benjamin Urfinus is especially commended, who applied Napier's logarithms to every ten seconds of the quadrant; but that he himself, encouraged by the noble inventor, devised other logarithms that were much easier and more excellent*. He says he put 10, with cyphers, for the logarithm of radius; 9 for the logarithm sine of $5^{\circ} 44'$, whose natural sine is one 10th of the radius; 8 for that of $34'$, whose natural sine is one 100th of the radius, &c: thereby making 1 the logarithm of the ratio of 10 to 1, which is the characteristic of his species of logarithms.

To construct the logarithmic sines, he directs first to divide the quadrant into 72 equal parts as before, and to find the logarithms of their natural sines, as in

* His words are "Ego verò, ipsius inventoris primi cohortatione adjutus, alios logarithmos applicandos censui, qui multo faciliorem usum habent, præstantiorémque. Logarithmus radii circularis vel sinus totius, a me ponitur 10 &c."

the 14th chapter of his *Arithmetica Logarithmica*; after which this number will be increased by quinquisection, first to 360, then to 1800, and lastly to 9000, or centesims of degrees. But if millesims of degrees be required, divide the quadrant first into 144 equal parts, and then by four quinquisections these will be extended to the following parts, 720, 3600, 18,000, and 90,000, or millesims of degrees. He remarks however that the logarithmic fines of only half the quadrant need be found in this manner; as the other half may be found by mere addition, or subtraction, by means of this theorem, as the sine of half an arc is to half the radius, so is the sine of the whole arc to the cosine of the said half arc. This theorem he illustrates with examples; and then adds a table of the logarithmic fines of the primary 72 parts of the quadrant, from which the rest are to be made out by quinquisection.

In the next chapter our author shews the construction of the natural tangents and secants more fully than he had done before, demonstrating and illustrating several curious theorems for the easy finding of them. He then concludes this chapter, and the book, with pointing out the very easy construction of the logarithmic tangents and secants by means of these three theorems.

- 1st, As cosine : sine :: radius : tangent,
- 2d, As tangent : radius :: radius : cotangent,
- 3d, As cosine : radius :: radius : secant.

So that in logarithms, the tangents are found by subtracting the cosines from the sines, adding always 10 or the radius; the cotangents are found by subtracting always the tangents from 20 or double the radius; and the secants are found by subtracting the cosines from 20 the double radius.

The 2d book, by Gellibrand, contains the use of the canon in plane and spherical trigonometry.

Besides Briggs's methods of constructing logarithms, above described, no others were given about that time. For as to the calculations made by Vlacq, his numbers being carried to comparatively but few places of figures, they were performed by the easiest of Briggs's methods, and in the manner which this ingenious man had pointed out in his two volumes. Thus, the 70 chiliads of logarithms, from 20,000 to 90,000, computed by Vlacq, and published in 1628, being extended only to 10 places, yield no more than two orders of mean differences, which are also the correct differences, in quinquisection, and therefore will be made out thus; namely, one-fifth of them by the mere addition of the constant logarithm of 5; and the other four-fifths of them by two easy additions of very small numbers, namely, of the 1st and 2d differences, according to the directions given in Briggs's *Arith. Log.* c. 13. p. 31. And as to Vlacq's logarithmic fines and tangents to every 10 seconds, they were easily computed thus: the fines for half the quadrant were found by taking the logarithms to the natural fines in Rheticus's canon; and then from these the logarithmic fines to the other half quadrant were found by mere addition and subtraction; and from these all the tangents by one single subtraction. So that all these operations might easily be performed by one person, as quickly as a printer could set up the types; and thus the computation and printing might both be carried on together. And hence it appears that there is no reason for admiration at the expedition with which these tables were said to have been brought out.

Of certain Curves related to Logarithms.

About this time the mathematicians of Europe began to consider some curves which have properties analogous to logarithms. Edmund Gunter, it has been said, first gave the idea of a curve, whose abscissæ are in arithmetical progression, while the corresponding ordinates are in geometrical progression, or whose abscissæ are the logarithms of their ordinates; but I cannot find it noticed in any part of his writings. The same curve was afterwards considered by others, and named the *Logarithmic* or *Logistic* curve by Huygens in his *Dissertatio de Causâ Gravitatis*, where he enumerates all the principal properties of this curve, shewing its analogy to logarithms *. Many other learned men have also treated of its properties; particularly Le Seur and Jacquier in their comment on Newton's Principia; Dr. John Keill in the elegant little tract on logarithms subjoined to his edition of Euclid's Elements; and Francis Maferes, Esq. Cursitor Baron of the Exchequer, in his ingenious treatise on Trigonometry; in which books the doctrine of logarithms is copiously and learnedly treated, and their analogy to the logarithmic curve, &c, fully displayed.—It is indeed rather extraordinary that this curve was not sooner announced to the public; since it results immediately from baron Napier's manner of conceiving the generation of logarithms, by only supposing the lines which represent the natural numbers to be placed at right angles to that upon which the logarithms are taken. This curve greatly facilitates the conception of logarithms to the imagination, and affords an almost intuitive proof of the very important property of their fluxions, or very small increments, to wit, that the fluxion of the number is to the fluxion of the logarithm, as the number is to the subtangent; as also of this property, that, if three numbers be taken very nearly equal, so that their ratios to each other may differ but a little from a ratio of equality, as for example, the three numbers 10,000,000, 10,000,001, 10,000,002, their differences will be very nearly proportional to the logarithms of the ratios of those numbers to each other: all which follows from the logarithmic arcs being very little different from their chords, when they are taken very small. And the constant subtangent of this curve is what was afterwards by Cotes called the *Modulus* of the system of logarithms: and since, by the former of the two properties above mentioned, this subtangent is a 4th proportional to the fluxion of the number, the fluxion of the logarithm, and the number; this property afforded occasion to Mr. Baron Maferes to give the following definition of the modulus, which is the same in effect as Cotes's, but more clearly expressed, namely, that it is the limit of the magnitude of a 4th proportional to these three quantities, to wit, the difference of any two natural numbers that are very nearly equal to each other, either of the said numbers, and the logarithm or measure of the ratio they have to each other. Or we may define the modulus to be the natural number at that part of the system of logarithms, where the fluxion of the number is equal to the fluxion of the logarithm, or where the numbers and logarithms have equal differences. And

* Mr. James Gregory also speaks of this curve in the highest terms of approbation, on account of its utility in illustrating the nature of ratios and logarithms. See his Preface, or Proœmium, to his *Geometriæ Pars Universalis*, published at Padua in Italy in the year 1668.

hence

hence it follows, that the logarithms of equal numbers or of equal ratios, in different systems, are to one another as the *moduli* of those systems. Moreover, the ratio whose measure, or logarithm, is equal to the modulus, and which is therefore by Cotes called the *ratio modularis*, is by calculation found to be the ratio of 2.718,281,828, 459, &c, to 1, or of 1 to .367,879,441,171, &c: the calculation of which number may be seen at full length in Mr. Baron Maferes's treatise on the Principles of Life-annuities, p. 274, 275*.

The hyperbolic curve also afforded another source for developing and illustrating the properties and construction of logarithms. For the hyperbolic areas lying between the curve and one asymptote, when they are bounded by ordinates parallel to the other asymptote, are analogous to the logarithms of their abscissæ or parts of the asymptote. And so also are the hyperbolic sectors; any sector bounded by an arc of the hyperbola and two radii, being equal to the quadrilateral space bounded by the same arc, the two ordinates to either asymptote from the extremities of the arc, and the part of the asymptote intercepted between them. And although Napier's logarithms are commonly said to be the same as hyperbolic logarithms, it is not to be understood that hyperbolas exhibit Napier's logarithms only, but indeed all other possible systems of logarithms whatever. For, like as the right-angled hyperbola, the side of whose square inscribed at the vertex is 1, gives us Napier's logarithms; so any other system of logarithms is expressed by the hyperbola whose asymptotes form a certain oblique angle, the side of the rhombus inscribed at the vertex of the hyperbola in this case also being still 1, the same as the side of the square in the right-angled hyperbola. But the areas of the square and rhombus, and consequently the logarithms of any one and the same number or ratio, will differ according to the sine of the angle of the asymptotes. And the area of the square or rhombus, or any inscribed parallelogram, is also the same thing as what was by Cotes called the modulus of the system of logarithms; which modulus will therefore be expressed by the numerical measure of the sine of the angle formed by the asymptotes, to the radius 1; as that is the same with the number expressing the area of the said square or rhombus, the side being 1: which is another definition of the modulus to be added to those we before remarked above, in treating of the logarithmic curve. And the evident reason of this is, that in the beginning of the generation of these areas from the vertex of the hyperbola, the nascent increment of the abscissæ drawn into the altitude 1, is to the increment of the area, as radius is to the sine of the angle of the ordinate and abscissæ, or of the asymptotes; and at the beginning of the logarithms, the nascent increment of the natural numbers is to the increment of the logarithms, as 1 is to the modulus of the system. Hence we easily discover that the angle formed by the asymptotes of the hyperbola exhibiting Briggs's system of logarithms, will be 25 deg. 44 min. 25½ sec. this being the angle whose sine is 0.434,294,481,9, &c, the modulus of this system.

Or indeed any one hyperbola, as has been remarked by Mr. Baron Maferes, will express all possible systems of logarithms whatever, namely, if the square or rhom-

* This calculation is also set forth in one of the tracts contained in this present volume, pages 358, 359, 360, and 371, 372, 373, 374, 375.

bus inscribed at the vertex, or, which is the same thing, any parallelogram inscribed between the asymptotes and the curve at any other point, be expounded by the modulus of the system; or, which is the same, by expounding the area, intercepted between two ordinates which are to each other in the ratio of 10 to 1, by the logarithm of that ratio in the proposed system.

As to the first remarks on the analogy between logarithms and the hyperbolic spaces, it having been shewn by Gregory St. Vincent, in his *Quadratura Circuli & Sectionum Coni*, published at Antwerp in 1647, that if one asymptote be divided into parts in geometrical progression, and from the points of division ordinates be drawn parallel to the other asymptote, they will divide the space between the asymptote and curve into equal portions; from hence it was shewn by Merfennus, that by taking the continual sums of those parts, there would be obtained areas in arithmetical progression, adapted to abscissies in geometrical progression, and which therefore were analogous to a system of logarithms. And the same analogy was remarked and illustrated soon after by Huygens, and many others, who shew how to square the hyperbolic spaces by means of the logarithms.

Of *Gregory's Computation of Logarithms.

On the other hand, Mr. James Gregory, in his *Vera Circuli et Hyperbolæ Quadratura*, first printed at Patavium, or Padua, in the north of Italy, in the year 1667, having approximated to the asymptotic spaces of an hyperbola by means of a series of inscribed and circumscribed polygons, from thence shews how to compute the logarithms, which are analogous to those areas: and thus the quadrature of the hyperbolic spaces became the same thing as the computation of the logarithms. He here also lays down various methods to abridge the computation, with the assistance of some properties of numbers themselves, by which we are enabled to compose the logarithms of all prime numbers under 1000, each by one multiplication, two divisions, and the extraction of the square root. And the same subject is farther pursued in his *Exercitationes Geometricæ*, to be described hereafter.

There are also innumerable other geometrical figures having properties analogous to logarithms; such as the equi-angular spiral (which is also called the *Logarithmic Spiral*), the figure of tangents, and the figure of secants, &c; which it is not to our purpose to distinguish more particularly.

Of †Mercator's Logarithmotechnia.

In 1668, Mr. Nicholas Mercator published his *Logarithmotechnia, sive Methodus*

* James Gregory was born at Aberdeen in Scotland in 1639, where he was educated. He was professor of Mathematics in the college of St. Andrew; and died of a fever in December 1675, being only 36 years of age.

† Nicholas Mercator, a learned mathematician, and an ingenious member of the Royal Society, was a native of Holstein in Germany, but spent most of his time in England, where he died in the year 1690, at about 50 years of age. He was the author of many other works in Geometry, Geography, Astronomy, Astrology, &c.

construendi Logarithmos nova, accurata, & facilis; in which he delivers a new and ingenious method for computing the logarithms upon principles purely arithmetical; which being curious, and very accurately performed, I shall here give a rather full and particular account of that little tract, as well as of the small specimen of the quadrature of curves by infinite series which is subjoined to it; and the rather, as this work gave occasion to the public communication of some of Sir Isaac Newton's earliest pieces, to evince that he had not borrowed them from this publication. So it appears that these two ingenious men had, independent of each other, in some instances fallen upon the same discoveries.

Our author begins this work with remarking that the word *Logarithm* is composed of the two Greek words *λογος* and *ἀριθμός*, which answer to the words *ratio* and *number*, being as much as to say, the *number of ratios*; which he observes is quite agreeable to the nature of them, for that a logarithm is nothing else but the number of *ratiunculae*, or very small ratios, contained in the ratio which any number bears to unity. He then makes a very learned and critical dissertation on the nature of ratios, their magnitude and measure, conveying a clearer idea of the nature of logarithms than had been given by either Napier or Briggs, or any other writer except the famous Kepler, in his work before described; although those other writers seem indeed to have had in their own minds the same ideas on the subject as Kepler and Mercator, but without having expressed them so clearly. Our author indeed pretty closely follows Kepler in his modes of thinking and expression, and after him in plain and express terms calls logarithms the measures of ratios; and, in order to the right understanding that definition of them, he explains what he means by the magnitude of a ratio. This he does pretty fully, but not too fully, considering the nicety and subtlety of the subject of ratios, and their magnitude, with their addition to, and subtraction from, each other, which are points that have often been misconceived by very learned mathematicians, who have thence been led into considerable mistakes. Witness the oversight of Gregory St. Vincent, which Huygens animadverted upon in the *Εξέτασις Cyclometriae Gregorii, à Sancto Vincentio*, and which arose from not understanding, or not adverting to, the nature of ratios, and their proportions one to another. And many other similar mistakes might here be adduced of other eminent writers. From all which we must commend the propriety of our author's attention, in so judiciously discriminating between the magnitude of a ratio, as of *a* to *b*, and the fraction $\frac{a}{b}$, or quotient arising from the division of one term of the ratio by the other; which latter method of considering ratios is always attended with danger of errors and confusion on the subject: though in the 5th definition of the 6th book of Euclid this quotient is accounted the quantity of the ratio; but this definition is probably not genuine, and therefore very properly omitted by professor Simson in his edition of the *Elements*. And in these ideas on the subject of logarithms, Kepler and Mercator have been followed by Dr. Halley, and Mr. Cotes, and most other eminent writers since that time.

Purely from the above idea of logarithms, namely, as being the measures of ratios, and as expressing the number of *ratiunculae* contained in any ratio, or into which it may be divided, the number of the like equal *ratiunculae* contained in
some

some one ratio, as of 10 to 1, being supposed given, our author shews how the logarithm, or measure, of any other ratio may be found. But this, however, he shews only by-the-by, as not being the principal method he intends to teach, as his last and best, and which we arrive not at till near the end of the book, as we shall see below. Having shewn, then, that these logarithms, or numbers of small ratios, or measures of ratios, may be all properly represented by numbers, and that (the logarithm, or measure, of the ratio of 1 to 1, or of the ratio of equality, or, as it is often called, the logarithm of 1, being always 0), the logarithm of 10, or the measure of the ratio of 10 to 1, is most conveniently represented by 1, with any number of cyphers: he then proceeds to shew how the measures of all other ratios may be found from this last supposition. And he explains the principles by the two following examples.

First, to find the logarithm of 100.5^* , or to find how many *ratiunculae* are contained in the ratio of 100.5 to 1 , the number of *ratiunculae* in the decuple ratio, or ratio of 10 to 1, being 10,000,000.

The given ratio 100.5 to 1 , he first divides into its parts, namely, the ratios of 100.5 to 100 , of 100 to 10 , and of 10 to 1 ; the last two of which being decuples, it follows that the characteristic of the logarithm of the whole ratio of 100.5 to 1 will be 2, and it only remains to find how many parts of the next decuple belong to the first of these three ratios, to wit, the ratio of 100.5 to 100 . Now if each term of this ratio be multiplied by itself, the products will be in the duplicate ratio of the first terms, or this last ratio will contain a double number of parts; and if these be multiplied by the first terms again, the ratio of the last products will contain three times the number of parts; and so on, the number of times of the first parts contained in the ratio of any like powers of the first terms, being always denoted by the exponent of the power. If therefore the first terms, 100.5 and 100 , be continually multiplied till the same powers of them have to each other a ratio whose measure is known, as suppose the decuple ratio 10 to 1 , whose measure is 10,000,000; then the exponent of that power shews what multiple this measure 10,000,000, of the decuple ratio, is of the required measure of the first ratio 100.5 to 100 ; and consequently, dividing 10,000,000, by that exponent, the quotient is the measure of the ratio 100.5 to 100 sought. The operation for finding this he sets down as here follows; where the several multiplications are all performed in the contracted way by inverting the figures of the multiplier, and retaining only the first number of decimals in each product.

* Mercator distinguishes his decimals from integers thus $100[5$, or thus $100]5$.

100.5000	-	-	1
5001	-	-	1
1005000			
5025			
1010025	-	-	2
5200101	-	-	2
1010025			
10100			
20			
5			
1020150	-	-	4
0510201	-	-	4
1020150			
20403			
102			
51			
1040706	-	-	8
6070401	-	-	8
1083068	-	-	16
8603801	-	-	16
1173035	-	-	32
5303711	-	-	32
1376011	-	-	64
1106731	-	-	64
1893406	-		128
6043981	-		128
3584985	-		256
5894853	-		256
12852116	-		512

This power being greater than the decuple of the like power of 100, which must always be 1 with ciphers, resume therefore the 256th power, and multiply it, not by itself, but by the next before it, viz. by the 128th, thus :

3584985	-	-	-	256
6043981	-	-	-	128
6787831	-	-	-	384
1106731	-	-	-	64
9340130	-	-	-	448
5303711	-	-	-	32
10956299	-	-	-	480

This power again exceeding the same power of 100 more than 10 times, I therefore draw the same 448th, not into the 32d, but the next preceding, thus :

9340130	-	-	-	448
8603801	-	-	-	16
10115994	-	-	-	464

This being again too much, instead of the 16th, draw it into the 8th or next preceeding, thus :

9340130	-	-	-	448
6070401	-	-	-	8
9720329	-	-	-	456
0510201	-	-	-	4
9916193	-	-	-	460
5200101	-	-	-	2
10015603	-	-	-	462

Which power again exceeds the limit; therefore draw the 460th into the 1st, thus :

9916193	-	-	-	460
5001	-	-	-	1
9965774	-	-	-	461

Since therefore the 462d power of 100.5 is greater, and the 461st power is less, than the decuple of the same power of 100; I find that the ratio of 100.5 to 100 is contained in the decuple more than 461 times, but less than 462 times. Again,

Since $\left\{ \begin{matrix} 460 \\ 461 \\ 462 \end{matrix} \right\}$ power is $\left\{ \begin{matrix} 9916193 \\ 9965774 \\ 10015603 \end{matrix} \right\}$ and the differences $\left\{ \begin{matrix} 49581 \\ 49829 \end{matrix} \right\}$ nearly equal;

therefore the proportional part which the exact power, or 10000000, exceeds the next less 9965774, will be easily and accurately found by the Golden Rule, thus :

The just power - - - 10000000
and the next less - - - 9965774
the difference - - - 34226; then

As 49829 the dif. between the next less and greater,
: To 34226 the dif. between the next less and just,
:: So is 10000 : to 6868, the decimal parts; and therefore the ratio of 100.5 to 100, is 461.6868 times contained in the decuple or ratio of 10 to 1. Dividing

now 1,0000000, the measure of the decuple ratio, by 461.6868, the quotient .002,165,97 is the measure of the ratio of 100.5 to 100; which being added to 2, the measure of 100 to 1, the sum 2.002,165,97 is the measure of the ratio of 100.5 to 1, that is, the log. of 100.5 is 2.002,165,97.

In the same manner he next investigates the log. of 99.5, and finds it to be 1.997,823,07.

A few observations are then added, calculated to generalize the consideration of ratios, their magnitude and affections. It is here remarked that he considers the magnitude of the ratio between two quantities as the same, whether the antecedent be the greater or the less of the two terms; so the magnitude of the ratio of 8 to 5, is the same as of 5 to 8; that is, by the magnitude of the ratio of either to the other, is meant the number of *ratiunculae* between them, which will evidently be the same whether the greater or less term be the antecedent. And he farther remarks that of different ratios, when we divide the greater term of each ratio by the less, that ratio is of the greater mass, or magnitude, which produces the greater quotient, *et vice versa*; although those quotients are not proportional to the masses, or magnitudes, of the ratios. But when he considers the ratio of a greater term to a less, or of a less to a greater, that is to say, the ratio of greater or less inequality, as abstracted from the magnitude of the ratio, he distinguishes it by the word *affection*, as much as to say, greater or less affection, something in the manner of positive and negative quantities, or such as are affected with the signs + and - The remainder of this work he delivers in several propositions, as follows.

Prop. 1. In subtracting from each other two quantities of the same affection, to wit, both positive or both negative; if the remainder be of the same affection with the two given, then is the quantity subtracted the less of the two, or expressed by the less number; but if the contrary, it is the greater.

Prop. 2. In any continued ratios, as $\frac{a}{a+b}$, $\frac{a+b}{a+2b}$, $\frac{a+2b}{a+3b}$, &c, (by which is meant the ratios of a to $a+b$, of $a+b$ to $a+2b$, of $a+2b$ to $a+3b$, &c) of equidifferent terms, the antecedent of each ratio being equal to the consequent of the next preceeding one, and proceeding from less terms to greater, the measure of each ratio will be expressed by a greater quantity than that of the next following; and the same through all their orders of differences, namely, the 1st, 2d, 3d, &c, differences; but the contrary when the terms of the ratios decrease from greater to less.

Prop.

Prop. 3. In any continued ratios of equi-different terms, if the 1st or least be a , the difference between the 1st and 2d b , and c, d, e , &c, the respective first terms of their 2d, 3d, 4th, &c, differences; then shall the several quantities themselves be as in the annexed scheme; where each term is composed of the first term, together with as many of the differences as it is distant from the first term, and to those differences joining, for coefficients, the numbers in the sloping or oblique lines contained in the annexed table of figurate numbers, in the same manner, he observes, as the same figurate numbers compleat the powers raised from a binomial root, as had long before been taught by others. He also remarks that this rule not only gives any one term, but also the sum of any number of successive terms from the beginning, making the 2d coefficient the first, the 3d the 2d, and so on; thus, the sum of the first 5 terms is $5a + 10b + 10c + 5d + e$.

1st term a
 2d - - $a + b$
 3d - - $a + 2b + c$
 4th - - $a + 3b + 3c + d$
 5th - - $a + 4b + 6c + 4d + e$
 &c. &c.

1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9
1	3	6	10	15	21	28	36	
1	4	10	20	35	56	84		
1	5	15	35	70	126			
1	6	21	56	126				
1	7	28	84					
1	8	36						
1	9							

In the 4th *prop.* it is shewn that if the terms decrease, proceeding from the greater to the less, the same theorems hold good, by only changing the sign of every other term, as in the margin.

1st term a
 2d - - $a - b$
 3d - - $a - 2b + c$
 4th - - $a - 3b + 3c - d$
 5th - - $a - 4b + 6c - 4d + e$
 &c. &c.

Prop. 6 and 7 treat of the approximate multiplication and division of ratios, or, which is the same thing, the finding nearly any powers or any roots of a given fraction, in an easy manner. The theorem for raising any power, when reduced to a simpler form, is this: the m power of $\frac{a}{b}$, or $\left(\frac{a}{b}\right)^m$, is $\frac{s \mp md}{s \pm md}$ nearly, where s is $= a + b$, and $d = a \cap b$, the sum and difference of the two numbers, and the upper or under signs take place according as $\frac{a}{b}$ is a proper or an improper fraction, that is, according as a is less or greater

than b . And the theorem for extracting the m th root of $\frac{a}{b}$ is $\sqrt[m]{\frac{a}{b}}$ or $\left(\frac{a}{b}\right)^{\frac{1}{m}} = \frac{ms \mp d}{ms \pm d}$ nearly; which latter rule is also the same as the former, as will be evident by substituting $\frac{1}{m}$ instead of m in the first theorem. So that universally $\left(\frac{a}{b}\right)^{\frac{1}{n}}$ is $\frac{ns \mp md}{ns \pm md}$ nearly. These theorems however are nearly true only in some certain cases, namely, when $\frac{a}{b}$ and $\frac{m}{n}$ do not differ greatly from unity. And in the 7th *prop.* the author shews how to find nearly the error of the theorems.

In the 8th *prop.* it is shewn that the measures of ratios of equidifferent terms, are nearly reciprocally as the arithmetical means between the terms of each ratio. So of the ratios $\frac{1}{18}, \frac{3}{32}, \frac{5}{52}$, the mean between the terms of the first ratio is 17, of the 2d 34, of the 3d 51, and the measures of the ratios are nearly as $\frac{1}{17}, \frac{1}{34}, \frac{1}{51}$.

From this property he proceeds, in the 9th *prop.* to find the measure of any ratio less than $\frac{99.5}{100.5}$, which has an equal difference (1) of terms. In the two examples, mentioned near the beginning, our author found the logarithm or measure of the ratio, of $\frac{99.5}{100}$, to be 21769, $\frac{3}{10}$, and that of $\frac{100}{100.5}$ to be 21659 $\frac{7}{10}$; therefore the sum 43429 is the logarithm of $\frac{99.5}{100.5}$, or $\frac{99.5}{100} \times \frac{100}{100.5}$; or the logarithm of $\frac{99.5}{100.5}$ is nearer 43430, as found by other more accurate computations.—

Now to find the logarithm of $\frac{100}{101}$, having the same difference of terms (1) with the former; it will be, by *prop.* 8, as 100.5 (the mean between 101 and 100) : 100 (the mean between 99.5 and 100.5) :: 43430 : 43213 the logarithm of $\frac{100}{101}$, or the difference between the logarithms of 100 and 101. But the log. of 100 is 2; therefore the logarithm of 101 is 2.004,321,3.—Again, to find the logarithm of 102, we must first find the logarithm of $\frac{101}{102}$; the mean between its terms being 101.5, therefore as 101.5 : 100 :: 43430 : 42788 the logarithm of $\frac{101}{102}$, or the difference of the logarithms of 101 and 102. But the logarithm of 101 was found above to be 2.004,321,3; therefore the logarithm of 102 is 2.008,600,1.—So that dividing continually 868596 (the double of 434298 the logarithm of $\frac{99.5}{100.5}$ or $\frac{199}{201}$) by each number of the series 201, 203, 205, 207, &c. then add 2 to the 1st quotient, to the sum add the 2d quotient, and so on, adding always the next quotient to the last sum, the several sums will be the respective logarithms of the numbers in this series 101, 102, 103, 104, &c.

The next, or *prop.* 10, shews that, of two pair of continued ratios whose terms have equal differences, the difference of the measures of the first two ratios, is to the difference of the measures of the other two, as the square of the common term in the two latter, is to the square of the common term in the two former, nearly. Thus in the four ratios $\frac{a}{a+b}, \frac{a+b}{a+2b}, \frac{a+3b}{a+4b}, \frac{a+4b}{a+5b}$, as the measure of $\frac{aa+2ab}{(a+b)^2}$ (the difference of the first two or the quotient of the two fractions) : the measure of $\frac{aa+8ab+15bb}{(a+4b)^2} :: \overline{a+4b}^2 : \overline{a+b}^2$, nearly.

In *prop.* 11, the author shews that similar properties take place among two sets of ratios consisting each of 3 or 4 &c continued numbers.

Prop. 12 shews that, of the powers of numbers in arithmetical progression, the orders of differences which become equal, are the 2d differences in the squares, the 3d differences in the cubes, the 4th differences in the 4th powers, &c. And from hence it is shewn how to construct all those powers by the continual addition of their differences. As had been long before more fully explained by Briggs.

In

In the next or 13th *prop.* our author explains his compendious method of raising the tables of logarithms, shewing how to construct the logarithms by addition only, from the properties contained in the 8th, 9th and 12th propositions. For this purpose he makes use of the quantity $\frac{a}{b-c}$, which by division he resolves into this infinite series $\frac{a}{b} + \frac{ac}{bb} + \frac{ac^2}{b^3} + \frac{ac^3}{b^4} \&c$ (*in infin.*). Putting then $a = 100$ the arithmetical mean between the terms of the ratio $\frac{99.5}{100.5}$, $b = 100000$, and c successively equal to 0.5 , 1.5 , 2.5 , &c. that so $b - c$ may be respectively equal to 99999.5 , 99998.5 , 99997.5 , &c, (the corresponding means between the terms of the ratios $\frac{99999}{100000}$, $\frac{99998}{99999}$, $\frac{99997}{99998}$, &c), it is evident that $\frac{a}{b-c}$ will be the quotient of the 2d term divided by the 1st in the proportions mentioned in the 8th and 9th propositions; and when each of these quotients are found, it remains then only to multiply them by the constant 3d term 43429 , or rather 43429.8 , of the proportion, to produce the logarithms of the ratios $\frac{99999}{100000}$, $\frac{99998}{99999}$, $\frac{99997}{99998}$, &c, till $\frac{10000}{10001}$; then adding these continually to 4 (the logarithm of 10000 the least number), or subtracting them from 5 (the logarithm of the highest term 100000), there will result the logarithms of all the absolute numbers from 10000 to 100000 . Now when c is $= 0.5$, then $\frac{a}{b} = .001$, $\frac{ac}{bb} = .000,000,005$, $\frac{ac^2}{b^3} = .000,000,000,000,025$, $\frac{ac^3}{b^4} = .000,000,000,000,000,000,125$, &c; therefore $\frac{a}{b-c} = \frac{a}{b} + \frac{ac}{bb} + \frac{ac^2}{b^3} \&c$ is $= .001,000,005,000,025,000,125$. In like manner, if $c = 1.5$, then $\frac{a}{b+c}$ will be $= .001,000,015,000,225,003,375$: and if $c = 2.5$, then $\frac{a}{b-c}$ will be $= .001,000,025,000,625,015,625$; &c. But instead of constructing all the values of $\frac{a}{b-c}$ in the usual way of raising the powers, he directs them to be found by addition only, as in the last proposition. Having thus found all the values of $\frac{a}{b-c}$, the author then shews that they may be drawn into the constant logarithm 43429 by addition only, by the help of the annexed table of the first nine products of it.

1	43429
2	86858
3	130287
4	173716
5	217145
6	260574
7	304003
8	347432
9	390861

The author then distinguishes which of the logarithms it may be proper to find in this way, and which from their component parts. Of these the logarithms of all even numbers need not be thus computed, being composed from the number 2 ; which cuts off one half of the numbers: neither are those numbers to be computed which end in 5 , because 5 is one of their factors: these last are $\frac{1}{10}$ of the numbers; and the two together $\frac{1}{2} + \frac{1}{10}$ make $\frac{3}{5}$ of the whole: and of the other $\frac{2}{5}$, the $\frac{1}{5}$ of them, or $\frac{1}{15}$ of the whole, are composed of 3 ; and hence $\frac{3}{5} + \frac{1}{15}$, or $\frac{11}{15}$ of the numbers, are made up of such as are composed of 2 , 3 , and 5 . As to the

the other numbers which may be composed of 7, of 11, &c; he recommends to find *their* logarithms in the general way, the same as if they were incompofites, as it is not worth while to separate them in fo easy a mode of calculation. So that of the 90 chiliads of numbers from 10000 to 100000, only 24 chiliads are to be computed. Neither indeed are all of these to be calculated from the foregoing series for $\frac{a}{b-c}$, but only a few of them in that way, and the rest by the proportion in the 8th proposition. Thus having computed the logarithms of 10003 and 10013, omitting 10023 as being divisible by 3, estimate the logarithms of 10033 and 10043, which are the 30th numbers from 10003 and 10013; and again omitting 10053, a multiple of 3, find the logarithms of 10063 and 10073. Then by prop. 8,

As 10048, the arithmetical mean between 10033 and 10063,
to 10018, the arithmetical mean between 10003 and 10033,
fo 13006, the difference between the logarithms of 10003 and 10033,
to 12967, the difference between the logarithms of 10033 and 10063;

That is, 1st - - As $\left\{ \begin{array}{l} 10048 \\ 10078 \\ 10108 \end{array} \right\} : 10018 :: 13006 : \left\{ \begin{array}{l} 12967 \\ \&c. \end{array} \right.$

Again, As $\left\{ \begin{array}{l} 10058 \\ 10088 \\ 10118 \end{array} \right\} : 10028 :: 12992 : \left\{ \begin{array}{l} 12953 \\ \&c. \end{array} \right.$

And 3dly, As $\left\{ \begin{array}{l} 10068 \\ 10098 \\ \&c. \end{array} \right\} : 10038 :: 12979 : \left\{ \begin{array}{l} 12940 \\ \&c. \end{array} \right.$

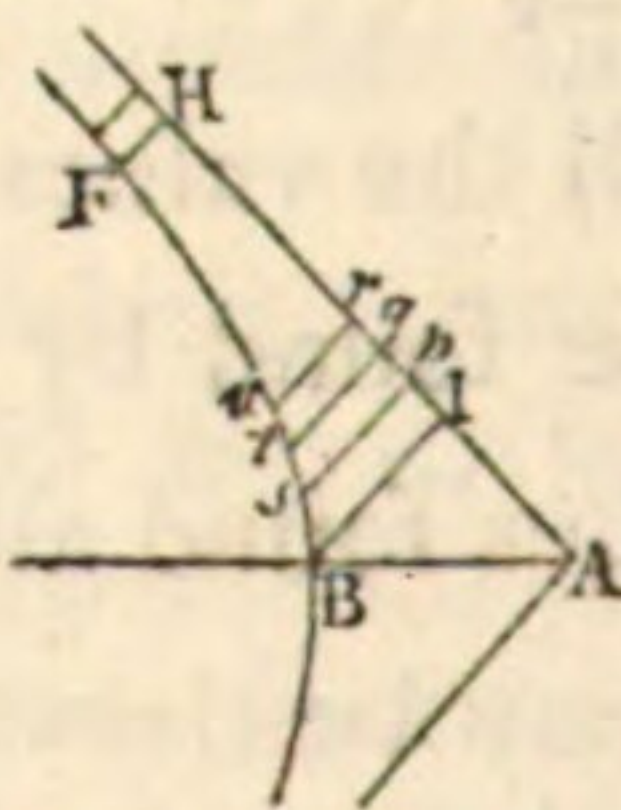
And with this our author concludes his compendium for constructing the tables of logarithms.

He afterwards shews some applications and relations of the doctrine of logarithms to geometrical figures: in order to which, in *prop.* 14, he proves algebraically that, in the right-angled hyperbola ABFH, if from the vertex B and from any other point F there be drawn the right lines BI, FH, perpendicular to the asymptote AH, or parallel to the other asymptote; then will $AH : AI :: BI : FH$. And,

In *prop.* 15, if $AI = BI = 1$, and $HI = a$; then will $FH = \frac{1}{1+a} = 1 - a + a^2 - a^3 + a^4 - a^5 \&c$, in infinitum, by a conti-

nual algebraic division, the process of which he describes step by step, as a thing that was new or uncommon. But that method of division had been taught before by Dr. Wallis, in his *Opus Arithmeticum*.

Prop. 16 is this: Any given number being supposed to be divided into innumerable small equal parts, it is required to assign the sum of any powers of the continual sums of those innumerable parts. For which purpose he lays down this rule: If the next higher power of the given number above that power whose sum is sought, be divided by its exponent, the quotient will be the sum of the powers sought. That is, if N be the given number, and a one of its innumera-



ble equal parts, then will $a^n + \overline{2a}^n + \overline{3a}^n + \overline{4a}^n \&c \dots N^n$ be $= \frac{N^{n+1}}{n+1}$: which theorem he demonstrates by a method of induction. And this, it is evident, is the finding the sum of any powers of an infinite number of arithmeticals, of which the greatest term is a given quantity, and the least indefinitely small. It is also remarkable that the above expression is similar to the rule for finding the fluent to the given fluxion of a power, as afterwards taught by Sir I. Newton.

Our author then applies this rule in *prop.* 17, to the quadrature of the hyperbola. Thus, putting $AI = 1$, conceive the asymptote to be divided from I into innumerable equal parts, namely $Ip = pq = qr = a$; then by the 14th and 15th,

$$\left. \begin{aligned} ps &= 1 - a + a^2 - a^3 \&c \\ qt &= 1 - 2a + 4a^2 - 8a^3 \&c \\ ru &= 1 - 3a + 9a^2 - 27a^3 \&c \end{aligned} \right\} \text{But the area BIru is = the sum } ps + qt + ru, \text{ which is =}$$

$3 - 6a + 14a^2 - 36a^3 \&c$, that is, equal to the number of terms contained in the line Ir, minus the sum of those terms, plus the sum of the squares of the same, minus the sum of their cubes, plus the sum of the 4th powers, &c. Putting now $IA = 1$, as before, and $Ip = 0.1$ the number of terms, to find the area BIPs; by *prop.* 16 the sum of the terms will be $\frac{0.1^2}{2} = 0.005$, the sum of their squares $= 0.000,333,333$, the sum of their cubes $= 0.000,025$, the sum of the 4th powers $= 0.000,002$, the sum of the 5th powers $= 0.000,000,166$, the sum of the 6th powers $= 0.000,000,014$, &c. Therefore the area BIPs is $= 1 - 0.005 + 0.000,333,333 - 0.000,025 + 0.000,002 - 0.000,000,166 + 0.000,000,014 \&c = 0.995,335,347 - 0.005,025,166 = 0.990,310,181 \&c$.

Again, putting $Iq = 0.21$ the number of terms, he finds in like manner the area BIqt $= 0.21 - 0.022,05 + 0.003,087 - 0.000,486,202 + 0.000,081,682 - 0.000,014,294 + 0.000,002,572 - 0.000,000,472 + 0.000,000,088 \&c = 0.213,171,345, - 0.022,550,984 = 0.190,620,361, \&c$.

He then adds, hence it appears that as the ratio of AI to Ap, or 1 to 1.1, is half, or subduplicate, of the ratio of AI to Aq, or 1 to 1.21; so the area BIPs is here found to be half of the area BIqt. These areas he computes to 44 places of figures, and finds them still in the ratio of 2 to 1.

The foregoing doctrine amounts to this, that if the rectangle $BI \times Ir$, which in this case is expressed by Ir only, be put $= A$, AI being $= 1$ as before; then the area BIru, or the hyperbolic logarithm of $1 + A$, or of the ratio of 1 to $1 + A$ will be equal to the infinite series $A - \frac{1}{2}A^2 + \frac{1}{3}A^3 - \frac{1}{4}A^4 + \frac{1}{5}A^5 \&c$; and which therefore may be considered as Mercator's quadrature of the hyperbola, or his general expression of an hyperbolic logarithm in an infinite series. And this method was farther improved by Dr. Wallis in the *Philos. Transf.* for the year 1668.

In *prop.* 18, our author compares the hyperbolic *areolæ* with the *ratiunculæ* of equidifferent numbers, and observes that

- the areola BIPs is the measure of the ratiuncula of AI to Ap,
- the areola spqt is the measure of the ratiuncula of Ap to Aq,
- the areola tqru is the measure of the ratiuncula of Aq to Ar, &c.

Finally, in the 19th *prop.* he shews how the sums of logarithms may be taken

after the manner of the sums of the *areolæ*. And from hence infers as a corollary, how the continual product of any given numbers in arithmetical progression may be obtained: for the sum of the logarithms is the logarithm of the continual product. He then remarks that from the premises it appears in what manner Mercennus's problem may be resolved, if not geometrically, at least in figures to any number of places. And thus closes this ingenious tract.

In the *Philos. Trans.* for 1668 are also given some farther illustrations of this work by the author himself. And in various places also in a similar manner are logarithms and hyperbolic areas treated of by Lord Brouncker, Dr. Wallis, Sir I. Newton, and many other learned persons.

Of Gregory's Exercitationes Geometricæ.

In the same year 1668 came out Mr. James Gregory's *Exercitationes Geometricæ*, in which are contained,

- 1, Appendicula ad veram circuli et hyperbolæ quadraturam:
- 2, Nicolai Mercatoris quadratura hyperbolæ geometricè demonstrata:
- 3, Analogia inter lineam meridianam planisphærii nautici et tangentes artificiales, geometricè demonstrata; seu quòd secantium naturalium additio efficiat tangentes artificiales:
- 4, Item quòd tangentium naturalium additio efficiat secantes artificiales:
- 5, Quadratura conchoidis:
- 6, Quadratura ciffoidis: &
- 7, Methodus facilis et accurata componendi secantes et tangentes artificiales.

The first of these pieces, or the *Appendicula*, contains some farther extension and illustration of his *Vera circuli et hyperbolæ quadratura*, occasioned by the animadversions made on that work by the famous mathematician and philosopher Huygens.

In the 2d is demonstrated geometrically the quadrature of the hyperbola, by which he finds a series similar to Mercator's for the logarithm, or the hyperbolic space beyond the first ordinate (*Bl, fig. pa. 94*). In like manner he finds another series for the space at an equal distance within that ordinate. These two series having all their terms alike, but all the signs of the one being plus, and those of the other being alternately plus and minus, it is evident that by adding the two together, every other term will be cancelled, and the double of the rest will denote the sum of both spaces. He then applies these properties to the logarithms; the conclusion from all which may be thus briefly expressed:

$$\text{since } A - \frac{1}{2}A^2 + \frac{1}{3}A^3 - \frac{1}{4}A^4 \text{ \&c.} = \text{the log. of } \frac{1+A}{1},$$

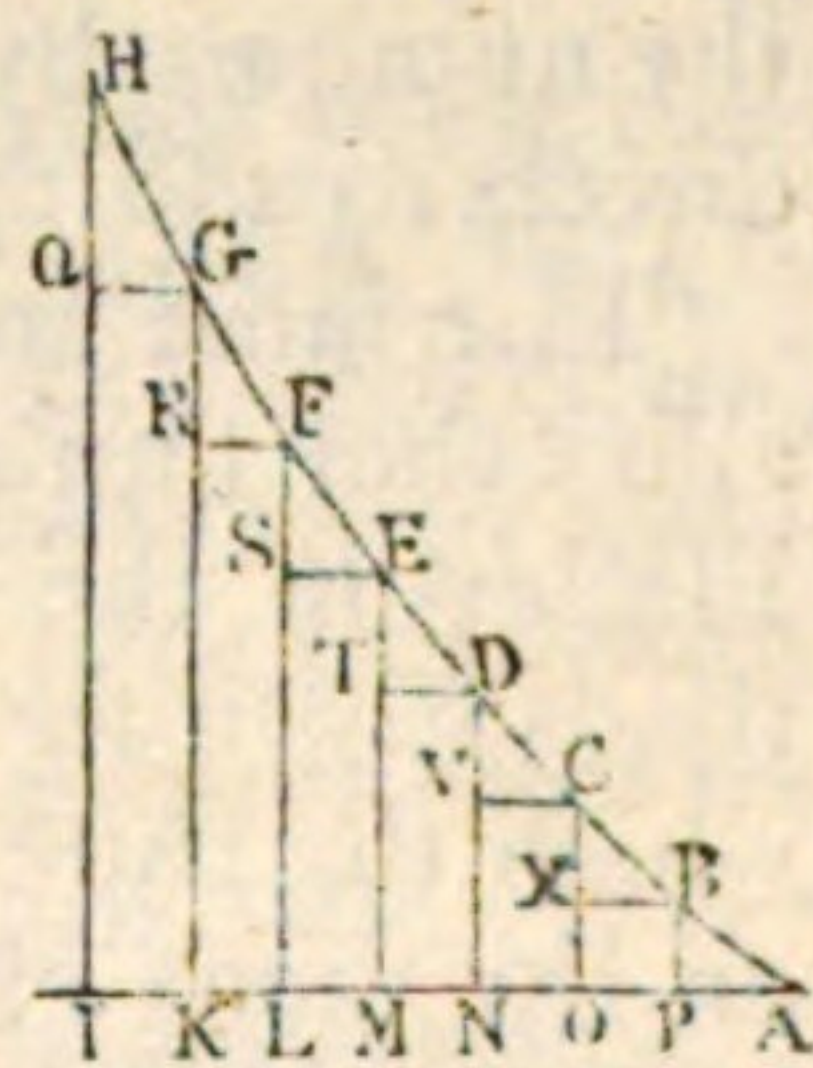
$$\text{and } A + \frac{1}{2}A^2 + \frac{1}{3}A^3 + \frac{1}{4}A^4 \text{ \&c.} = \text{the log. of } \frac{1}{1-A},$$

therefore $2A + \frac{2}{3}A^3 + \frac{2}{5}A^5 + \frac{2}{7}A^7 \text{ \&c.} = \text{the log. of } \frac{1+A}{1-A}$, or of the ratio of $1 - A$ to $1 + A$. Which may be accounted Mr. James Gregory's method of making logarithms.

The remainder of this little volume is chiefly employed about the nautical meridian,

meridian, and the logarithmic tangents and secants. It does not appear by whom, nor by what accident, was discovered the analogy between a scale of logarithmic tangents and Wright's protraction of the nautical meridian line, which consisted of the sums of the secants. It appears, however, to have been first published, and introduced into the practice of navigation, by Mr. Henry Bond, who mentions this property in an edition of Norwood's Epitome of Navigation, printed about 1645; and he again treats of it more fully in an edition of Gunter's works printed in 1653, where he teaches, from this property, to resolve all the cases of Mercator's sailing by the logarithmic tangents, independent of the table of meridional parts. This analogy had only been found to be nearly true by trials, but not demonstrated to be a mathematical property. Such demonstration seems to have been first discovered by Mr. Nicholas Mercator, who, desirous of making the most advantage of this and another concealed invention of his in navigation, invited the publick, by a paper in the Philosophical Transactions for June 4, 1666, to enter into a wager with him on his ability to prove the truth or falsehood of the supposed analogy. This mercenary proposal, however, seems not to have been taken up by any one, and Mercator reserved his demonstration. The proposal, however, excited the attention of mathematicians to the subject itself, and a demonstration was not long wanting. The first was published about two years after by Gregory in the tract now under consideration, and from thence and other similar properties here demonstrated, he shews in the last article how the tables of logarithmic tangents and secants may easily be computed from the natural tangents and secants. The substance of which is as follows:

Let AI be the arc of a quadrant extended in a right line, and let the figure AHI be composed of the natural tangents of every arc from the point A erected perpendicular to AI at their respective points: let AP, PO, ON, NM, &c, be the very small equal parts into which the quadrant is divided, namely, each $\frac{1}{60}$, or $\frac{1}{180}$, of a degree; and let PB, OC, ND, ME, &c be drawn perpendicular to AI, and equal to the tangents of the several arcs AP, AO, AN, AM, &c, respectively. Then it is manifest from what had been demonstrated, that the figures ABP, ACO, &c, are the artificial secants of the arcs AP, AO, &c, putting 0 for the artificial radius. It is also manifest that the rectangles BO, CN, DM, &c, will be found from the multiplication of the small part AP of the quadrant by each natural tangent. But, he proceeds, there is a little more difficulty in measuring the figures ABP, BCX, CDV, &c; for if the first differences of the tangents be equal, AB, BC, CD, &c, will not differ from right lines, and then the figures ABP, BCX, CDV, &c, will be right-angled triangles, and therefore any one, as HQG, will be $= \frac{1}{2}QH \times QG$: but if the second differences be equal, the said figures will be portions of trilineal quadratrices; for example HQG will be a portion of a trilineal quadratrix, whose axis is parallel to QH; and each of the last differences being Z, it will be $QHG = \frac{1}{2}QH \times QG - \frac{1}{12}Z \times QG$: and if the 3d differences be equal, the said figures will be portions of trilineal cubics, and then shall QHG be equal $\frac{1}{2}QH \times QG - \sqrt{\frac{1}{12}QH \times Z \times QG^2 - \frac{1}{1728}Z^2 \times QG^2}$: when the 4th differences are equal, the said figures are portions of trilineal quadrato-quadratrices, and the 4th differences are equal to 24 times the 4th power of QG



divided by the cube of the latus rectum ; also when the 5th differences are equal, the said figures are portions of trilineal surfolids, and the 5th differences are equal to 120 times the surfolid of QG divided by the 4th power of the latus rectum : and so on *in infinitum*. What has been here said of the composition of artificial secants from the natural tangents, it is remarked, may in like manner be understood of the composition of artificial tangents from the natural secants, according to what was before demonstrated. It is also observed that the artificial tangents and secants are computed, as above, on the supposition that 0 is the logarithm of 1, and 1,000,000,000,000,000 the radius, and 2,302,585,092,994,045,624,017,870 the logarithm of 10 ; but that they may be more easily computed, namely, by addition only, by putting $\frac{1}{5730}$ of a degree = QG = AP = 1, and the logarithm of 10 = 7,915,704,467,897,819 ; for by this means $\frac{1}{2}QH \times QG$ is = $\frac{1}{2}QH = QHG$, and $\frac{1}{2}QH \times QG - \frac{1}{12}Z \times QG = \frac{1}{2}QH - \frac{1}{12}Z = QHG$, also $\frac{1}{2}QH \times QG - \sqrt{\frac{1}{72}QH \times Z \times QG^2 - \frac{1}{1728}Z^2 \times QG^2} = \frac{1}{2}QH - \sqrt{\frac{1}{72}QH \times Z - \frac{1}{1728}Z^2} = QHG$: And finally by one division only are found the artificial tangents and secants to 1,000,000,000,000,000 the logarithm of 10, putting still 1 for radius, which are the differences of the artificial tangents and secants in the table from that artificial radius ; and to make the operations easier in multiplying by the number 7,915,704,467,897,819, or logarithm of 10, a table is set down of its products by the first 9 figures. But if AP or QG be = $\frac{1}{5730}$ of a degree, the artificial tangents and secants will answer to 13,192,840,779,829,703 as the logarithm of 10, whose first 9 multiples are also placed in the table. But to represent the numbers by the artificial radius rather than by the logarithm of 10, the author directs to add cyphers, &c.—And so much for Gregory's *Exercitationes Geometricæ*.

The same analogy between the logarithmic tangents and the meridian line, as also other similar properties, were afterwards more elegantly demonstrated by Dr. Halley in the Philosophical Transactions for February, 1696, and various methods given for computing the same, by examining the nature of the spirals into which the rhumbs are transformed in the stereographical projection of the sphere on the plane of the equator : the doctrine of which was rendered still more easy and elegant by the ingenious Mr. Cotes in his *Logometria*, first printed in the Philosophical Transactions for 1714, and afterwards in the collection of his works published in 1782, by his cousin Dr. Robert Smith, who succeeded him in the Plumian professorship of philosophy in the University of Cambridge.

The learned Dr. Isaac Barrow also, in his *Lectiones Geometricæ*, *Lect. XI. Append.* first published in 1672, delivers a similar property, namely, that the sum of all the secants of any arc, is analogous to the logarithm of the ratio of $r + s$ to $r - s$, or radius plus sine to radius minus sine ; or (which is the same thing) that the meridional parts answering to any degree of latitude, are as the logarithms of the ratios of the versed sines of the distances from the two poles.

Mr. Gregory's method for making logarithms was farther exemplified in numbers, in a small tract on this subject, printed in 1688, by one Euclid Speidell, a simple and illiterate person, and son of John Speidell before mentioned among the first writers on logarithms.

Mr. Gregory also invented many other infinite series, and among them these here following, viz. a being an arc, t its tangent, and s the secant, to the radius r ; then is

$$a = t$$

$$\begin{aligned} a &= t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} \&c. \\ t &= a + \frac{a^3}{3r^2} + \frac{2a^5}{15r^4} + \frac{17a^7}{315r^6} + \frac{62a^9}{2835r^8} \&c. \\ s &= r + \frac{a^2}{2r} + \frac{5a^4}{24r^3} + \frac{61a^6}{720r^5} + \frac{277a^8}{8064r^7} \&c. \end{aligned}$$

And if τ and σ be the artificial, or logarithmic, tangent and secant of the same arc a , the whole quadrant being q , and $e = 2a - q$; then

$$\begin{aligned} e &= \tau - \frac{\tau^3}{6r^2} + \frac{\tau^5}{24r^4} - \frac{61\tau^7}{5040r^6} + \frac{277\tau^9}{72576r^8} \&c. \\ \tau &= e + \frac{e^3}{6r^2} + \frac{e^5}{24r^4} + \frac{61e^7}{5040r^6} + \frac{277e^9}{72576r^8} \&c. \\ \sigma &= \frac{a^2}{2r} + \frac{a^4}{12r^3} + \frac{a^6}{45r^5} + \frac{17a^8}{2520r^7} + \frac{62a^{10}}{28350r^9} \&c. \end{aligned}$$

Also if f be the artificial secant of 45° , and $f + l$ the artificial secant of any arc a , the artificial radius being o ; then is

$$a = \frac{1}{2}q + l - \frac{l^2}{r} + \frac{4l^3}{3r^2} - \frac{7l^4}{3r^3} + \frac{14l^5}{3r^4} - \frac{45l^6}{45r^5} \&c.$$

The investigation of all which series may be seen at pages 298 *et seq.* vol. 1. of Dr. Horsley's learned and elegant commentary on Sir Isaac Newton's works, they having been given without demonstration, in the *Commercium Epistolicum*, No. xx. where the number 2 is also wanting in the denominator of the first term of the series expressing the value of σ .

Such then were the ways in which Mercator and Gregory applied these their very simple series $A - \frac{1}{2}A^2 + \frac{1}{3}A^3 - \frac{1}{4}A^4 \&c.$ and $A + \frac{1}{2}A^2 + \frac{1}{3}A^3 + \frac{1}{4}A^4 \&c.$ for the purpose of computing logarithms. But they might, as I apprehend, have applied them to this purpose in a shorter and more direct manner, by computing, by their means, only a few logarithms of small ratios, in which the terms of the series would have decreased by the powers of 10 or some greater number, the numerators of all the terms being unity, and their denominators the powers of 10 or some greater number, and then employing these few logarithms, so computed, to the finding of the logarithms of other and greater ratios by the easy operations of mere addition and subtraction. This might have been done for the logarithms of the ratios of the first ten numbers, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11, to 1, in the following manner, communicated by Mr. Baron Maseres.— In the first place the logarithm of the ratio of 10 to 9, or of 1 to $\frac{9}{10}$, or of 1 to $1 - \frac{1}{10}$, is equal to the series $\frac{1}{10} + \frac{1}{2 \times 100} + \frac{1}{3 \times 1000} + \frac{1}{4 \times 10000} + \frac{1}{5 \times 100000} \&c.$ In like manner are easily found the logarithms of the ratios of 11 to 10; and then, by the same series, those of 121 to 120, and of 81 to 80, and of 2401 to 2400; in all which cases the series would converge still faster than in the two first cases. We may then proceed by mere addition and subtraction of logarithms, as follows.

$$\begin{array}{l} \text{Log. } \frac{11}{9} = \text{L. } \frac{11}{10} + \text{L. } \frac{10}{9}, \quad \left| \text{L. } \frac{120}{110} = \text{L. } \frac{3}{2}, \quad \left| \text{L. } \frac{80}{16} = \text{L. } \frac{81}{16} - \text{L. } \frac{81}{80}, \right. \\ \text{L. } \frac{121}{81} = 2\text{L. } \frac{11}{9}, \quad \left| \text{L. } \frac{9}{4} = 2\text{L. } \frac{3}{2}, \quad \left| \text{L. } \frac{5}{1} = \text{L. } \frac{80}{16}, \right. \\ \text{L. } \frac{121}{80} = \text{L. } \frac{121}{81} + \text{L. } \frac{81}{80}, \quad \left| \text{L. } \frac{10}{4} = \text{L. } \frac{10}{9} + \text{L. } \frac{9}{4}, \quad \left| \text{L. } \frac{5}{2} = \text{L. } \frac{10}{4}, \right. \\ \text{L. } \frac{120}{80} = \text{L. } \frac{121}{80} - \text{L. } \frac{121}{120}, \quad \left| \text{L. } \frac{81}{16} = 2\text{L. } \frac{9}{4}, \quad \left| \text{L. } \frac{2}{1} = \text{L. } \frac{5}{1} - \text{L. } \frac{5}{2} \right. \end{array}$$

O 2

Having

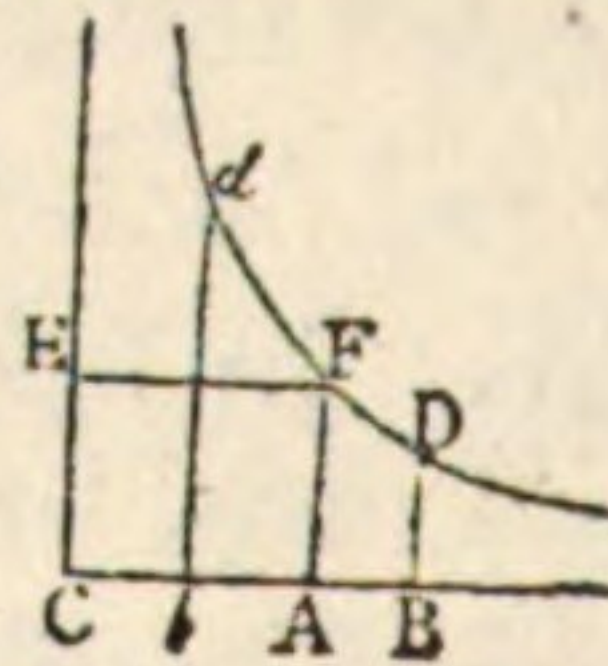
Having thus got the logarithm of the ratio of 2 to 1, or, in common language, the logarithm of 2, the logarithms of all sorts of even numbers may be derived from those of the odd numbers which are their coefficients with 2 or its powers. I would then proceed as follows.

$$\begin{array}{l} \text{L. } 4 = 2\text{L. } 2, \\ \text{L. } 10 = \text{L. } \frac{10}{4} + \text{L. } 4, \\ \text{L. } 9 = \text{L. } \frac{9}{4} + \text{L. } 4, \\ \text{L. } 3 = \frac{1}{2}\text{L. } 9 \end{array} \quad \begin{array}{l} \text{L. } 100 = 2\text{L. } 10, \\ \text{L. } 8 = 3\text{L. } 2, \\ \text{L. } 24 = \text{L. } 8, + \text{L. } 3, \\ \text{L. } 2400 = \text{L. } 100 + \text{L. } 24, \end{array} \quad \begin{array}{l} \text{L. } 2401 = \text{L. } \frac{2401}{4} + \text{L. } 2400, \\ \text{L. } 7 = \frac{1}{4}\text{L. } 2401, \\ \text{L. } 11 = \text{L. } \frac{11}{9} + \text{L. } 9, \\ \text{L. } 6 = \text{L. } 2 + \text{L. } 3. \end{array}$$

Thus we have got the logarithms of 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11. And this is, upon the whole, perhaps the best method of computing logarithms that can be taken*. There have been indeed some methods discovered by Dr. Halley, and other mathematicians, for computing the logarithms of the ratios of prime numbers to the next adjacent even numbers, that are still shorter than the application of the foregoing series. But those methods are less simple and easy to understand and apply than these series; and the computation of logarithms by these series, when the terms of them decrease by the powers of 10, or of some greater number, is so very short and easy (as we have seen in the foregoing computations of the logarithms of the ratios of 10 to 9, 11 to 10, 81 to 80, 121 to 120, &c,) that it is not worth while to seek for any shorter methods of computing them. And this method of computing logarithms is very nearly the same with that of Sir Isaac Newton in his second letter to Mr. Oldenburg, dated October 1676, as will be seen in the following article.

Of Sir Isaac Newton's Methods.

The excellent Sir I. Newton greatly improved the quadrature of the hyperbolic asymptotic spaces by infinite series derived from the general quadrature of curves by his method of fluxions; or rather indeed he invented that method himself, and the construction of logarithms derived from it, in the year 1665 or 1666, before the publication of either Mercator's or Gregory's books, as appears by his letter to Mr. Oldenburg, dated Oct. 24, 1676, printed in p. 634 *et seq.* vol. 3. of Wallis's works, and elsewhere. The quadrature of the hyperbola, thence translated, is to this effect. Let dFD be an equilateral hyperbola, whose center is C, vertex F, and interposed square CAFE = 1. In CA take AB and Ab, on each side of the point A, = $\frac{1}{10}$ or 0.1: And, erecting the perpendiculars BD, bd; half the sum of the spaces AD and Ad will be = $0.1 + \frac{0.001}{3} + \frac{0.000,01}{5} + \frac{0.000,000,1}{7}$ &c. and the



* This method of making logarithms is very copiously explained, and illustrated by examples, in one of the tracts contained in this volume, intitled, "*Remarks on the two infinite Serieses* $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \text{\&c.}$ and $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \text{\&c.}$ which were found by Mr. Nicholas Mercator and Dr. John Wallis, &c," in pages 235, 236, &c—344.

half-difference will be $= \frac{0.01}{2} + \frac{0.0001}{4} + \frac{0.000,001}{6} + \frac{0.000,000,01}{8} \&c.$ Which reduced will stand thus,

1.000,000,000,000,0	0.005,000,000,000,0	The sum of these, to wit, 0.105,360,515,657,7, is Ad ;
333,333,333,3	25,000,000,0	and the difference, to wit, 0.095,310,179,804,3, is AD.
2,000,000,0	166,666,6	In like manner putting AB and Ab
14,285,7	1,250,0	each = 0.2, there is obtained
111,1	10,0	Ad = 0.223,143,551,314,2, and
9	1	AD = 0.182,321,556,793, 9.
0.100,335,347,731,0	0.005,025,167,926,7	

Having thus the hyperbolic logarithms of the four decimal numbers 0.8, 0.9, 1.1, and 1.2 ; and since $\frac{1.2}{0.8} \times \frac{1.2}{0.9} = 2$, and 0.8 and 0.9 are less than unity ; add their logarithms to double the logarithm of 1.2, and you will have 0.693,147,180,559,7 for the hyperbolic logarithm of 2. To the triple of this add the logarithm of 0.8, because $\frac{2 \times 2 \times 2}{0.8} = 10$, and you will have 2.302,585,092,993,3 the logarithm of 10. Hence by one addition are found the logarithms of 9 and 11 : And thus the logarithms of all the prime numbers 2, 3, 5, 11 are prepared. Moreover, by only depressing the numbers, above computed, lower in the decimal places, and adding, are obtained the logarithms of the decimals 0.98, 0.99, 1.01, 1.02 ; as also of these, 0.998, 0.999, 1.001, 1.002. And hence by addition and subtraction will arise the logarithms of the primes 7, 13, 17, 37, &c. All which logarithms being divided by the above logarithm of 10, give the common logarithms to be inserted in the table.

And again a few pages farther on in the same letter he resumes the construction of the logarithms, thus : Having found, as above, the hyperbolic logarithms of 10, 0.98, 0.99, 1.01, 1.02, which may be effected in an hour or two, divide the last four logarithms by the logarithm of 10, and adding the index 2, you will have the tabular logarithms of 98, 99, 100, 101, 102. Then by interpolating nine means between each of these, will be obtained the logarithms of all numbers between 980 and 1020 ; and, again interpolating 9 means between every two numbers from 980 to 1000, the table will be so far constructed. Then from these will be collected the logarithms of all the primes under 100, together with those of their multiples : all which will require only addition and subtraction ; for

$$\begin{aligned} \sqrt[1]{\frac{9984 \times 1020}{9945}} &= 2, \frac{10}{2} = 5, \sqrt{\frac{98}{2}} = 7, \frac{99}{9} = 11, \frac{1001}{7 \times 11} = 13, \frac{102}{6} = 17, \frac{988}{4 \times 13} \\ &= 19, \frac{9936}{16 \times 27} = 23, \frac{986}{2 \times 17} = 29, \frac{992}{3^2} = 31, \frac{999}{27} = 37, \frac{984}{24} = 41, \frac{989}{23} = 43, \frac{987}{27} \\ &= 47, \frac{9911}{11 \times 17} = 53, \frac{9971}{13 \times 13} = 59, \frac{9882}{2 \times 81} = 61, \frac{9849}{3 \times 49} = 67, \frac{994}{14} = 71, \frac{9928}{8 \times 17} = \\ 73, \frac{9954}{7 \times 18} &= 79, \frac{996}{12} = 83, \frac{9968}{7 \times 16} = 89, \frac{9894}{6 \times 17} = 97. \end{aligned}$$

This quadrature of the hyperbola, and its application to the construction of logarithms, are still farther explained by our celebrated author in his treatise on Fluxions, published by Mr. Colson, in 1736, where he gives all the three series for

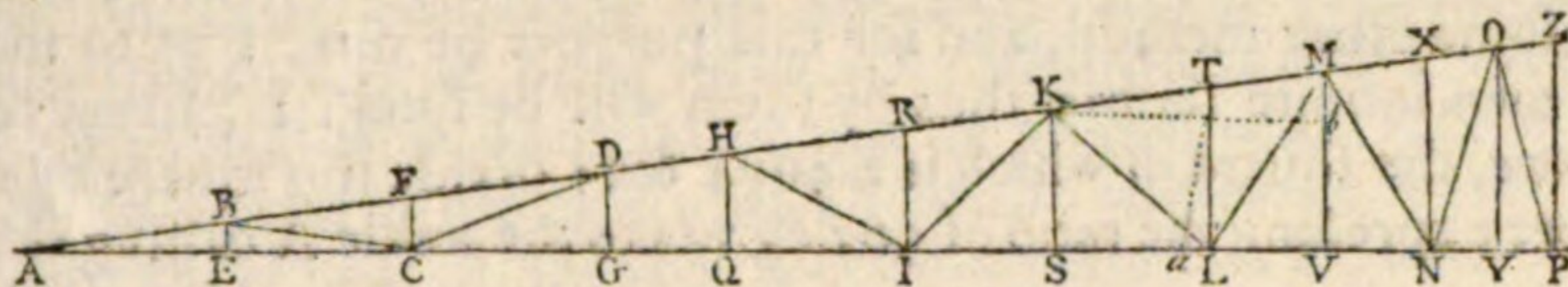
for the areas AD, Ad, Bd, in general terms, the two former of which are the same as those published by Mercator and Wallis, and the last is the same as that published by Gregory; and he explains the manner of deriving the last series from the two former, namely, by uniting together the two series for the spaces on each side of an ordinate, bounded by other ordinates at equal distances, by which means every 2d term of each series is cancelled, and the result is a series converging much quicker than either of the former. And, in this treatise on fluxions, as well as in the letter before quoted, he recommends this as the most commodious method of constructing a canon of logarithms, computing by the series the hyperbolic spaces answering to the prime numbers 2, 3, 5, 7, 11, &c, and dividing them by 2.302,585,092,994,045,7, which is the area corresponding to the number 10, or else multiplying them by its reciprocal 0.434,294, 481,903, 251,8, for the common, or Briggs's, logarithms. "Then the logarithms of all the numbers in the canon which are made by the multiplication of these, are to be found by the addition of their logarithms, as is usual. And the void places are to be interpolated afterwards by the help of this theorem: Let n be a number to which a logarithm is to be adapted, x the difference between that and the two nearest numbers equally distant on each side, whose logarithms are already found, and let d be half the difference of the logarithms; then the required logarithm of the number n will be obtained by adding $d + \frac{dx}{2n} + \frac{dx^3}{12n^3}$ &c, to the logarithm of the less number." This theorem he demonstrates by the hyperbolic areas, and then proceeds thus; "The two first terms $d + \frac{dx}{2n}$ of this series I think to be accurate enough for the construction of a canon of logarithms, even though they were to be produced to 14 or 15 figures; provided the number whose logarithm is to be found be not less than 1000. And this can give little trouble in the calculation, because x is generally an unit, or the number 2. Yet it is not necessary to interpolate all the places by the help of this rule. For the logarithms of numbers which are produced by the multiplication or division of the number last found, may be obtained by the numbers whose logarithms were had before, by the addition or subtraction of their logarithms. Moreover by the differences of the logarithms, and by their 2d and 3d differences, if there be occasion, the void places may be more expeditiously supplied; the foregoing rule being to be applied only when the continuation of some full places is wanted, in order to obtain those differences, &c." So that Sir Isaac Newton of himself discovered all the series for the above quadrature which were found out, and afterwards published, partly by Mercator and Dr. Wallis, and partly by Gregory; and these we may here exhibit in one view all together, and that in a general manner for any hyperbola; namely, putting $CA = a$, $AF = b$, and $AB = Ab = x$; then will $BD = \frac{ab}{a+x}$, and $bd = \frac{ab}{a-x}$; whence the area

$$AD \text{ will be } = bx - \frac{bx^2}{2a} + \frac{bx^3}{3a^2} - \frac{bx^4}{4a^3} + \frac{bx^5}{5a^4} \&c.$$

$$\text{and the area Ad will be } = bx + \frac{bx^2}{2a} + \frac{bx^3}{3a^2} + \frac{bx^4}{4a^3} + \frac{bx^5}{5a^4} \&c.$$

$$\text{and the area Bd will be } = 2bx + \frac{2bx^3}{3a^2} + \frac{2bx^5}{5a^4} + \frac{2bx^7}{7a^6} + \frac{2bx^9}{9a^8} \&c.$$

In the same letter also, above quoted, to Mr. Oldenburg, our illustrious author reaches a method of constructing the trigonometrical canon of sines by an easier method of multiple angles than that before delivered by Briggs for the same purpose, because that in Sir Isaac's way the radius, or 1, is the first term, and double the sine or cosine of the first given angle is the second term of all the proportions by which the several successive multiple sines or cosines are found. The substance of this method is thus. The best foundation for the construction of the table of sines, is the continual addition of a given angle to itself or to another given angle. As if the angle A be to



be added; inscribe HI, IK, KL, LM, MN, NO, OP, &c, each equal to the radius AB; and to the opposite sides draw the perpendiculars BE, HQ, IR, KS, LT, MV, NX, OY, &c; so shall the angle A be the common difference of the angles HIQ, IKH, KLI, LMK, &c; their sines HQ, IR, KS, &c; and their cosines IQ, KR, LS, &c. Now let any one of them LMK, be given, and the rest will be thus found: Draw Ta and Kb perpendicular to SV and MV; then because of the equiangular triangles ABE, TLa, KMb, ALT, AMV, &c, it will be $AB : AE :: KT : Sa (= \frac{1}{2}LV + \frac{1}{2}LS) :: LT : Ta (= \frac{1}{2}MV + \frac{1}{2}KS)$, and $AB : BE :: LT : La (= \frac{1}{2}LS - \frac{1}{2}LV) :: KT (= \frac{1}{2}KM) : \frac{1}{2}Mb (= \frac{1}{2}MV - \frac{1}{2}KS)$. Hence are given the sines and cosines KS, MV, LS, LV. And the method of continuing the progressions is evident. Namely,

$$AB : 2 AE :: \begin{cases} LV : MT + MX :: MX : NV + NY, & \&c. \\ MV : NX + LT :: NX : OY + MV, & \&c. \end{cases}$$

$$\text{or } AB : 2 BE :: \begin{cases} LV : NX - LT :: MX : OY - MV, & \&c. \\ MV : MT - MX :: NX : NV - NY, & \&c. \end{cases}$$

And on the other hand, $AB :: 2 AE :: LS : KT + KR \&c$. Therefore put $AB = 1$, and make $BE \times LT = La$, $AE \times KT = Sa$, $Sa - La = LV$, $2AE \times LV - TM = MX$, &c.

The sense of these general theorems is this, that if P be any one among a series of angles in arithmetical progression, the angle d being their common difference, then as radius or

$$1 : 2 \cos. d :: \begin{cases} \cos. P : \cos. P + d + \cos. P - d \\ \sin. P : \sin. P + d + \sin. P - d \end{cases}$$

$$1 : 2 \sin. d :: \begin{cases} \cos. P : \sin. P + d - \sin. P - d \\ \sin. P : \cos. P + d - \cos. P - d \end{cases}$$

where the fourth terms of these proportions are the sums or differences of the sines or cosines of the two angles next less and greater than any angle P in the series; and therefore, subtracting the less extreme from the sum, or adding it to the difference, the result will be the greater extreme, or next sine or cosine beyond that of

of the term P . And in the same manner are all the rest to be found. This method it is evident, is equally applicable whether the common difference d , or angle A , be equal to one term of the series or not; when it is one of the terms, then the whole series of sines and cosines becomes thus; As $1 : 2 \cos. d :$

$$\begin{aligned} \sin. d : \sin. 2d : : \sin. 2d : \sin. d + \sin. 3d : : \sin. 3d : \sin. 2d + \sin. 4d : : \sin. 4d : \sin. 3d + \sin. 5d \&c, \\ \cos. d : 1 + \cos. 2d : : \cos. 2d : \cos. d + \cos. 3d : : \cos. 3d : \cos. 2d + \cos. 4d : : \cos. 4d : \cos. 3d + \cos. 5d \&c, \end{aligned}$$

which is the very method contained in the directions given by Mr. Abraham Sharp for constructing the canon of sines.

Sir Isaac Newton remarks that it only remains to find the sine and cosine of a first angle A by some other method, and for this purpose he directs us to make use of some of his own infinite series: thus by them will be found 1.57079 , &c, for the quadrantal arc, the square of which is 2.4694 &c; divide this square by the square of the number expressing the ratio of 90 degrees to the angle A calling the quotient z ; then three or four terms of this series $1 - \frac{z}{2} + \frac{z^2}{24} - \frac{z^3}{720} + \frac{z^4}{40320}$ &c, will give the cosine of that angle A . Thus we may first find an angle of 5 degrees, and thence the table computed to the series of every 5 degrees; then these interpolated to degrees or half degrees by the same method; and these interpolated again; and so on as far as necessary. But two-thirds of the table being computed in this manner, the remaining third will be found by addition or subtraction only, as is well known.

Various other improvements in logarithms and trigonometry are owing to the same excellent personage; such as the series for expressing the relation between circular arcs and their sines, cosines, versed sines, tangents, &c; namely, the arc being a , the sine s , the versed-sine v , cosine c , tangent t , radius 1, then is

$$\begin{aligned} a &= s + \frac{1}{6}s^3 + \frac{3}{40}s^5 + \frac{5}{112}s^7 + \frac{35}{1152}s^9 + \frac{63}{2816}s^{11} \&c. \\ a &= v^{\frac{1}{2}} + \frac{1}{6}v^{\frac{3}{2}} + \frac{3}{40}v^{\frac{5}{2}} + \frac{5}{112}v^{\frac{7}{2}} + \frac{35}{1152}v^{\frac{9}{2}} + \frac{63}{2816}v^{\frac{11}{2}} \&c. \\ a &= t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \frac{1}{9}t^9 - \frac{1}{11}t^{11} \&c. \\ s &= a - \frac{1}{6}a^3 + \frac{1}{120}a^5 - \frac{1}{5040}a^7 + \frac{1}{362880}a^9 - \frac{1}{39916800}a^{11} \&c. \\ c &= 1 - \frac{1}{2}a^2 + \frac{1}{24}a^4 - \frac{1}{720}a^6 + \frac{1}{40320}a^8 - \frac{1}{3628800}a^{10} \&c. \\ v &= \frac{1}{2}a^2 - \frac{1}{24}a^4 + \frac{1}{720}a^6 - \frac{1}{40320}a^8 + \frac{1}{3628800}a^{10} - \frac{1}{479001600}a^{12} \&c. \\ t &= a + \frac{1}{3}a^3 + \frac{1}{15}a^5 + \frac{1}{315}a^7 + \frac{62}{2835}a^9 + \frac{1302}{155520}a^{11} \&c. \end{aligned}$$

Of Dr. Halley's Method.

Many other improvements in the construction of logarithms are also derived from the same doctrine of fluxions, as we shall shew hereafter. In the mean time proceed we to the ingenious method of the learned Dr. Edmund Halley, Secretary to the Royal Society, and the second Astronomer Royal, having succeeded Mr. Flamsteed in that honourable office in the year 1719 at the Royal Observatory at Greenwich, where he died the 14th of January 1742, in the 86th year of his age. His method was first printed in the Philosophical Transactions, for the year 1695, and it is entitled, "A most compendious and facile method for constructing

constructing the logarithms, exemplified and demonstrated from the nature of numbers, without any regard to the hyperbola, with a speedy method for finding the number from the given logarithm."

Instead of the more ordinary definition of logarithms, *numerorum proportionalium æquidifferentes comites*, in this tract our learned author adopts this other, *numeri rationum exponentes*, as being better adapted to the principle on which logarithms are here constructed, where those quantities are not considered as the logarithms of the numbers, for example, of 2, or of 3, or of 10, but as the logarithms of the ratios of 1 to 2, or of 1 to 3, or of 1 to 10. In this consideration he first pursues the idea of Kepler and Mercator, remarking that any such ratio is proportional to, and is measured by, the number of equal *ratiunculæ* contained in each; which *ratiunculæ* are to be understood as in a continued scale of proportionals, infinite in number, between the two terms of the ratio; which infinite number of mean proportionals is to that infinite number of the like and equal *ratiunculæ* between any other two terms, as the logarithm of the one ratio is to the logarithm of the other: thus, if there be supposed between 1 and 10 an infinite scale of mean proportionals, whose number is 100000 &c *in infinitum*; then between 1 and 2 there will be 30102 &c of such proportionals; and between 1 and 3 there will be 47712 &c of them; which numbers therefore are the logarithms of the ratios of 1 to 10, 1 to 2, and 1 to 3. But for the sake of *his* mode of constructing logarithms, he changes this idea of *equal* *ratiunculæ* for that of other *ratiunculæ*, so constituted as that the *same* infinite number of them shall be contained in the ratio of 1 to every other number whatever; and that therefore these latter *ratiunculæ* will be of *unequal* or different magnitudes in all the different ratios, and in such sort that, in any one ratio, the *magnitude* of each of the *ratiunculæ* in this latter case, will be as the *number* of them in the former. And therefore, if between 1 and any number proposed, there be taken any infinity of mean proportionals, the infinitely small ratio of the first term 1 to the said first term, together with the infinitely small augment or decrement of the first of those means from the first term 1, will be a *ratiuncula* of the ratio of 1 to the said number; and as the number of all the *ratiunculæ* in these continued proportionals is the same, their sum, or the whole ratio, will be directly proportional to the magnitude of one of the said *ratiunculæ* in each ratio. But it is also evident that the first of any number of means between 1 and any number, is always equal to such root of that number whose index is expressed by the number of those proportionals from 1; so if *m* denote the number of proportionals from 1, then the first term after 1 will be the *m*th root of that number. Hence the indefinite root of any number being extracted, the *differentiola* of the said root from unity, shall be as the logarithm of that number. So if there be required the logarithm of the ratio of 1 to $1 + q$;

the first term after 1 will be $1 + q^{\frac{1}{m}}$, and therefore the required logarithm will be as $1 + q^{\frac{1}{m}} - 1$. But by the binomial theorem, $1 + q^{\frac{1}{m}}$ is $= 1 + \frac{1}{m} q + \frac{1}{m} \cdot \frac{1-m}{2m} q^2 + \frac{1}{m} \cdot \frac{1-m}{2m} \cdot \frac{1-2m}{3m} q^3$ &c; or, by omitting the 1 in the compound numerators, as infinitely small in respect of the infinite number *m*, the same series

will become $1 + \frac{1}{m}q + \frac{1}{m} \cdot \frac{-m}{2m} q^2 + \frac{1}{m} \cdot \frac{-m}{2m} \cdot \frac{-2m}{3m} q^3$ &c, or, by abbreviation, it is $1 + \frac{1}{m}q - \frac{1}{2m}q^2 + \frac{1}{3m}q^3 - \frac{1}{4m}q^4$ &c; and hence, finding the differentiola by subtracting 1, the logarithm of the ratio of 1 to $1 + q$ will be as $\frac{1}{m}$ into $q - \frac{1}{2}q^2 + \frac{1}{3}q^3 - \frac{1}{4}q^4 + \frac{1}{5}q^5 - \frac{1}{6}q^6$ &c. Now the index m may be taken equal to any infinite number, and thus all the varieties of scales of logarithms may be produced: so, if m be taken 1000000 &c, the theorem will give Napier's logarithms; but if m be taken equal to 2302585 &c, there will arise Briggs's logarithms.

This theorem being for the increasing ratio of 1 to $1 + q$; if that for the decreasing ratio of 1 to $1 - q$ be also sought, it will be obtained by a proper change of the signs, by which the decrement of the first of the infinite number of proportionals will be found to be $\frac{1}{m}$ into $q + \frac{1}{2}q^2 + \frac{1}{3}q^3 + \frac{1}{4}q^4$ &c, which therefore is as the logarithm of the ratio of 1 to $1 - q$.

Hence the terms of any ratio being a and b , q becomes $\frac{b-a}{a}$, or the difference divided by the less term, when it is an increasing ratio; or $q = \frac{b-a}{b}$ when the ratio is decreasing, or as b to a . Wherefore the logarithm of the same ratio may be doubly expressed; for putting x for the difference $b - a$ of the terms, it will be

$$\begin{aligned} \text{either } \frac{1}{m} \text{ into } \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} \text{ \&c,} \\ \text{or } \frac{1}{m} \text{ into } \frac{x}{b} + \frac{x^2}{2b^2} + \frac{x^3}{3b^3} + \frac{x^4}{4b^4} \text{ \&c.} \end{aligned}$$

But if the ratio of a to b be supposed divided into two parts, namely, into the ratio of a to $\frac{1}{2}a + \frac{1}{2}b$ or $\frac{1}{2}z$, and the ratio of $\frac{1}{2}z$ to b , then will the sum of the logarithms of those two ratios be the logarithm of the ratio of a to b . Now by substituting in the foregoing series, the logarithms of those two ratios will be

$$\begin{aligned} \frac{1}{m} \text{ into } \frac{x}{z} + \frac{x^2}{2z^2} + \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} \text{ \&c.} \\ \text{and } \frac{1}{m} \text{ into } \frac{x}{z} - \frac{x^2}{2z^2} + \frac{x^3}{3z^3} - \frac{x^4}{4z^4} + \frac{x^5}{5z^5} \text{ \&c; and so the sum} \\ \frac{1}{m} \text{ into } \frac{2x}{z} + \frac{2x^3}{3z^3} + \frac{2x^5}{5z^5} + \frac{2x^7}{7z^7} \text{ \&c will be the log. of the ratio of } a \text{ to } b. \end{aligned}$$

Moreover, if from the logarithm of the ratio of a to $\frac{1}{2}z$ be taken that of $\frac{1}{2}z$ to b , we shall have the logarithm of the ratio of ab to $\frac{1}{4}z^2$; and the half of this gives that of $\sqrt{ab}$ to $\frac{1}{2}z$, or of the geometrical mean to the arithmetical mean. And consequently the logarithm of this ratio will be equal to half the difference of that of the above two ratios, and will therefore be $\frac{1}{m}$ into $\frac{x^2}{2z^2} + \frac{x^4}{4z^4} + \frac{x^6}{6z^6} + \frac{x^8}{8z^8}$ &c.

The above series are similar to some that were before given by Newton and Gregory

Gregory for the same purpose, deduced from the consideration of the hyperbola. But the rule which is properly our author's own, is that which follows, and is derived from the series above given for the logarithm of the sum of two ratios. For the ratio of ab to $\frac{1}{4}z^2$ or $\frac{1}{4}a^2 + \frac{1}{2}ab + \frac{1}{4}b^2$, having the difference of its terms $\frac{1}{4}a^2 - \frac{1}{2}ab + \frac{1}{4}b^2$ or $\frac{1}{4}(b - a)^2$ or $\frac{1}{4}x^2$, which in the case of finding the logarithms of prime numbers is always 1, if we put the sum of the terms $\frac{1}{4}z^2 + ab = y^2$, the logarithm of the ratio of $\sqrt{ab}$ to $\frac{1}{2}a + \frac{1}{2}b$, or $\frac{1}{2}z$, will be found to be

$$\frac{1}{m} \text{ into } \frac{1}{y^2} + \frac{1}{13y^6} + \frac{1}{5y^{10}} + \frac{1}{7y^{14}} + \frac{1}{9y^{18}} \&c.$$

And these rules our learned author exemplifies by some cases in numbers, to shew the easiest mode of application in practice.

Again by means of the same binomial theorem he resolves with equal facility the reverse of the problem, namely, from the logarithm given, to find its number or ratio: For as the logarithm of the ratio of 1 to $1 + q$ was proved to be

$\frac{1}{1 + q} \frac{1}{m} = 1$, and that of the ratio of 1 to $1 - q$ to be $\frac{1}{1 - q} \frac{1}{m}$; hence calling the given logarithm L , in the former case

$$\text{it will be } 1 + q \frac{1}{m} = 1 + L,$$

$$\text{and in the latter } 1 - q \frac{1}{m} = 1 - L;$$

$$\left. \begin{array}{l} \text{and therefore } 1 + q = \frac{1 + L}{\frac{1}{m}} \\ \text{and } 1 - q = \frac{1 - L}{\frac{1}{m}} \end{array} \right\}, \text{ that is by the binomial theorem}$$

$$\begin{array}{l} 1 + q = 1 + mL + \frac{1}{2}m^2L^2 + \frac{1}{6}m^3L^3 + \frac{1}{24}m^4L^4 + \frac{1}{120}m^5L^5 \&c, \\ \text{and } 1 - q = 1 - mL + \frac{1}{2}m^2L^2 - \frac{1}{6}m^3L^3 + \frac{1}{24}m^4L^4 - \frac{1}{120}m^5L^5 \&c. \end{array}$$

m being any infinite index whatever, differing according to the scale of logarithms, being 1000,000 &c in Napier's or the hyperbolic logarithms, and 2302,585 &c. in Briggs's.

If one term of the ratio, of which L is the logarithm, be given, the other term will be easily obtained by the same rule. For, if L be Napier's logarithm of the ratio of a , the less term, to b , the greater, then, according as a or b is given, we shall have

$$\begin{array}{l} b = a \text{ into } 1 + L + \frac{1}{2}L^2 + \frac{1}{6}L^3 + \frac{1}{24}L^4 \&c, \\ \text{or } a = b \text{ into } 1 - L + \frac{1}{2}L^2 - \frac{1}{6}L^3 + \frac{1}{24}L^4 \&c. \end{array}$$

Whence, by the help of the logarithms contained in the tables, may easily be found the number to any given logarithm to a great extent; For if the small difference between the given logarithm L and the nearest tabular logarithm, either greater or less, be called l , and the number answering to the tabular logarithm a when it is less than the given logarithm, but b when greater; it will follow that the number answering to the logarithm L will be either

$$\begin{array}{l} a \text{ into } 1 + l + \frac{1}{2}l^2 + \frac{1}{6}l^3 + \frac{1}{24}l^4 + \frac{1}{120}l^5 \&c, \\ \text{or } b \text{ into } 1 - l + \frac{1}{2}l^2 - \frac{1}{6}l^3 + \frac{1}{24}l^4 - \frac{1}{120}l^5 \&c. \end{array}$$

which

which series converge so quick, l being always very small, that the first two terms $1 \pm l$ are generally sufficient to find the number to 10 places of figures.

Dr. Halley subjoins also an easy approximation for these series, by which it appears that the number answering to the log. is nearly

$$\frac{1+\frac{1}{2}l}{1-\frac{1}{2}l} \times a, \text{ or } \frac{1-\frac{1}{2}l}{1+\frac{1}{2}l} \times b, \left\{ \begin{array}{l} \text{in Napier's} \\ \text{logs.} \end{array} \right. \text{ and } \left\{ \begin{array}{l} \frac{n+\frac{1}{2}l}{n-\frac{1}{2}l} \times a, \text{ or } \frac{n-\frac{1}{2}l}{n+\frac{1}{2}l} \times b, \\ \text{in Briggs's} \\ \text{logarithms;} \end{array} \right.$$

where n is $= 0.434,294,481,903, \&c = \frac{1}{m}$.

Of Mr. Sharp's Methods.

The labours of Mr. Abraham Sharp of Little-Horton, near Bradford in Yorkshire, in this branch of mathematics, were very great and meritorious. His merit, however, consisted rather in the improvement and illustration of the methods of former writers, than in the invention of any new ones of his own. In this way he greatly extended and improved Dr. Halley's method above-described, as also those of Mercator and Wallis; illustrating these improvements by extensive calculations, and by them computing our table 5, consisting of the logarithms of all numbers to 100, and of all prime numbers to 1100, each to 61 places. He also composed a neat compendium of the best methods for computing the natural sines, tangents, and secants, chiefly from the rules before given by Newton; and by Newton's, or Gregory's, series $a \pm t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 \&c$, for the arc in terms of the tangent, he computed the circumference of the circle to 72 places, namely, from the arc of 30 degrees, whose tangent t is $= \sqrt{\frac{1}{3}}$ to the radius 1. Other astonishing instances of his industry and labour appear in his *Geometry Improved*, printed in 1717, and signed *A. S. Philomath*, from whence the said table of logarithms was extracted. This ingenious man was some time assistant at the Royal Observatory to Mr. Flamsteed, the first Astronomer Royal; and being one of the most accurate and indefatigable computers that ever existed, he was for many years the common resource for Mr. Flamsteed, Sir Jonas Moore, Dr. Halley, &c. in all intricate and troublesome calculations. He afterwards retired to his native place at Little-Horton, where, after a life spent in intense study and calculations, he died the 18th of July 1742, in the 91st year of his age.

Of the Construction of Logarithms by Fluxions.

It appears by the very definition and description given by Napier of his logarithms, as stated in page 46 of this introduction, that the fluxion of his, or the hyperbolic logarithm, of any number, is a fourth proportional to that number, its logarithm, and unity; or which is the same, that it is equal to the fluxion of the number divided by the number: For the description shews that $z1 : za$ or

$1 : : z1$ the fluxion of $z1 : za$, which therefore is $= \frac{z1}{z1}$; but za is also equal to

the fluxion of the logarithm A &c by the description ; therefore the fluxion of the logarithm is equal to $\frac{\dot{z}_1}{z_1}$, the fluxion of the quantity divided by the quantity itself. The same thing appears again at art. 2 of that little piece in the appendix to his *Constructio Logarithmorum*, intituled *Habitudines Logarithmorum & suorum naturalium numerorum invicem*, where he observes that, as any greater quantity is to a less, so is the velocity of the increment or decrement of the logarithms at the place of the less quantity, to that at the greater. Now this velocity of the increment or decrement of the logarithms being the same thing as their fluxions, that proportion is this, $x : a :: \text{flux. log. } a : \text{flux. log. } x$; hence if a be $= 1$, as at the beginning of the table of numbers, where the fluxion of the logs. is the index or characteristic c , which is also 1 in Napier's, or the hyperbolic, logarithms, and 43429 &c, in Briggs's, the same proportion becomes $x : 1 :: c : \text{flux. log. } x$; but the constant fluxion of the numbers is also 1, and therefore that proportion is also this, $x : \dot{x} :: c : \frac{c\dot{x}}{x} =$ the fluxion of the logarithm of x ; and in the hyper-

bolic logarithms, where c is $= 1$, it becomes $\frac{\dot{x}}{x} =$ the fluxion of Napier's, or the hyperbolic, logarithm of x . This same property has also been noticed by many other authors since Napier's time. And the same, or a similar, property is evidently true in all the systems of logarithms whatever, namely, that the modulus of the system is to any number as the fluxion of its logarithm is to the fluxion of the number.

Now from this property, by means of the doctrine of fluxions, are derived other ways for making logarithms, which have been illustrated by many writers on this branch, as Craig, Jo. Bernoulli, and almost all the writers on fluxions. And this method chiefly consists in expanding the reciprocal of the given quantity in an infinite series, then multiplying each term by the fluxion of the said quantity, and lastly taking the fluents of the terms ; by which there arises an infinite series of terms for the logarithm sought. So, to find the logarithm of any number N ; put any compound quantity for N , as suppose $\frac{n+x}{n}$;

then the fluxion of the logarithm, or $\frac{\dot{N}}{N}$, being $\frac{\dot{x}}{n+x} = \frac{\dot{x}}{n} - \frac{x\dot{x}}{nn} + \frac{x^2\dot{x}}{n^3} - \frac{x^3\dot{x}}{n^4} \&c$,

the fluents give log. of N , or log. of $\frac{n+x}{n} = \frac{x}{n} - \frac{x^2}{2n^2} + \frac{x^3}{3n^3} - \frac{x^4}{4n^4} \&c$.

And writing $-x$ for x gives log. $\frac{n-x}{n} = -\frac{x}{n} - \frac{x^2}{2n^2} - \frac{x^3}{3n^3} - \frac{x^4}{4n^4} \&c$.

Also because $\frac{n}{n \pm x} = 1 \div \frac{n \pm x}{n}$, or log. $\frac{n}{n \pm x} = 0 - \log. \frac{n \pm x}{n}$

we have log. $\frac{n}{n+x} = -\frac{x}{n} + \frac{x^2}{2n^2} - \frac{x^3}{3n^3} + \frac{x^4}{4n^4} \&c$,

and log. $\frac{n}{n-x} = +\frac{x}{n} + \frac{x^2}{2n^2} + \frac{x^3}{3n^3} + \frac{x^4}{4n^4} \&c$.

And

And by adding and subtracting any of these series to or from one another, and multiplying or dividing their corresponding numbers, various other series for logarithms may be found, converging much quicker than these do.

In like manner by assuming quantities otherwise compounded for the value of N , various other forms of logarithmic series may be found by the same means.

Of Mr. Cotes's Logometria.

Mr. Roger Cotes was elected the first Plumian professor of astronomy and experimental philosophy in the university of Cambridge, January 1706, which appointment he filled with the greatest credit, till he died the 5th of June 1716, in the prime of life, having not quite compleated the 34th year of his age. His early death was a great loss to the mathematical world, as his genius and abilities were of the brightest order, as is manifest by the specimens of his performance given to the public. Among these are his *Logometria*, first printed in Number 338 of the Philosophical Transactions, and afterwards in his *Harmonia Mensurarum*, published in 1722 with his other works, by his cousin, and successor in the Plumian professorship, Dr. Robert Smith. In this piece he first treats in a general way of measures of ratios, which measures, he observes, are quantities of any kind whose magnitudes are analogous to the magnitudes of the ratios, these magnitudes mutually increasing and decreasing together in the same proportion. He remarks that the ratio of equality has no magnitude, because it produces no change by adding and subtracting; that the ratios of greater and less inequality, are of different affections; and therefore, if the measure of the one of these be considered as positive, that of the other will be negative; and the measure of the ratio of equality nothing; that there are endless systems of these, which have all their measures of the same ratios proportional to certain given quantities, called *moduli*, which he defines afterwards; and the ratio of which they are the measures, each in its peculiar system, is called the modular ratio, *ratio modularis*, which ratio is the same in all systems. He then adverts to logarithms, which he considers as the *numerical* measures of ratios; and he describes the method of arranging them in tables, with the use of them in multiplication and division, raising of powers and extracting of roots, by means of the corresponding operations of addition and subtraction, multiplication and division.

After this introduction, which is only a flight abridgment of the doctrine long before very amply treated of by others, and particularly by Kepler and Mercator, we arrive at the first proposition, which has justly been censured as obscure and imperfect, seemingly through an affectation of brevity, intricacy, and originality without sufficient room for a display of this qualification. The reasoning in this proposition, such as it is, seems to be something between that of Kepler and the principles of fluxions, to which the quantities and expressions are nearly allied. However, as it is my duty rather to narrate than explain, I shall here exhibit it exactly as it stands. This proposition is "to determine the measure of any ratio," as, for instance, that of AC to AB , and which is effected in this manner. Conceive the

the

the difference BC to be divided into innumerable very small particles as PQ, and the ratio $\frac{A}{B}$ into as many such very small ratios, as between AQ and AP. Then, if the magnitude of the ratio between AQ and AP be given, by dividing there will also be given that of PQ to AP; and therefore, this being given, the magnitude of the ratio between AQ and AP may be expounded by the given quantity $\frac{PQ}{AP}$; for, AP remaining constant, conceive the particle PQ to be augmented or diminished in any proportion, and in the same proportion will the magnitude of the ratio between AQ and AP be augmented or diminished: Also, taking any determinate quantity M, the same may be expounded by $M \times \frac{PQ}{AP}$; and therefore the quantity $M \times \frac{PQ}{AP}$ will be the measure of the ratio between AQ and AP. And this measure will have divers magnitudes, and be accommodated to divers systems, according to the divers magnitudes of the assumed quantity M, which therefore is called the *modulus* of the system. Now like as the sum of all the ratios AQ to AP is equal to the proposed ratio AC to AB, so the sum of all the measures $M \times \frac{PQ}{AP}$ (found by the known methods) will be equal to the required measure of the said proposed ratio.

The general solution being thus dispatched, from the general expression he next deduces other forms of the measure in several corollaries and scholia; as 1st, The terms AP, AQ, approach the nearer to equality as the small difference PQ is less; so that either $M \times \frac{PQ}{AP}$ or $M \times \frac{PQ}{AQ}$ will be the measure of the ratio between AQ and AP to the modulus M. 2d, That hence the modulus M is to the measure of the ratio between AQ and AP, as either AP or AQ is to their difference PQ. 3d, The ratio between AC and AB being given, the sum of all the $\frac{PQ}{AP}$ will be given; and the sum of all the $M \times \frac{PQ}{AP}$ is as M: therefore the measure of any given ratio is as the modulus of the system from which it is taken. 4th, Therefore, in every system of measures, the modulus will always be equal to the measure of a certain determinate and immutable ratio; which therefore he calls the *modular ratio*. 5th, To illustrate the solution by an example: let z be any determinate and permanent quantity, x a variable or indeterminate quantity, and $\dot{x}$ its fluxion; then to find the measure of the ratio between $z + x$ and $z - x$, put this ratio equal to the ratio between y and 1, expounding the number y by AP, its fluxion $\dot{y}$ by PQ, and 1 by AB: then the fluxion of the required measure of the ratio between y and 1 is $M \times \frac{\dot{y}}{y}$. Now for y restore its val.

$\frac{z+x}{z-x}$, and for $\dot{y}$ the fluxion of that value $\frac{2\dot{z}x}{z^2-x^2}$, so shall the fluxion of the measure

become $2 M \times \frac{\dot{z}x}{z^2-x^2}$ or $2 M$ into $\frac{\dot{x}}{z} + \frac{x\dot{x}}{z^3} + \frac{x\dot{x}^2}{z^5}$ &c. And therefore that measure will be $2 M$ into $\frac{x}{z} + \frac{x^3}{3z^3} + \frac{x^5}{5z^5}$ &c. In like manner the measure of

the

the ratio between $1 + v$ and 1 will be found to be $- - - M$ into $v - \frac{1}{2}v^2 + \frac{1}{3}v^3 - \frac{1}{4}v^4 \&c.$ And hence, to find the number from the logarithm given, he reverts the series in this manner: If the last measure be called m , we

$$\begin{aligned} \text{shall have } \frac{m}{M} \text{ or } Q &= v - \frac{1}{2}v^2 + \frac{1}{3}v^3 - \frac{1}{4}v^4 + \frac{1}{5}v^5 \&c, \\ \text{and therefore } Q^2 &= \dots v^2 - v^3 + \frac{1}{12}v^4 - \frac{1}{6}v^5 \&c, \\ \text{and } Q^3 &= \dots v^3 - \frac{1}{2}v^4 + \frac{7}{24}v^5 \&c, \\ \text{and } Q^4 &= \dots v^4 - 2v^5 \&c, \\ \text{and } Q^5 &= \dots v^5 \&c; \end{aligned}$$

then by adding continually we shall have

$$\begin{aligned} Q + \frac{1}{2}Q^2 &= v - \frac{1}{6}v^3 + \frac{5}{24}v^4 - \frac{1}{60}v^5 \&c, \\ Q + \frac{1}{2}Q^2 + \frac{1}{6}Q^3 &= v - \frac{1}{24}v^4 + \frac{3}{40}v^5 \&c, \\ Q + \frac{1}{2}Q^2 + \frac{1}{6}Q^3 + \frac{1}{24}Q^4 &= v - \frac{1}{120}v^5 \&c, \\ Q + \frac{1}{2}Q^2 + \frac{1}{6}Q^3 + \frac{1}{24}Q^4 + \frac{1}{120}Q^5 &= v \&c, \end{aligned}$$

that is $v = Q + \frac{1}{2}Q^2 + \frac{1}{6}Q^3 + \frac{1}{24}Q^4 + \frac{1}{120}Q^5 \&c.$ And therefore the required ratio of $1 + v$ to 1 is equal to the ratio of $1 + Q + \frac{1}{2}Q^2 \&c$ to 1 . Put now $m = M$, or $Q = 1$, and the above will become the ratio of $1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} \&c.$ to 1 for the constant modular ratio. In like manner, if the ratio between 1 and $1 - v$ be proposed, the measure of this ratio will come out M into $v + \frac{1}{2}v^2 + \frac{1}{3}v^3 + \frac{1}{4}v^4 \&c$; which being called m , and $\frac{m}{M} = Q$, that ratio will be the ratio of 1 to $1 - Q + \frac{1}{2}Q^2 - \frac{1}{6}Q^3 + \frac{1}{24}Q^4 \&c.$ And hence taking $m = M$, or $Q = 1$, the said modular ratio will also be the ratio of 1 to $1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} \&c.$ And the former of these expressions for the modular ratio comes out the ratio of 2.718,281,828,459, &c. to 1 , and the latter the ratio of 1 to 0.367,879,441,171, &c.

In the 2d proposition our learned author gives directions for constructing Briggs's canon of logarithms, namely, first by the general series $2M$ into $\frac{x}{z} + \frac{x^3}{3z^3} + \frac{x^5}{5z^5} \&c$, finding the logarithms of a few such ratios as that of 126 to 125, 225 to 224, 2401 to 2400, 4375 to 4374, &c, from whence the logarithm of 10 will be found to be 2.302,585,092,994, &c, when M is 1; but since Briggs's log. of 10 is 1, therefore as 2.302,585, &c, is to the modulus 1, so is 1 (Briggs's log. of 10) to 0.434,294,481,903, &c, which therefore is the modulus of Briggs's logarithms. Hence he deduces the logarithms of 7, 5, 3, and 2. In like manner are the logarithms of other prime numbers to be found, and from them the logarithms of composite numbers by addition and subtraction only.

He then remarks that the first term of the general series $2M$ into $\frac{x}{z} + \frac{x^3}{3z^3} + \frac{x^5}{5z^5} \&c$ will be sufficient for the logarithms of intermediate numbers between those in the table, or even for numbers beyond the limits of the table. Thus, to find the logarithm answering to any intermediate number; let a and e be two numbers, the one the given number, and the other the nearest tabular number, a

* The computation of these numbers may be seen in one of the tracts contained in this present volume, pages 358, 359, 360, and 371, 372, 373, 374, and 375.

being the greater, and e the less, of them; put $z = a + e$ their sum, $x = a - e$ their difference, $\lambda =$ the logarithm of the ratio of a to e , that is, the excess of the logarithm of a above that of e : so shall the said difference of their logarithms be $\lambda = 2M \times \frac{x}{z}$ very nearly.—And, if there be required the number answering to any given intermediate logarithm, because λ is $= \frac{2Mx}{z} = \frac{2Mx}{2a-x}$ or $\frac{2Mx}{2e+x}$, therefore x will be $= \frac{\lambda a}{M + \frac{1}{2}\lambda}$, or $\frac{\lambda e}{M - \frac{1}{2}\lambda}$, very nearly.

In the 3d proposition our ingenious author teaches how to convert the canon of logarithms to logarithms of any other system, by means of their *moduli*. And in several more propositions he exemplifies the canon of logarithms in the solution of various important problems in geometry and physics; such as the quadrature of the hyperbola, the description of the logistica, the equi-angular spiral, the nautical meridian, &c; the descent of bodies in resisting mediums, the density of the atmosphere at any altitude, &c, &c.

Of Dr. Taylor's Construction of Logarithms.

Dr. Brook Taylor (a very learned mathematician, and secretary to the Royal Society, who died at Somerset-House, Nov. 1731) gave the following method of constructing logarithms in number 352 of the Philosophical Transactions. His method is founded on these three considerations: 1st, That the sum of the logarithms of any two numbers is the logarithm of the product of those numbers; 2d. That the logarithm of 1 is nothing, and consequently that the nearer any number is to 1, the nearer will its logarithm be to 0; 3d, That the product of two numbers or factors, of which the one is greater and the other less than 1, is nearer to 1 than that factor is which is on the same side of 1 with itself; so of the two numbers $\frac{2}{3}$ and $\frac{4}{3}$, the product $\frac{8}{9}$ is less than 1, but yet nearer to it than $\frac{2}{3}$ is, which is also less than 1. On these principles he founds the present approximation, which he explains by the following example. To find the relation between the logarithms of 2 and 10: In order to this he assumes two fractions as $\frac{128}{100}$ and $\frac{8}{10}$, or $\frac{2^7}{10^2}$ and $\frac{2^3}{10}$, whose numerators are powers of 2, and their denominators powers of 10, the one fraction being greater and the other less than unity or 1. Having set these two down, in the form of decimal fractions, below each other in the first column of the following table, and in the second column A and B for their logarithms, expressing by an equation how they are composed of the logarithms of 2 and 10, the numbers in question, those logarithms being denoted thus, $l 2$ and $l 10$. Then

1,280,000,000,000	A = . . . =	$7\frac{1}{2}-$	$2\frac{1}{10}\frac{1}{2}$	$\Delta 0,28$
0,800,000,000,000	B = . . . =	$3\frac{1}{2}-$	$\frac{1}{10}$	$\nabla 0,33$
1,024,000,000,000	C = A + B =	$10\frac{1}{2}-$	$3\frac{1}{10}$	$\Delta 0,300$
0,990,352,031,429	D = B + 9C =	$93\frac{1}{2}-$	$28\frac{1}{10}$	$\nabla 0,301,07$
1,004,336,277,664	E = C + 2D =	$169\frac{1}{2}-$	$59\frac{1}{10}$	$\Delta 0,301,020$
0,998,959,536,107	F = D + 2E =	$485\frac{1}{2}-$	$146\frac{1}{10}$	$\nabla 0,301,030,9$
1,000,162,894,165	G = E + 4F =	$2136\frac{1}{2}-$	$643\frac{1}{10}$	$\Delta 0,301,029,96$
0,999,936,281,874	H = F + 6G =	$13301\frac{1}{2}-$	$4004\frac{1}{10}$	$\nabla 0,301,029,997$
1,000,035,441,215	I = G + 2H =	$28738\frac{1}{2}-$	$8651\frac{1}{10}$	$\Delta 0,301,029,995,1$
0,999,971,720,830	K = H + I =	$42039\frac{1}{2}-$	$12655\frac{1}{10}$	$\nabla 0,301,029,995,9$
1,000,007,161,046	L = I + K =	$70777\frac{1}{2}-$	$21306\frac{1}{10}$	$\Delta 0,301,029,995,62$
0,999,993,203,514	M = K + 3L =	$254370\frac{1}{2}-$	$76573\frac{1}{10}$	$\nabla 0,301,029,995,67$
1,000,000,364,511	N = L + M =	$325147\frac{1}{2}-$	$97879\frac{1}{10}$	$\Delta 0,301,029,995,663,5$
0,999,999,764,687	O = M + 18N =	$6107016\frac{1}{2}-$	$1838335\frac{1}{10}$	$\nabla 0,301,029,995,664,0$
comp. ar. 235313				
$0 = 364511 O + 235313 N = 2302585825187\frac{1}{2} - 693147400972\frac{1}{10}$				$\Delta 0,301,029,995,663,987$

multiplying the two numbers in the first column together, there is produced a third number 1,024, against which is written C, for its logarithm, expressing likewise by an equation in what manner C is formed of the foregoing logarithms A and B. And in the same manner the calculation is continued throughout; only observing this compendium, that before multiplying the two last numbers already entered in the table, to consider what power of one of them must be used to bring the product the nearest that can be to unity. Now, after having continued the table a little way, this is found by only dividing the differences of the numbers from unity one by the other, and taking the nearest quotient for the index of the power sought. Thus, the second and third numbers in the table being 0,8 and 1,024, their differences from unity are 0,200 and 0,024; hence $0,200 \div 0,024$ gives 9 for the index; and therefore multiplying the 9th power of 1,024 by 0,8 produces the next number 0,990,352,031,429, whose logarithm is $D = B + 9C$.

When the calculation is continued in this manner till the numbers become small enough, or near enough to 1, the last logarithm is supposed equal to nothing, which gives an equation expressing the relation of the logarithms, and from whence the required logarithm is determined. Thus supposing $G = 0$, we have $2136\frac{1}{2} - 643\frac{1}{10} = 0$, and hence, because the logarithm of 10 is 1, we obtain $\frac{1}{2} = \frac{643}{2136} = 0,301,029,96$, too small in the last figure; which so happens because the number corresponding to G is greater than 1. And in this manner are all the numbers in the third or last column obtained, which are continual approximations to the logarithms of 2.

There is another expedient which renders this calculation still shorter, and it is founded on this consideration; that when x is small, $\overline{1+x}^n$ is nearly $= 1 + nx$. Hence if $1+x$ and $1-z$ be the two last numbers already found in the first

first column of the table, the product of their powers $\overline{1+x}^m \times \overline{1-z}^n$ will be nearly $= 1$; and hence the relation of m and n may be thus found, $\overline{1+x}^m \times \overline{1-z}^n$ is nearly $= 1 + mx \times 1 - nz = 1 + mx - nz - mnxz = 1 + mx - nz$ nearly, which being also $= 1$ nearly, therefore $m : n :: z : x :: l. 1 - z : l. 1 + x$; whence $x l. 1 - z + z l. 1 + x = 0$. For example, let 1,024 and 0,990352 be the last numbers in the table, their logarithms being C and D: here we have $1,024 = 1 + x$, and $0,990352 = 1 - z$; consequently $x = 0,024$, and $z = 0,009648$; and hence the ratio $\frac{z}{x}$ in small numbers is $\frac{201}{500}$. So that for finding the logarithms proposed, we may take $500 D + 201 C = 48510 l_2 - 14603 l_{10} = 0$; which gives $l_2 = 0,3010307$. And in this manner are found the numbers in the last line of the table.

Cf Mr. Long's Method.

In number 339 of the Philosophical Transactions, are given a brief table and method for finding the logarithm to any number, and the number to any logarithm, by Mr. John Long, B. D. Fellow of C. C. C. Oxon. This table and method are similar to those described in chap. 14 of Briggs's *Arithmetica Logarithmica*, differing only in this, that in this table by Mr. Long, the logarithms, in each class, are in arithmetical progression, the common difference being 1; but in Briggs's little table the column of natural numbers has the like common difference. The table consists of eight classes of logarithms and their corresponding numbers as follows :

Lo.	Nat. Num.	Log.	Nat. Num.	Log.	Nat. Num.	Log.	Nat. Num.
,9	7,943282347	,009	1,020939484	,00009	1,000207254	,0000009	1,000002072
,8	6,309573445	8	1,018591388	8	1,000184224	8	1,000001842
,7	5,011872336	7	1,016248694	7	1,000161194	7	1,000001611
,6	3,981071706	6	1,013911386	6	1,000138165	6	1,000001381
,5	3,162277660	5	1,011579454	5	1,000115136	5	1,000001151
,4	2,511886432	4	1,009252886	4	1,000092106	4	1,000000921
,3	1,995262315	3	1,006931669	3	1,000069080	3	1,000000690
,2	1,584893193	2	1,004615794	2	1,000046053	2	1,000000460
,1	1,258925412	1	1,002305238	1	1,000023026	1	1,000000230
,09	1,230268771	,009	1,002074475	,00009	1,000020724	,0000009	1,000000207
8	1,202264435	8	1,001843766	8	1,000018421	8	1,000000184
7	1,174897555	7	1,001613109	7	1,000016118	7	1,000000161
6	1,148153621	6	1,001382506	6	1,000013816	6	1,000000138
5	1,122018454	5	1,001151956	5	1,000011513	5	1,000000115
4	1,096478196	4	1,000921459	4	1,000009210	4	1,000000092
3	1,071519305	3	1,000691015	3	1,000006908	3	1,000000069
2	1,047128548	2	1,000460623	2	1,000004605	2	1,000000046
1	1,023292992	1	1,000230285	1	1,000002302	1	1,000000023

where, because the logarithms in each class are the continual multiples 1, 2, 3, &c of the lowest, it is evident that the natural numbers are so many scales of geometrical proportionals, the lowest being the common ratio, or the ascending numbers are the 1, 2, 3, &c powers of the lowest, as expressed by the figures 1, 2, 3, &c of their corresponding logarithms. Also the last number in the first, second, third, &c class, is the 10th, 100th, 1000th, &c root of 10; and any number in any class is the 10th power of the corresponding number in the next following class.

To find the logarithm of any number, as suppose of 2000, by this table: Look in the first class for the number next less than the first figure 2, and it is 1,995262315, against which is 3 for the first figure of the logarithm sought. Again, dividing 2 the number proposed by 1,995262315 the number found in the table, the quotient is 1,002374467; which being looked for in the second class of the table, and finding neither its equal nor a less, 0 is therefore to be taken for the second figure of the logarithm; and the same quotient 1,002374467 being looked for in the third class, the next less is there found to be 1,002305238, against which is 1 for the third figure of the logarithm; and dividing the quotient 1,002374467 by the said next less number 1,002305238, the new quotient is 1,000069070; which being sought in the fourth class gives 0, but sought in the fifth class gives 2, which are the fourth and fifth figures of the logarithm sought: again, dividing the last quotient by 1,000046053 the next less number in the table, the quotient is 1,000023015, which gives 9 in the 6th class for the 6th figure of the logarithm sought: and again dividing the last quotient by 1,000020724 the next less number, the quotient is 1,000002291, the next less than which in the 7th class gives 9 for the 7th figure of the logarithm: and dividing the last quotient by 1,000002072, the quotient is 1,000000219, which gives 9 in the 8th class for the 8th figure of the logarithm: and again, the last quotient 1,000000219 being divided by 1,000000207 the next less, the quotient 1,000000012 gives 5 in the same 8th class, when one figure is cut off, for the 9th figure of the logarithm sought. All which figures collected together give 3,301029995 for Briggs's logarithm of 2000, the index 3 being supplied; which logarithm is true in the last figure.

To find the number answering to any given logarithm, as, suppose, to 3.301,030,0: omitting the characteristic, against the other figures 3, 0, 1, 0, 3, 0, 0, as in the first column in the margin, are the several numbers as in the 2d column, found from their respective 1st, 2d, 3d, &c classes; the effective numbers of which multiplied continually together, the last product is 2,000000019966, which, because the characteristic is 3, gives 2000,000019966, or 2000 only, for the required number answering to the given logarithm.

3	1,995262315
0	0
1	1,002305238
0	0
3	1,000069080
0	0
0	0

Of Mr. Jones's Method.

In the 61st volume of the Philosophical Transactions, is a small paper on logarithms, which had been drawn up and left unpublished, by the learned and ingenious William Jones, Esq. The method contained in this memoir, depends on an application of the doctrine of fluxions, to some properties drawn from the nature of the exponents of powers. Here all numbers are considered as some certain powers of a constant determinate root: so any number x may be considered as the z power of any root r , or that $x = r^z$ is a general expression of all numbers in terms of the constant root r and a variable exponent z . Now the index z being the logarithm of the number x , therefore to find this logarithm, is the same thing as to find what power of the radical r is equal to the number x .

From this principle, the relation between the fluxions of any number x and its logarithm z is thus determined: put $r = 1 + n$; then is $x = r^z = \overline{1 + n}^z$, and $x + \dot{x} = \overline{1 + n}^{z+\dot{z}} = \overline{1 + n}^z \times \overline{1 + n}^{\dot{z}} = x \times \overline{1 + n}^{\dot{z}} =$ (by expanding $\overline{1 + n}^{\dot{z}}$, omitting the 2d, 3d, &c powers of $\dot{z}$, and writing q for $\frac{n}{1+n}$) $x + x\dot{z} \times : q + \frac{1}{2}q^2 + \frac{1}{3}q^3 + \frac{1}{4}q^4$ &c; therefore $\dot{x} = ax\dot{z}$, putting a for the series $q + \frac{1}{2}q^2 + \frac{1}{3}q^3$ &c, or $f \dot{x} = x\dot{z}$, putting $f = \frac{1}{a}$.

Now when $r = 1 + n = 10$, as in the common logarithms of Briggs's form; then $n = 9$, $q = 0.9$, and the series $q + \frac{1}{2}q^2 + \frac{1}{3}q^3$ &c gives $a = 2.302,585$ &c, and therefore its reciprocal f is $= 0.434,294$ &c. But, if $a = 1 = f$, the form will be that of Napier's logarithms.

From the above form $x\dot{z} = f\dot{x}$ or $\dot{z} = \frac{f\dot{x}}{x}$, are then deduced many curious and general properties of logarithms, with the several series heretofore given by Gregory, Mercator, Wallis, Newton, and Halley. But of all these series, that one which our author selects for constructing the logarithms, is this; putting $N = \frac{r-p}{r+p}$, the logarithm of $\frac{r}{p}$ is $= 2f \times : N + \frac{1}{3}N^3 + \frac{1}{5}N^5 + \frac{1}{7}N^7$ &c in the case in which $r - p$ is $= 1$, and consequently then $N = \frac{1}{2r-1}$ or $\frac{1}{2p+1}$; which series will then converge very fast.

Hence, having given any numbers, p, q, r , &c, and as many ratios a, b, c , &c, composed of them, the difference between the two terms of each ratio being 1; as also the logarithms A, B, C , &c of those ratios given: to find the logarithms P, Q, R , &c of those numbers; supposing $f = 1$. For instance, if $p = 2, q = 3, r = 5$; and $a = \frac{19}{8} = \frac{3^2}{2^3}, b = \frac{16}{15} = \frac{2^4}{3 \cdot 5}, c = \frac{25}{24} = \frac{5^2}{3 \cdot 2^3}$. Now the logarithms A, B, C , of these ratios a, b, c , being found by the above series, from the nature of powers we have these three equations.

$$\left. \begin{aligned} A &= 2Q - 3P \\ B &= 4P - Q - R \\ C &= 2R - Q - 3P \end{aligned} \right\} \text{ which equations reduced give } \begin{cases} P = 3A + 4B + 2C = \log. \text{ of } 2, \\ Q = 5A + 6B + 3C = \log. \text{ of } 3, \\ R = 7A + 9B + 5C = \log. \text{ of } 5. \end{cases}$$

And hence $P + R = 10A + 13B + 7C$ is = the logarithm of 2×5 or 10.

An elegant tract on logarithms, as a comment on Dr. Halley's method, was also given by Mr. Jones in his *Synopsis Palmariorum Matheseos*, published in the year 1706. And in the Philosophical Transactions he communicated various improvements in goniometrical properties, and the series relating to the circle and to trigonometry.

The memoir above described was delivered to the Royal Society by their then librarian, Mr. John Robertson, a worthy, ingenious, and industrious man; who also communicated to the Society several little tracts of his own relating to logarithmical subjects: he was also the author of an excellent Treatise on the Elements of Navigation, in two volumes; and he was successively mathematical master to Christ's hospital in London; head master to the royal naval academy at Portsmouth; and librarian, clerk, and housekeeper to the Royal Society; at whose house, in Crane-Court, Fleet-Street, he died in 1776, aged 64 years.

And among the papers of Mr. Robertson, I have, since his death, found one containing the following particulars relating to Mr. Jones, which I here insert, as I know of no other account of his life, &c. and as any true anecdotes of such extraordinary men must always be acceptable to the learned. This paper is not in Mr. Robertson's hand writing, but in a kind of running law-hand, and is signed R. M. 12 Sept. 1771.

"William Jones, Esq. F. R. S. was born at the foot of Bodavon mountain [Myriadd Bodafon], in the parish of Llanfihangel tre'r Bardd, in the isle of Anglesey, North Wales, in the year 1675. His father John George * was a farmer, of a good family, being descended from Hwfa ap Cynddelw, one of the 15 tribes of North Wales. He gave his two sons the common school education of the country, reading, writing and accounts, in English, and the Latin grammar. Harry his second son took to the farming business; but William the eldest, having an extraordinary turn for mathematical studies, determined to try his fortune abroad from a place where the same was but of little service to him; he accordingly came to London, accompanied by a young man, Rowland Williams, afterwards an eminent perfumer in Wych Street. The report in the country is, that Mr. Jones soon got into a merchant's counting house, and so gained the esteem of his master, that he gave him the command of a ship for a West India voyage; and that upon his return he set up a mathematical school, and published his book of navigation †; and that upon the death of the merchant he married his widow: that Lord Macclesfield's son being his pupil, he was made secretary

* "It is the custom in several parts of Wales for the name of the father to become the surname of his children. John George the father was commonly called Sion Sors, of Llanbabo, to which parish he moved, and where his children were brought up."

† This tract on navigation, intituled, "A New Compendium of the whole Art of Practical Navigation," was published in 1702, and dedicated "to the reverend and learned Mr. John Harris, M. A. and F. R. S." the author (as I apprehend) of the "Universal Dictionary of Arts and Sciences," under whose roof Mr. Jones says he composed the said treatise on Navigation.

to the chancellor, and one of the Deputy-tellers of the exchequer—and they have a story of an Italian wedding which caused great disturbance in Lord Macclesfield's family, but was compromised by Mr. Jones; which gave rise to a saying, that Macclesfield was the making of Jones, and Jones the making of Macclesfield."

Mr. Jones died July 3, 1749, being a vice-president of the Royal Society; and left one daughter, and his widow big with another child, which proved a son, who is the present Sir William Jones, at this time, 1791, one of the judges in India, and highly esteemed for his great abilities, extensive learning, and eminent patriotism.

Of Mr. Andrew Reid and others.

Andrew Reid, Esq. published in 1767 a quarto tract under the title of *An Essay on Logarithms*, in which he also shews the computation of logarithms from principles depending on the binomial theorem and the nature of the exponents of powers, the logarithms of numbers being here considered as the exponents of the powers of 10. He hence brings out the usual series for logarithms, and largely exemplifies Dr. Halley's most simple construction.

Besides the authors whose methods have been here particularly described, many others have treated on the subjects of logarithms, and of the sines, tangents, secants, &c; among the principal of whom are Leibnitz, Euler, Maclaurin, Wolfius, and professor Simson in an accurate geometrical tract on logarithms, contained in his posthumous works, which were elegantly printed in 4to. at Glasgow, in the year 1776, at the expence of that very learned nobleman, the late Earl Stanhope, and by his lordship disposed of in presents among gentlemen most eminent for mathematical learning.

Of Mr. Dodson's Anti-logarithmic Canon.

The only remaining considerable work of this kind published, that I know of, is the Anti logarithmic Canon of Mr. James Dodson, a very ingenious mathematician, which work he published in folio in the year 1742; a very great performance, containing all logarithms under 100,000, and their corresponding natural numbers to 11 places of figures, with all their differences and the proportional parts; the whole arranged in the order contrary to that used in the common tables of numbers and logarithms, the exact logarithms being here placed first, and increasing continually by 1, from 1 to 100,000, and their corresponding nearest numbers in the columns opposite to them; and by means of the differences and proportional parts, the logarithm to any number, or the number to any logarithm, each to 11 places of figures, is readily found. This work contains also, besides the construction of the natural numbers to the given logarithms, "precepts and examples, shewing some of the uses of logarithms, in facilitating the most difficult operations in common arithmetic, cases of interest, annuities, mensuration, &c; to which is prefixed an introduction, containing a short account of logarithms, and of the most considerable improvements made, since their invention, in the manner of constructing them."

The

The manner in which these numbers were constructed, consists chiefly in imitations of some of the methods before described by Briggs, and is nothing more than generating a scale of 100000 geometrical proportionals from 1 the least term to 10 the greatest, each continued to 11 places of figures; and the means of effecting this are such as easily flow from the nature of a series of proportionals, and are briefly as follows. First between 1 and 10 are interposed 9 mean proportionals; then between each of these 11 terms there are interposed 9 other means, making in all 101 terms; then between each of these a 3d set of 9 means, making in all 1001 terms; again between each of these a 4th set of 9 means, making in all 10001 terms; and lastly between each two of these terms, a 5th set of 9 means, making in all 100001 terms, including both the 1 and the 10. The first four of these 5 sets of means, are found each by one extraction of the 10th root of the greater of the two given terms, which root is the least mean, and then multiplying it continually by itself according to the number of terms in the section or set; and the 5th or last section is made by interposing each of the 9 means by help of the method of differences before taught. Namely, putting 10 the greatest term = A , $A^{\frac{1}{10}} = B$, $B^{\frac{1}{10}} = C$, $C^{\frac{1}{10}} = D$, $D^{\frac{1}{10}} = E$, and $E^{\frac{1}{10}} = F$; now extracting the 10th root of A or 10, it gives 1,2589254118 = $B = A^{\frac{1}{10}}$ for the least of the 1st set of means; and then multiplying it continually by itself, we have $B, B^2, B^3, B^4, \&c$ to $B^{10} = A$ for all the 10 terms: 2dly, the 10th root of 1,2589254118 gives 1,0232929923 = $C = B^{\frac{1}{10}} = A^{\frac{1}{100}}$ for the least of the 2d class of means, which being continually multiplied gives $C, C^2, C^3, \&c$ to $C^{100} = B^{10} = A$ for all the 2d class of 100 terms: 3dly, the 10th root of 1,0232929923 gives 1,0023052381 = $D = C^{\frac{1}{10}} = B^{\frac{1}{1000}} = A^{\frac{1}{10000}}$ for the least of the 3d class of means, which being continually multiplied gives $D, D^2, D^3, \&c$ to $D^{10000} = C^{1000} = B^{1000} = A$ for the 3d class of 1000 terms: 4thly, the 10th root 1,0023052381 gives 1,0002302850 = $E = D^{\frac{1}{10}} = C^{\frac{1}{10000}} = B^{\frac{1}{100000}} = A^{\frac{1}{1000000}}$ for the least of the 4th class of means, which being continually multiplied gives $E, E^2, E^3, \&c$ to $E^{1000000} = D^{100000} = C^{10000000} = B^{100000000} = A$ for the 4th class of 10000 terms. Now these 4 classes of terms thus produced, require no less than 11110 multiplications of the least means by themselves; which however are much facilitated by making a small table of the first 10 or even 100 products of the constant multiplier, and from thence only taking out the proper lines and adding them together: and these 4 classes of numbers always prove themselves at every 10th term, which must always agree with the corresponding successive terms of the preceeding class. The remaining 5th class is constructed by means of differences, being much easier than the method of continual multiplication, the 1st and 2d differences only being used, as the 3d difference is too small to enter the computation of the sets of 9 means between each two terms of the 4th class. And the several 2d differences for each of these sets of 9 means, are found from the properties of a set of proportionals 1, $r, r^2, r^3, \&c$, as disposed

posed in the 1st column of the annexed table, and their several orders of differences as in the other columns of the table; where it is evident that each column, both that of the given terms of the progression, and those of their orders of differ-

Terms	1st dif.	2d. dif.	3d dif.	&c
$1 \times$	$r - 1 \times$	$(r - 1)^2 \times$	$(r - 1)^3 \times$	
1	1	1	1	∞
r	r	r	r	
r^2	r^2	r^2	r^2	
r^3	r^3	r^3	r^3	
&c	&c	&c	&c	

ences, forms a scale of proportionals, having the same common ratio, and that each horizontal line, or row, forms a geometrical progression, having all the same common ratio $r - 1$, which is also the 1st difference of each set of means; so $(r - 1)^2$ is the 1st of the 2d differences, and which is constantly the same, as the 3d differences become too small in the required terms of our progression to be regarded, at least near the beginning of the table: hence, like as 1, $r - 1$, and $(r - 1)^2$ are the 1st term with its 1st and 2d differences; so r^n , $r^n \cdot r - 1$, and $r^n \cdot (r - 1)^2$ are any other term with its 1st and 2d differences. And by this rule the 1st and 2d differences are to be found for every set of 9 means, viz. multiplying the 1st term of any class (which will be the several terms of the series E, E^2 , E^3 , &c, or every 10th term of the series F, F^2 , F^3 , &c) by $r - 1$ or $F - 1$ for the 1st difference, and this multiplied by $F - 1$ again for the true 2d difference at the beginning of that class. Thus the 10th root of 1,0002302850, or E, gives 1,000023026116 for F or the 1st mean of the lowest class; therefore, $F - 1 = r - 1 = ,000023026116$ is its 1st difference, and the square of it is $(r - 1)^2 = ,0000000005302$ its 2d difference; then is $,000023026116 F^{10^n}$ or $,000023026116 E^n$ the 1st difference, and $,0000000005302 F^{20^n}$ or $,0000000005302 E^{2n}$, is the 2d difference at the beginning of the n th class of decades. And this 2d difference is used as the constant 2d difference through all the 10 terms, except towards the end of the table, where the differences increase fast enough to require a small correction of the 2d difference, and which Mr. Dodson effects by taking a mean 2d difference among all the 2d differences in this manner; having found the series of 1st differences $F - 1 \cdot E^n$, $F - 1 \cdot E^{n+1}$, $F - 1 \cdot E^{n+2}$ &c, take the differences of these, and $\frac{1}{10}$ of them will be the mean 2d differences to be used, namely, $\frac{F-1}{10} \cdot E^{n+1} - E^n$, $\frac{F-1}{10} \cdot E^{n+2} - E^{n+1}$, &c, are the mean 2d differences. And this is not only the more exact, but also the easier, way. The common 2d difference and the successive 1st differences are then continually added through the whole decade, to give the successive terms of the required progression.

JOANNIS KEPLERI
IMP. CÆS. FERDINANDI II. MATHEMATICI,
CHILIAS
LOGARITHMORUM

AD TOTIDEM NUMEROS ROTUNDOS;

PRÆMISSA

DEMONSTRATIONE LEGITIMA

ORTUS LOGARITHMORUM EORUMQUE USUS,

QUIBUS

NOVA TRADITUR ARITHMETICA, SEU COMPENDIUM,

Quo, post numerorum notitiam, nullum nec admirabilius, nec utilius, solvendi pleraque
Problemata Calculatoria, præsertim in Doctrina Triangulorum, citrà Multi-
plicationis, Divisionis, Radicúmque extractionis, in Numeris prolixis,
labores molestissimos.

AD

ILLUSTRISSIMUM PRINCIPEM ET DOMINUM,

DN. PHILIPPUM, LANDGRAVIUM HASSIÆ, &c.

CUM PRIVILEGIO AUTHORIS CÆSAREO.

Prima hujus tractatús editio impressa fuit Marpurgi, et excusa typis Casparis Chemlini, Anno
Domini MDCXXIV,

A D

ILLUSTRISSIMUM PRINCIPEM ET DOMINUM,
DN. PHILIPPUM, LANDGRAVIUM HASSIÆ,
COMITEM CHATTIMELIBOCCI, SIGENÆ, DITH, ET NIDDÆ,
DOMINUM MEUM CLEMENTISSIMUM.

De Munere ejus Celsit. amplissimo, deque usitata Cels. suæ Allusione, &
Alliteratione ad Nomen PHILIPS, BILLIEBS.

TRIGINTA expensis vendit φίλα πολλὰ Philippis
Princeps PHILIPPUS Hassiæ.

Sed facies rerum versa est; pauperculus emptor
Mercemque pretium retulit

Triginta acceptis, sed vendidit ille Magistrum
Mortalium immanissimus.

I nunc & dubita, Tharsensi interprete, num sit
Dare quàm accipere beatius.

Triginta tamen en! penso totidem ἀργυρὰ * verbis;
φίλα πολλὰ penso Chiliade.

Corde, manu, promissis redolentia munera fontes:
Mentemque redolent quæ accipis.

Corde φιλεῖς, manibus πωλεῖς, sed Mente θεωρεῖς;
Tibi cor, manus, mentem dico.

Illustr. Cels. T.

Subiectissimus Cultor Gratusque Hospes,

JOHAN. KEPLERUS.

* Propositionibus.

D E M O N S T R A T I O

S T R U C T U R Æ L O G A R I T H M O R U M.

P O S T U L A T U M I.

OMNES proportiones inter se æquales, quacunque varietate binorum unius, & binorum alterius terminorum, eadem quantitate metiri seu exprimere.

A X I O M A I.

Si fuerint quantitates quotcunque ejusdem generis, quocunque ordine sibi invicem succedentes, ut si ordine magnitudinis sibi invicem succedant: proportio extremarum composita esse intelligitur ex omnibus proportionibus intermediis binarum, & binarum inter se vicinarum.

Seu, quod eodem redit, proportio minuitur aucto minori termino, vel diminuto majori; augetur rationibus contrariis.

I. P R O P O S I T I O.

Medium proportionale inter duos terminos dividit proportionem terminorum in duas proportiones inter se æquales.

Nam si sunt duo termini, eorumque medium proportionale: est ergò inter tres quantitates Analogia seu Proportionalitas.

At Analogia definitur æqualitate τῶν λόγων, proportionum: quare proportionum sectione constitutæ, utpote partes proportionis totius propositæ, sunt inter se æquales.

A X I O M A S E U N O T I T I A C O M M U N I S II.

Si fuerint quantitates quotcunque crescentes ordine: proportio extremarum divisa est per intermedias in partes unâ plures quàm sunt intermediae, divisionem facientes.

Sic quatuor interstitia digitorum arguunt quinque digitos. Sic quinque corpora regularia sibi invicem inserta ordine interpositis orbibus inscriptis & circumscriptis, arguunt orbium talium sex esse.

POSTULATUM II.

Proportionem inter datos duos terminos quoscunque dividere in partes quotcunque (ut in partes numero continuè multiplici progressione binariæ) & eousque donec partes oriantur minores quantitate proposita.

Proportio enim est etiam una ex quantitatibus continuis in infinitum dividuis.

Vide hic typum divisæ proportionis inter 10 & 7, per triginta proportionales medias.

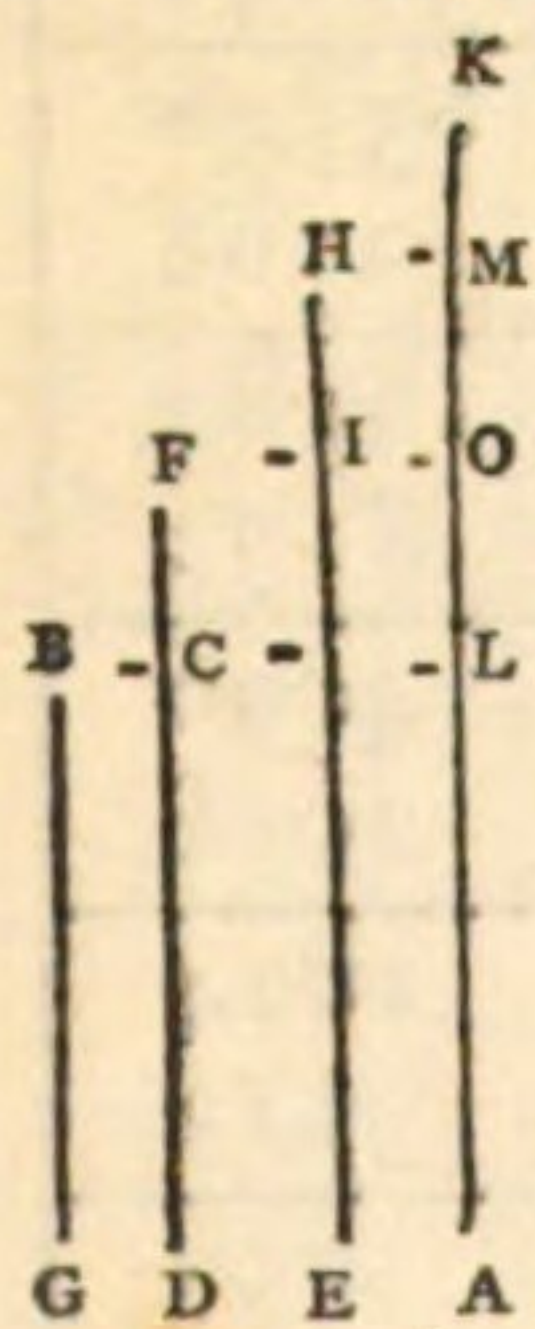
Hic Typus sic intelligatur: Inter Terminos, majorem 100, &c. & minorem 70 &c. quærat media proportionalis, hæc erit $83\frac{2}{3}$ &c. Sunt ergo proportionis inter dictos terminos constitutæ partes duæ, una inter 100 & $83\frac{2}{3}$, altera inter $83\frac{2}{3}$ & 70. Hæ per 1 prop. sunt inter se æquales. Quærat secundò, media proportionalis inter 100 & $83\frac{2}{3}$ &c. hæc erit $91\frac{1}{2}$ &c. rursumque erunt partes inter se æquales, una inter 100 &c. & $91\frac{1}{2}$ &c. & altera inter $91\frac{1}{2}$ &c. & $83\frac{2}{3}$ &c. Ita prior semiffis proportionis totius hîc est divisus in duas quartas partes ejusdem totius. Et intelligitur semiffis alter, qui erat inter $83\frac{2}{3}$ & 70, per fociam ipsius $91\frac{1}{2}$, seu secundarum trium minimam (quæ in hoc typo non exprimitur) similiter divisus esse in alias duas quartas totius. Quærat tertiò media proportionalis inter 100, &c. & $91\frac{1}{2}$ &c. hæc erit $95\frac{2}{3}$ &c. determinans cum 100 partem totius octavam, quod indicat numerus illi ad dextram exterius respondens. Et sic deinceps.

POSTULATUM III.

Minimum, proportionis elementum quantum pro minimo placuerit, metiri seu signare per quantitatem quamcunque; ut per excessum terminorum hujus Elementi.

II. PROPOSITIO.

Cùm fuerint tres continuè proportionales, quæ est proportio primæ ad secundam, vel secundæ ad tertiam, eadem est proportio differentiæ priorum, ad differentiam posteriorum.



Sint continuè proportionales AK, EH, DF, differentia priorum KM, posteriorum HI: Dico, ut est AK ad EH, sic esse KM ad HI. Est enim HE ad FD, ut KA ad HE ex hypothesi. Sed HE est æqualis ipsi MA, et FD æqualis ipsi IE, rursum ex hypothesi. Quare etiam MA erit ad IE, ut KA ad HE; quare per 17 quinti Eucl. etiam residua KM ad residuam HI, erit ut tota KA ad totam HE.

III. PROPOSITIO.

Cùm fuerint aliquot quantitates in proportionem continuâ, minimarum minima erit differentia, maximarum maxima.

Nam per 2 prop. sicut est maxima ad vicinam minorem, sic est differentia inter maximas, ad differentiam succedentium: Minor igitur est quælibet differentiarum succedentium, quàm antecedens. Minima igitur est ultima differentia, quæ scilicet est inter minimas.

IV. PRO-

IV. PROPOSITIO.

Cùm fuerint aliquot quantitates in proportione continuâ : si differentia maximarum statuitur mensura proportionis illarum : differentiaë quarumcunque duarum deinceps erunt minores mensurâ proportionis illarum iustâ.

Nam quia ponuntur continuè proportionales, igitur æqualis est proportio inter duas maximas, & proportio inter quascunque duas minores deinceps fitas per 1 Prop. Major verò est differentia inter maximas, differentiis inter quascunque alias deinceps fitas per 3. Si ergò major differentia statuitur mensura proportionis inter duas maximas per 3 postulatum : tunc eadem duarum maximarum differentia statuitur etiam mensura proportionis inter duas minores deinceps sequentes. At differentia inter duas minores, minor etiam est differentia inter maximas per 3 Prop. quare etiam minor est, quàm ut fuorum terminorum proportionem metiatur.

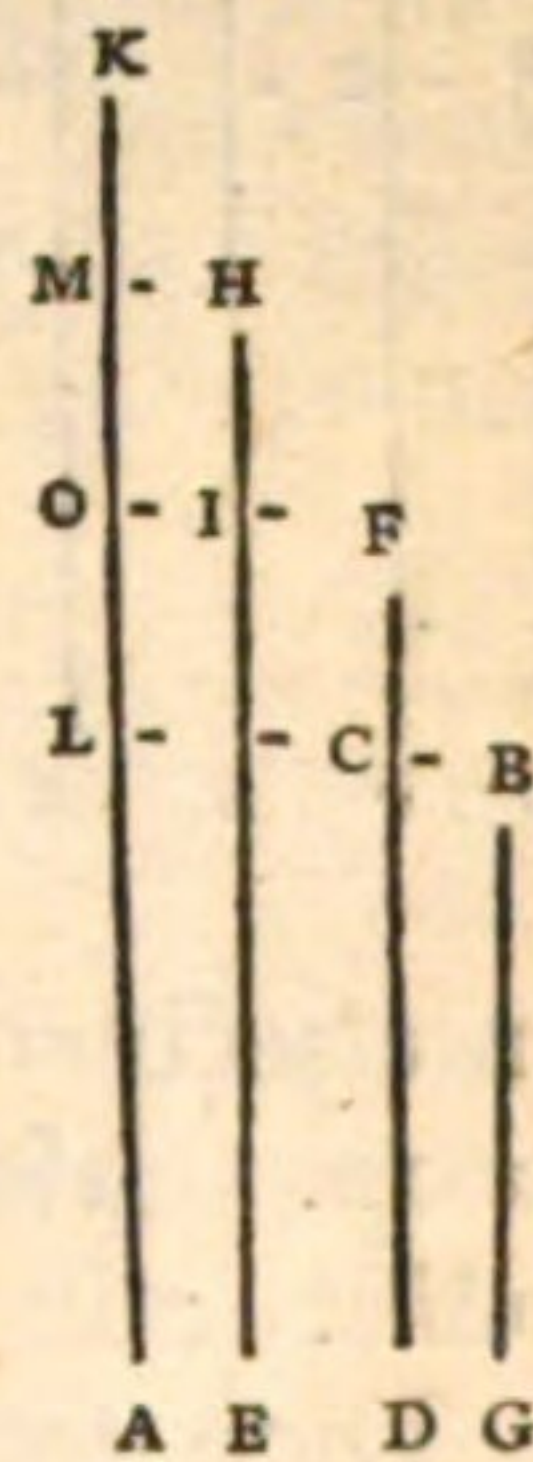
V. PROPOSITIO.

In continuè proportionalibus si differentia maximarum statuitur mensura proportionis illarum : omnes reliquæ proportiones, quæ sunt inter maximam & unamquamlibet reliquarum minorum, sortientur mensuras, majores differentiis fuorum terminorum.

Nam proportio maximæ AK ad minimam GB componitur ex proportionibus binarum, & binarum deinceps usque ad minimam, per Axioma 1. At omnes binarum deinceps fitarum proportiones inter se sunt æquales per 1 prop. mensuras igitur etiam æquales habent per postulat. 1. Quare quot sunt elementa proportionis inter maximam AK & minimam GB, toties differentia KM, maximarum KA, HE, vel MA, continetur in mensurâ proportionis maximæ KA ad minimam BG.

Jam verò maximæ KA & minimæ BG, differentia KL componitur ex differentiis KM, MO, OL binarum, & binarum deinceps fitarum omnibus.

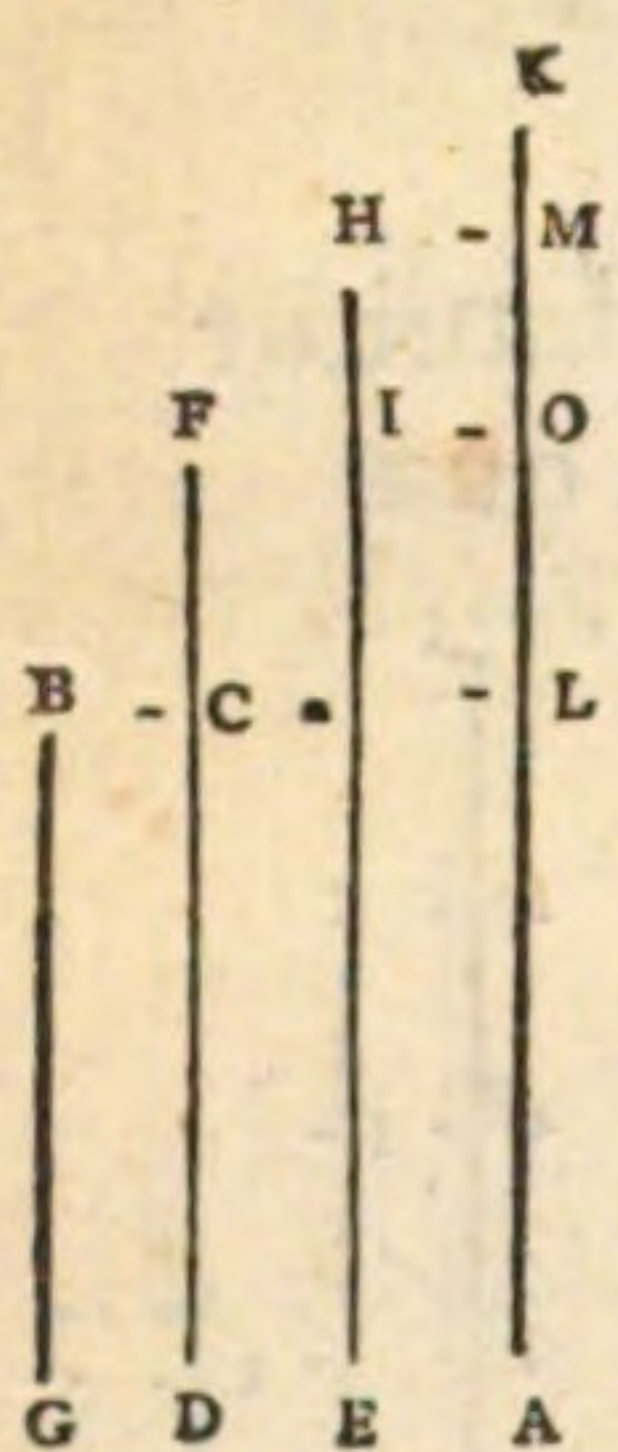
Sed quælibet differentia MO, OL, binarum deinceps seorsim minor est differentia KM maximarum per 3 Prop. quare etiam totidem junctæ differentiaë binarum deinceps, id est, differentia KL maximæ KA, & minimæ BG erit minor quàm multiplex ipsius KM differentiaë maximarum, secundum numerum elementorum proportionis sectæ. Sed multiplex ista est mensura proportionis inter maximam KA & minimam BG, ut jam ostensum. Ergò differentia KL maximæ & minimæ non æquat mensuram proportionis earum, positis quæ sunt posita.



In numeris sic. Sint numeri 1000, 900, 810, 729, continuè proportionales. Maximi sunt 1000, 900. Eorum differentia est 100. Sit hæc mensura arbitraria proportionis 1000, 900. Erit igitur etiam mensura hæc proportionis 900, 810, & proportionis 810, 729. Composita igitur proportionis inter 1000 & 729 mensura erit 300, quia elementa æqualia proportionis tria sunt, per duos medios proportionales per axioma 2. Atqui terminorum hujus proportionis, scilicet 1000 & 729, differentia est 271, multò minor quàm 300, triplum ipsius 100.

VI. PROPOSITIO.

In continuè proportionalibus, si differentia maximæ & unius minorum non deinceps sequentis statuitur mensura proportionis illarum: reliquæ proportiones, quæ quidem sunt inter maximam & unam prius ascitâ majorem, fortientur mensuram minorem differentiâ suorum terminorum: quæ verò proportiones sunt inter quantitatem maximam & unamquamlibet, prius ascitâ minorem, nanciscentur mensuram majorem, quàm est differentia suorum terminorum.



Sint proportionales AK , EH , DF , GB , & assumantur maxima quidem AK , & una minorum non deinceps DF , sitque illarum differentia KO , & KO mensuret quantitatem proportionis inter AK & DF . Et sint aliæ quantitates, EH quidem major prius ascitâ DF ; GB verò illâ minor: & sit MK differentia ipsarum AK , EH ; sic HI vel MO sit differentia ipsarum EH , DF ; denique, FC sit differentia ipsarum DF , GB . Dico primò mensuram proportionis AK , EH fore minorem differentiâ terminorum MK .

Nam quia proportio AK , DF , mensuram accipit KO , eadem verò proportio habet partes æquales constitutas per mediam proportionalem EH , per 1 prop. Proportionis igitur AK , EH mensura erit dimidia ipsius KO per 1 postul. (vel pars alia aliquota, secundum numerum interjectarum EH .) At differentia MK major est quàm MO residua de KO per 3 prop. Major igitur quàm dimidium ipsius KO . Ergò & major, quàm ut possit esse mensura proportionis inter KA & HE majorem prius ascitâ FD .

Dico secundò mensuram proportionis AK , GB fore majorem differentiâ LK . Rursum enim proportionis EH , DF , quæ semissis est ipsius AK , DF , mensura erit semissis ipsius KO per 1 postul. vel pars alia aliquota, &c. At differentia HI vel MO minor est quàm MK , residua de KO per 3 prop. Minor igitur HI , quàm dimidium ipsius KO . Ergò minor quàm mensura proportionis EH , DF . Sed differentia FC vel OL rursum minor est, quàm differentia MO per 3 prop. Plus igitur deficit OL mensura proportionis DF , GB , quàm MO , à mensura proportionis EH , DF . Est verò proportio DF , GB , æqualis proportioni EH , DF , quia hac quidem vice ipsi DF æquè propinqua est GB versus minora, atque EH versus majora, & si hæc non esset, alia sumi possit, quippe inter continuè proportionales. Deficit igitur OL à mensurâ proportionis DF , GB : sed KO statuitur esse ipsa mensura proportionis AK , DF .

Cùm

Cùm ergò proportio AK, GB, composita fit ex proportione AK, DF, & proportione DF, GB, per ax. 1. Mensura etiam illius composita erit ex mensuris harum; ut si proportio DF, GB semissis est ipsius proportionis AK, DF, & proportio AK, GB sesquialtera ipsius AK, DF; erit etiam ipsius KO susceptæ mensuræ, sesquialtera pro ipsius AK, GB proportionis mensura habenda.

Similiter verò & KL componitur ex KO, OL; OL verò demonstrata est esse minor dimidiâ ipsius KO: Tota igitur KL minor est quàm sesquialtera ipsius KO, minor igitur quàm mensura proportionis suæ inter AK, GB.

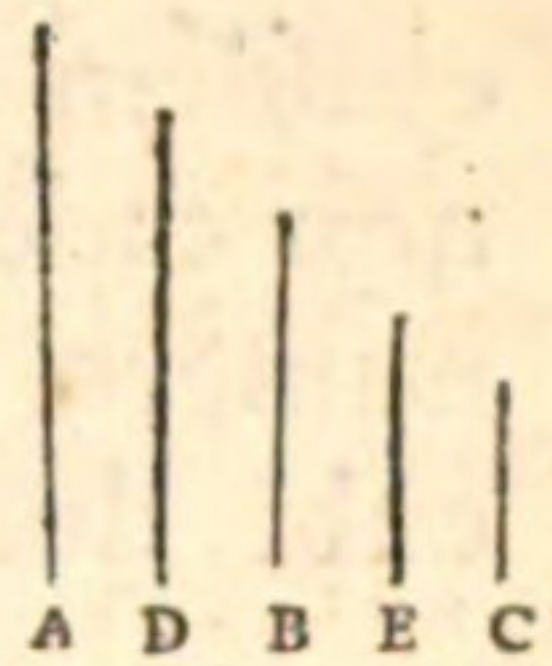
Per numeros sic: Sint continuè proportionales 100000, 90000, 81000, 72900, 65610, 59049. Eligantur 100000, 72900, earumque differentia 27100 statuatur mensura proportionis terminorum. Cùm igitur hæc proportio habeat partes tres æquales, quarum una 100000, 90000, cum differentiâ 10000: altera 90000, 81000, cum differentia 9000: tertia 81000, 72900, cum differentia 8100. Quælibet vero proportionum harum sit pars tertia suæ totius, valebit etiam tertiam partem mensuræ illius, scilicet $9033\frac{1}{3}$, hæc vero mensura Elementorum proportionis minor quidem est, quàm primorum terminorum differentia 10000, major vero quam secunda differentia 9000, major etiam multo quam tertia 8100. Consequenter etiam major quam differentia 7290, numerorum 72900, 65610: quam & differentia 6561 numerorum 65610, 59049. Ac proinde proportionis 100000, 9000 terminorum differentia 10000, excedit mensuram susceptam $9033\frac{1}{3}$. Nec minus etiam proportionis 100000, 81000 differentia 19000 excedit mensuram suam, scilicet duplum ipsius $9033\frac{1}{3}$, scilicet $18066\frac{2}{3}$, quia termini minores 90000, & 81000 adhuc stant ante primò adjunctum 72900. E contrario proportionis 100000, 65610 differentia 34390 minor est quam illius mensura $36133\frac{1}{3}$, quadruplum scilicet ipsius $9033\frac{1}{3}$. Sic etiam proportionis 100000, 59049 differentia 40951 minor est quintuplo ipsius $9033\frac{1}{3}$, scilicet $45166\frac{2}{3}$, quia termini minores 65610 & 59049 stant post primo adjunctum 72900 utpote minores illo.

Proportionales.	Differentia.
100000	-10000
90000	-9000
81000	-8100
72900	-7290
65610	-6561
59049	

VII. PROPOSITIO.

Si quantitates aliquot ordine magnitudinis deinceps collocentur, binæ deinceps proportionales æquales facientes, ipsæ quantitates continuè proportionales erunt.

Collocentur deinceps A, D, B, E, C, hoc ordine minores à primâ, sintque AB & BC proportionales æquales, dico B esse mediam proportionalem inter A, C. Si enim non: erit vel major vel minor mediâ proportionali; sit major illâ, & sit verbi causa E ipsa media proportionalis. Erit igitur AE proportio major quam AB, per 1 ax. quare EC minor quam AE. At si E media proportionalis, tunc AE & EC proportionales sunt æquales per 1 prop. Non igitur minor erit media proportionalis quam B.



Sit major, & sit D, rursus idem absurdum sequetur.

Si ergo AB, BC æquales proportionales, ipsa B est media proportionis, &c.

C

VIII. PRO-

VIII. PROPOSITIO.

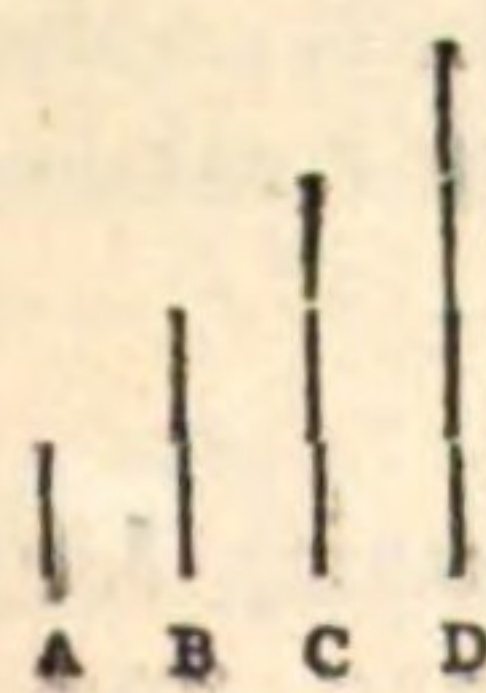
Si quantitates quæcunque deinceps collocentur, ordine magnitudinis, quarum, quæ intermediæ, non sint inter proportionales, medias, proportionis cujuscunque, five actu continuatæ, five potestate continuandæ interpositione omiffarum: intermediæ tales proportionem extremarum non dividunt in commensurabilia.

Definitiones.

1. Commensurabilia enim ex eo dicuntur, quod habeant unam communem mensuram, quam quodlibet contineat secundum certum numerum aliquoties exactè, sic ut nihil, quod eâ mensurâ minus sit, restet residuum. 2. Jam vero mensura proportionum communis, est & ipsa aliqua proportio, minor utrâque mensurandâ. 3. Omnis vero proportio est inter duos terminos. 4. Et Proportio repetitione sui, mensurans aliam proportionem, incipit ab uno mensurandæ termino, eique sociat alium, pro ratione quantitatis suæ minoris: tum illo jam pro antecedenti sumpto, statuit alium consequentem, hoc identidem, quoadusque permeatur proportionis mensurandæ quantitas: non aliter, quam cum intervallo pedum Circini metimur lineam, fixo pede Circini in unâ lineæ extremitate, pede altero punctum signamus, deinde pede priore in hoc punctum translato, punctum aliud altero pede metamur, versus ulteriora, donec emensi fuerimus totam lineam. 5. Et Proportio proportionem exactè mensurare dicitur, quando in hâc continuâ terminorum interpositione & coaptatione tandem ultimus terminus proportionis mensurantis, cum secundo termino mensuratæ coincidit in quantitate. Igitur identitas illa proportionis mensurandis continuè repetitæ efficit, terminos continuè proportionales per 7 prop. Ergo si proportio aliqua duas proportiones exactè metitur, necesse est, ut termini quos ipsa mensurans interponit, sint cum ipsius mensurandæ terminis continuè proportionales. Si ergò nulla unquam, quantumvis parva proportio potest inveniri, quæ repetitione sui, terminos ultimos assequatur proportionum mensurandarum, sic ut tam major communis terminus, quàm duo minores proportionum mensurandarum sint cum mensurantis terminis interpositis continuè proportionales: proportionem illæ sunt inter se incommensurabiles.

IX. PROPOSITIO.

Cum duæ longitudines effabiles non fuerint ad invicem, ut duo numeri ejusdem speciei figurativæ, verbi causâ, duo quadrati, aut duo cubi: non cadent inter illas, longitudines aliæ effabiles, mediæ proportionales, numero tot quot ipsa species postulat, verbi causâ, quadrati unam, cubi duas, biquadrati tres, &c.



Sint enim duæ longitudines AD habentes, se quidem ad invicem, ut numerus ad numerum, at non ut numerus cubicus ad cubicum: & quia de cubo agimus, de duabus igitur mediis proportionalibus erit dicendum, sint eæ B & C. Dico B & C non esse longitudines effabiles.

Si enim quis contendat esse effabiles, esto hoc positum. Sunt igitur ut Numeri. Sunt autem simul mediæ proportionales inter A, D , ex hypothesi. Et quia etiam A, D sunt ut numeri, quippe effabiles, & ipsæ supponuntur, habent verò duas medias B & C , ut numeros, quare per 21 octavi Eucl. A & D similes erunt solidi: quare per 27 ejusdem erunt ad invicem, ut Numerus ad numerum cubicum. Hoc verò est contra primam propositionis hypothesin. Falsum igitur positum fuit, B & C esse longitudine effabiles. Vera igitur est negatio in propositione comprehensa.

Eodem modo etiam de quadratis, & de unâ mediâ proportionali ratiocinari possumus, deductione ad impossibile: nec minus & de cæteris speciebus, post quadratum & cubum sequentibus.

X. PROPOSITIO.

Si ex aliquot quantitatibus effabilibus ordine magnitudinis invicem sequentibus duæ extremæ non fuerint ad invicem, ut duo numeri quadrati, aut duo cubi, aut duo alii ejusdem speciei; intermediarum nulla dividet proportionem in commensurabilia.

Nam nisi duæ quantitates effabiles recipiant media proportionalia effabilia, earum proportio non dividetur per effabilem intermediam in commensurabilia per 8 prop. At si duæ quantitates non fuerint inter se, ut duo numeri ejusdem speciei figurativæ, non recipiunt media proportionalia effabilia, per 9 prop. Quare illæ intermediae quas propositio admittit, cum sint effabiles, non erunt ex proportionalibus mediis. Non igitur dividunt proportionem extremarum in commensurabilia.

XI. PROPOSITIO.

Omnes proportionales deinceps ordinatæ, quæ sunt inter terminos effabiles æqualitate Arithmetica se invicem excedentes, inter se sunt incommensurabiles.

Nam termini extremi effabiles vel recipiunt effabilem mediam proportionalem quantitatem unam pluresve, vel non recipiunt. Si non recipiunt, nulla igitur effabili, & sic neque medio arithmetico dividitur eorum proportio in commensurabilia per 8 prop. Recipiunt verò effabile medium proportionale, ut termini 8 & 18, recipiunt enim 12 effabilem, cum sint ut 4 ad 9, quadratus ad quadratum. Est verò inter 8, 18, medium Arithmeticum 13, ideo proportio 8, 13 major est quàm 8, 12 & 13, 18 minor, quàm 12, 18, quantitate utrinque parvæ proportionis inter terminos 12, 13, sed proportio 12, 13 nulli reliquarum est commensurabilis. Nam termini 8, 13, quia non sunt ad invicem, ut numerus figuratus ad alium ejusdem figurationis per 10 prop. non

capiunt mediam vel medias proportionales effabiles, quare numerus 12 non est unus ex iis numeris, qui inter 8 & 13 intercidunt in continua proportionem: non est igitur commensurabilis proportio 8, 13 proportioni 12, 13 vel 8, 12. Sic ex iisdem fundamentis, quia termini 12, 18 non capiunt effabilem mediam proportionalem: ergo 12, 13 & 13, 18 sunt incommensurabiles; ad commensurabilem igitur 8, 12 ipsi 8, 18 est apposita incommensurabilis 12, 13. Tota ergo 8, 13 est ipsi 8, 18 incommensurabilis. Sic in 13, 18 de commensurabili 12, 18 dempta est pars incommensurabilis 12, 13, residua ergo 13, 18 est incommensurabilis ipsi 12, 18. At neque proportio inter 8, 13 est commensurabilis ipsi proportioni inter 13, 18, quia tota inter 8, 18 alterutri parti inter 8, 13 est incommensurabilis, ergo & partes invicem per 16 decimi Eucl. Item si essent partes commensurabiles, mensuram utraque communem haberet, atque illa etiam compositam inter 8, 18 emetiretur, & sic partes cum tota commensurabiles essent, atqui jam demonstratum est, neutram partem toti esse commensurabilem. Non sunt igitur partes, quas hîc facit medium arithmeticum inter se commensurabiles.

XII. PROPOSITIO.

Si quantitates quæcunque deinceps collocentur ordine magnitudinis, proportionis verò inter maximas mensura statuatur differentia inter eas, differentia inter quascunque alias, ex positis, minor erit mensurâ suæ proportionis; & si proportionis inter minimas mensura statuatur differentia minimarum: differentiæ reliquæ erunt majores mensurâ proportionis suorum terminorum.

Aut enim continuè proportionales sunt quantitates collocatæ actu, vel potestate supplendi omittas, & tunc patet propositum per vi & per iii. Aut non sunt in proportionem continua, sic ut partes constituent incommensurabiles: & tum conceptione mentis in infinitas particulas æquales secari intelligeretur per interpositas infinitas medias proportionales: ita rediguntur cum iis, quæ actu sunt continuè proportionales ad eandem vim demonstrationis.

COROLLARIUM.

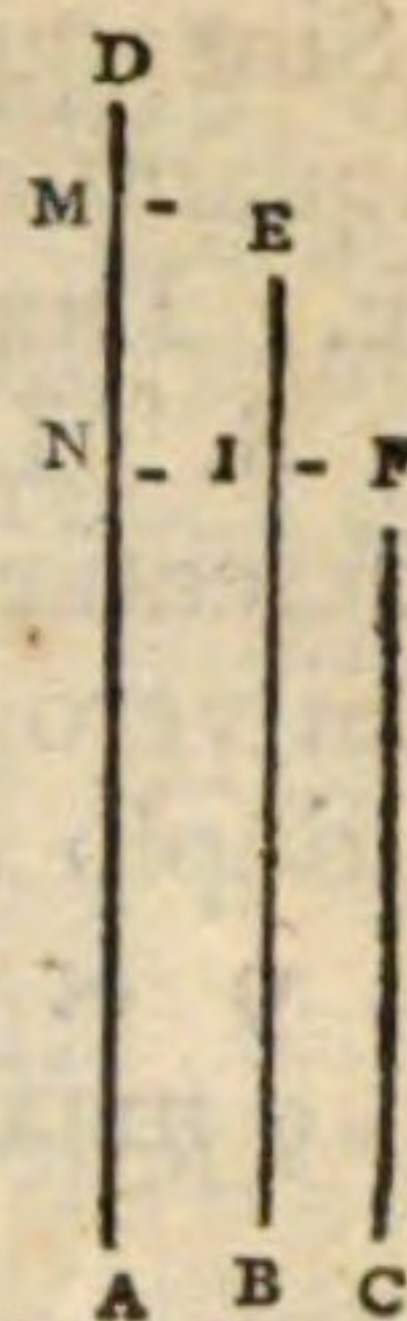
Quod si superet mensura proportionis inter maximas, differentiam earum: hujus mensuræ ad hanc differentiam proportio minor erit quàm sequentis mensuræ ad differentiam suam: cum proportionalium eadem sit ratio.

XIII. PROPOSITIO.

Si quantitates tres ordine magnitudinis se insequantur, proportio minimarum duarum in proportionem extremarum continebitur rarius, quàm differentia minimarum in differentiâ extremarum: & vicissim proportio maximarum in proportionem extremarum continebitur sæpius, quàm differentia illarum in differentiâ harum.

Con-

Contineatur enim in adjecto diagrammate differentia minimarum EI, vel MN, in differentia DN extremarum aliquoties licet non exactè : & capiat proportio minimarum BE, CF mensuram differentiam MN, erit igitur proportionis AD, CF mensura minor quàm differentia DN per 12; rarius igitur continebitur MN, in mensura ipsius AD, CF proportionis, quàm in ND longiore; rarius igitur & ipsa proportio in proportionem. Vicissim contineatur DM, differentia in DN aliquoties : & fit DM mensura proportionis AD, BE. Erit proportionis AD, CF mensura major quàm DN, sæpius igitur erit DM, in mensura ipsius AD, CF, quàm in DN. Ergo, &c.



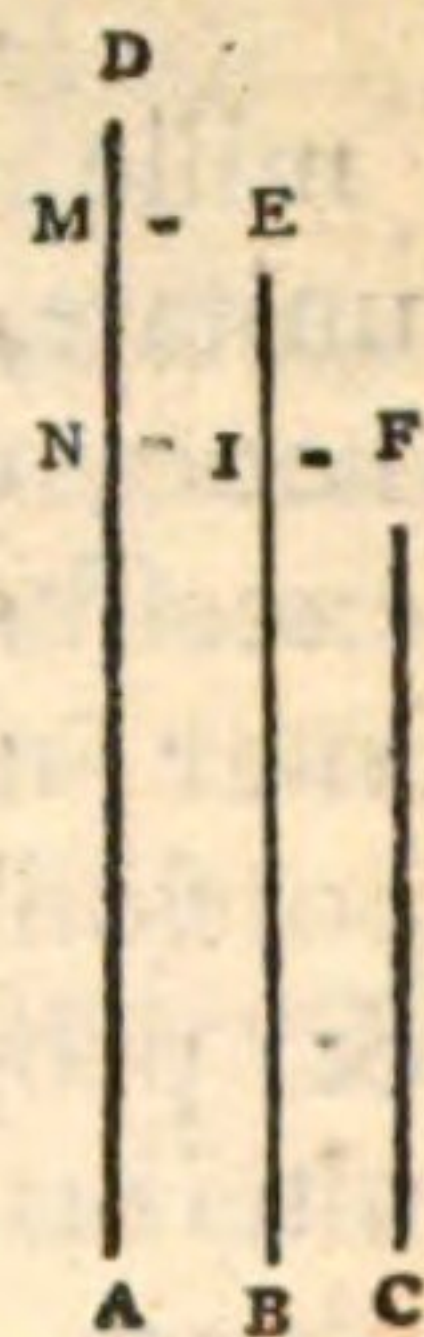
COROLLARIUM.

Differentia eadem inter terminos binos hinc majores, inde minores existente : proportio inter majores minor erit, inter minores major.

XIV. PROPOSITIO.

Si quantitates tres ordinentur deinceps, æqualibus differentiis invicem excedentes : proportio inter extremas est major quàm dupla proportionis maximarum.

Sint tres quantitates, AD maxima, BE media, CF minima, & fit excessus primæ super secundam DM, æqualis excessui secundæ super tertiam EI, vel MN. Dico proportionem inter AD, CF, majorem esse quàm duplam ipsius inter AD ad BE. Mensuretur enim proportio AD, BE per differentiam DM per 3 postul. Erit igitur mensura proportionis BE, CF major quàm differentia IE, vel NM per 12. Sed MN æquat DM mensuram proportionis inter AD, BE. Ergo mensura proportionis BE, CF est major mensura proportionis AD, BE; Et sic ipsa proportio BE, CF major est proportionem AD, BE. Sed proportio AD, CF componitur ex proportionem AD, BE & ex proportionem BE, CF per axioma 1. Ergo proportio AD, CF partes habet AD, BE, & eam majorem BE, CF, major igitur dupla ipsius AD, BE.



COROLLARIUM.

Hinc sequitur, femissem proportionis extremarum esse majorem proportionem maximarum, minorem proportionem minimarum.

XV. PROPOSITIO.

Si duæ quantitates proportionem constituerint, dimidium verò quantitatis majoris dematur de quantitate utraque, residuæ quantitates proportionem constituent majorem duplâ prioris.

Sint

Sint quantitates 10, 9 & ablato dimidio ipsius 10, hoc est 5, ex utraque relinquantur 5, 4. Dico proportionem inter 5, 4 dupla ipsius 10, 9 majorem esse. Duplicentur enim 5, 4 fient 10, 8, eritque proportio eadem 5, 4 quæ & 10, 8. Differentia vero 10, 8, hoc est 2, dupla erit differentiæ 5, 4, hoc est differentiæ 10, 9, scilicet 1.

Si vero tres quantitates ordinentur 10, 9, 8, quarum prima 10 excedat tertiam 8, duplo ejus, quod excedit secundam 9, seu in quibus æquales sunt excessus 10, 9 & 9, 8, proportio extremarum 10, 8 (id est 5, 4) maior est dupla ipsius 10, 9 maximarum per 14. Ergo.

XVI. PROPOSITIO.

Incommensurabilium proportionum partes aliquotæ sunt inter se incommensurabiles.

Nam pars aliquota est sui toti commensurabilis, at tota illa perhibetur toti sociæ incommensurabilis, ergo & pars unius toti alteri erit incommensurabilis per 14 decimi Eucl. & pro eadem, & parti aliquotæ alterius.

XVII. PROPOSITIO.

Si mille numeri invicem succedant ordine naturali, differentes bini unitate, initio facto à maximo 1000; deinde proportio inter maximos 1000, 999, bisectione continuâ, secetur in partes minutiores, quàm est excessus proportionis inter proximos 999, 998, super proportionem inter maximos 1000, 999: minimum verò illud elementum proportionis inter 1000, 999, capiat mensuram differentiam inter 1000 & proportionalem illam mediam, quæ alter elementi terminus est: ulterius si proportio inter 1000, 998 seorsim secetur in partes duplo plures quàm prior proportio inter 1000, 999, & hujus separatæ divisionis minimum elementum seorsim capiat mensuram, suorum terminorum (quorum alter sit 1000) differentiam, eodemque modo quælibet proportio ipsius 1000 ad sequentes numeros, ut 997, &c. bisectione continuâ secetur in particulas tantæ magnitudinis, ut versentur inter sesquiplum & dodrantem elementi, quod emerferat ex sectione proportionis primæ inter 1000 & 999, singulisque elementis mensura detur à suorum terminorum differentiâ, maximo existente 1000, & si hoc facto, cuicunque ex mille proportionibus, mensura constituatur ex tanto numero mensurarum elementi sui, in quot elementa ipsa divisa fuit: proportionibus omnes, ad omnem calculi subtilitatem, emendatas exactasque habebunt mensuras.

Nam

Nam succedant invicem numeri 1000, 999, 998, &c. ordine naturali, differentes unitate: erit inter maximas 1000, 999, minima proportio: major inter proximos 999, 998: hæc iterum major proximâ inter 998 & 997 sic semper & hoc per 14 prop. Donec 500, 499 fiat major quàm dupla ipsius 1000, 999 per prop. 15. Dico secundò excessum secundæ super primam fore minimum: sic ut semper excessus sequentis super præcedentem sit major priore excessu: ut quoties sequens proportio duplo longius distiterit à prima, quàm aliqua præcedentium, toties excessus sequentis super primam amplius quàm duplo major sit excessu præcedentis. Fiat enim ut 1000 ad 999, sic 999 ad $998\frac{1}{1000}$. Igitur proportio 999 ad $998\frac{1}{1000}$ est eadem quæ 1000 ad 999. Aufer illam à proportionem 999, 998, relinquitur excessus 998001, 998000.

Fiat etiam ut 1000 ad 999, sic 998 ad $997\frac{2}{1000}$. Igitur proportio 998 ad $997\frac{2}{1000}$ est eadem quæ 1000 ad 999. Aufer illam à proportionem 998, 997, relinquitur excessus, proportio inter 997002 & 997000. At in priori excessu proportionis 998001, 998000, termini differebant per 1, in hoc verò excessu termini 997002, 997000 differunt per 2. Atqui si æquales fuissent majores termini hîc & illîc, proportio sequens, ubi dupla differentia prioris, fuisset major dupla ipsius per 14. Multò igitur major erit proportio, ubi etiam minor fuerit terminus, qui proportioni sequenti est loco majoris scilicet 997002.

Lucis causa sint numeri minores & pauciores, & qui etiam unitate differant, ut 10, 9, 8, 7, 6, 5, 4, 3, 2, 1. Dico excessum proportionis 8, 7 super proport. 10, 9, amplius quàm duplo majorem esse quàm excessum proportionis 9, 8 super eandem 10, 9. Reducatur enim ut prius prima proportio inter 10, 9 cum singulis sequentium, quas bini deinceps numeri constituunt, reducantur, inquam, ad communem terminum maximum. Hic etiam excessuum termini differunt magis, magisque, ut quia 72, 70 est loco secundo, differentia 2 (ut prioris 1, inter 81, 80, dupla) indicat excessum 72, 70 esse majorem duplo ipsius 81, 80. Sic 54, 50 loco quarto major duplo est ipsius 72, 70 loco secundo. Sic 18, 10 loco octavo major duplo ipsius 54, 50, indice differentia 8, dupla ipsius 4, inter terminos minores.

	Est	At propor- tio inter 10, 9 valet	Excessus il- larum.
10, 9	100, 90	100, 90	
9, 8	90, 80	90, 81	81, 80
8, 7	80, 70	80, 72	72, 70
7, 6	70, 60	70, 63	63, 60
6, 5	60, 50	60, 54	54, 50
5, 4	50, 40	50, 45	45, 40
4, 3	40, 30	40, 36	36, 30
3, 2	30, 20	30, 27	27, 20
2, 1	20, 10	20, 18	18, 10

Cùm

Cùm igitur in priori exemplo mille numerorum, minimus excessus sit 998001, 998000, facilè secamus proportionem primam 1000, 999 in partes æquales, minores, posito excessu numero. Reducatur enim ille excessus ad terminum minorem dividendæ, fiat scilicet ut 998000 ad 998001, sic 999000000000 ad 999001002004, &c. unde apparet differentiam terminorum fieri 1002004, &c. quæ est paulò major quàm millesima pars de differentia terminorum proportionis primæ. Ergò per 13 non totupla pars proportionis 1000, 999 est in excessu proportionis sequentis 999, 998.

Quæraturn ergò medium proportionale inter 1000, 999. Id secabit proportionem terminorum in æquales duas partes, per 1 prop. quæraturn secundò inter hanc inventam mediam, & 1000 alia media proportionalis, ut ita prior media intelligatur, circumdata duabus mediis proportionalibus aliis, quarum tamen unius solum, versus terminum 1000 investigatione opus est. Per has igitur tres medias (ut est in ax. 2.) secabiturn proportio in partes 4. Sic proportionalis tertia inter prius inventam, & 1000 secabit in partes 8 quarta in 16, & sic consequenter (per postul. 2) in 32, 64, 128, 256, 512, 1024, quod fit actu decimo. Hic itaque numerus 1024 certò constituit particulas proportionis 1000, 999 minores supra investigato excessu: quia proportio capit hujus excessus minus quàm mille; hic verò constituuntur mille viginti quatuor particule. Hic igitur particula millesima vicesima quarta capiat loco mensuræ, differentiam termini sui minoris, seu mediæ proportionalis decimæ à termino majore 1000 per 3 postulat.

Pergimus ad similem sectionem proportionis inter 1000, 998. Hæc igitur est incommensurabilis priori 1000, 999. Nam per 11 prop. proportio 999, 998 est incommensurabilis proportioni 1000, 999, qui termini bini differunt unitate æqualiter, sed proportio 1000, 998 componitur ex illis inter se incommensurabilibus per axiom. 1. Cum verò totum in incommensurabilia secatur, ipsum singulis est incommensurabile per 17 decimi Eucl.

Est verò hæc proportio 1000, 998 major quàm dupla prioris 1000, 999 excessu incommensurabili, ut supra allegatione præmissæ 14 indicatum. Pars igitur ejus millesima vigesima quarta plus quàm duplo major est minimo elemento prioris; bisecetur igitur, ut totius fiant partes 2048 per inquisitionem undecimæ proportionalis. Tunc fanè elementum hoc ejus erit proximè æquale elemento minimo prioris, majus tamen illo etiamnum, & illi incommensurabile per 16 præmissam.

Si ergò illud accepit mensuram, differentiam suorum terminorum, hoc jam elementum proportionis, quippe majus illo, mensuram habebit majorem differentiâ suorum terminorum, per 12 prop. Ac proinde si prioris elementi terminorum differentia multiplicetur 1024ies pro mensurâ proportionis inter 1000, 999 tunc posterioris elementi terminorum differentia, multiplicata 2048ies, adhuc minor erit mensurâ proportionis inter 1000, 998.

Verumtamen si attendamus ad quantitatem hujus defectus, illa est omninò subtilissima, & nullâ calculi diligentia observabilis. Quia enim proportio inter 1000 & 999 secta est in particulas plus quàm mille, & elementi tam parvi mensura constituta est differentia terminorum 1000000, & 999999, scilicet 1, & minor adhuc certè proportio proxima inter 999999, 999998 major priori per 12 prop. mensuram habebit, quæ excedat terminorum differentiam primam
vix

vix millies millesimâ fui; ac proinde composita proportio 1000000, 999998 major est dupla prioris vix millies millesimâ prioris particulâ. At jam elementum secundæ proportionis inter 1000, 998 in superioribus constitutum, nequaquam est majus duplo prioris proportionis elemento; sed ob idipsum factæ sunt partes non 1024, sed 2048, ut esset pars ista proximè æqualis priori, quomodo excedit illam rursus vix millesima parte illius, si ergò totum superadderet differentiæ suæ millies millesimam, ad constituendam proportionis mensuram, pars utique millesima totius non plus addet, quàm millies millesimæ prioris elementi millesimam.

Transeamus ad sectionem sequentis proportionis inter 1000, 997. Hæc verò est paulò major quàm tripla primæ inter 1000, 999. Quare si secetur in æquales 1024 particulas, particula talis etiam erit paulò major quàm tripla elementi primi. Sin ulterius illæ bisecentur per undecimam proportionalem, existent numero 2048, eritque unaquælibet major quàm sesquialtera primi elementi. Nam triplæ dimidia est ipsa sesquialtera totius. Quia ergò superant primum elementum plus quàm dimidio ipsius, bisecentur denuò constitutione duodecimæ proportionalis, ut fiant partes 4096: quælibet major dodrante prioris elementi, quia sesquialteri dimidium est dodrans. Satis igitur appropinquat elemento priori. Si igitur huic elemento proportionis tertiæ mensura seorsim ponatur, differentia terminorum ipsius peccabit quidem illa excessu, per 12 præmissam, at peccato nunquam æstimabili, ob causas in secundæ sectione dictas.

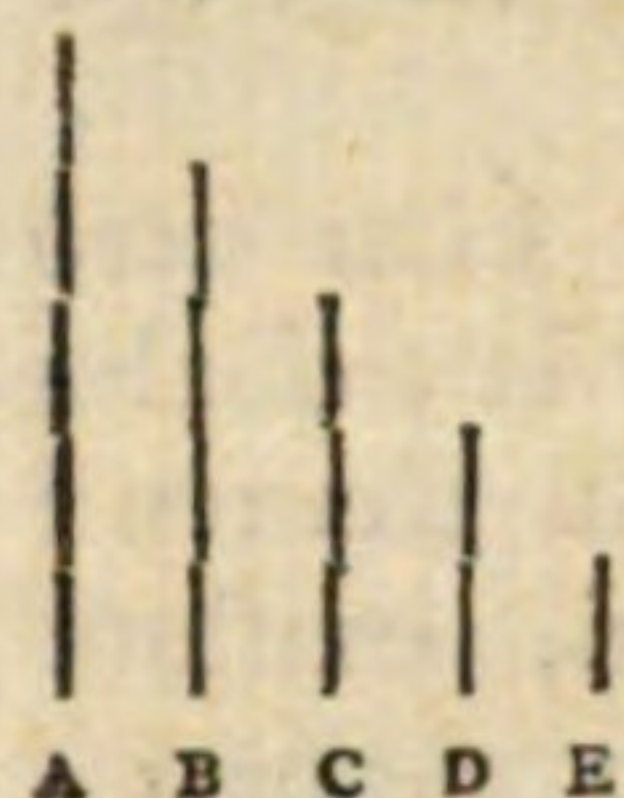
Hoc pacto, proportio inter 1000, 996, per sectionem in 4096, paulò majus fortitur elementum quàm est elementum primæ, & proportio inter 1000, 995 paulò majus quinque quantis primi elementi. At proportio inter 1000, 994 jam per tredecimam proportionalem, capere debet partes 8192 ut sint rursus paulò majores dodrante primi elementi. Et proportio inter 1000, 993 paul majores habebit septem octavis primi, & proportio inter 1000, 992 rursus paulò majores primo elemento.

Hoc igitur tantisper obtinet, quoad numeri ordinis millenarii proximi invicem appropinquaverint magis ipsi 500 dimidio primi, quàm ipsi primo 1000.

Nam quia proportio inter 500, 499 major est quàm dupla proportionis inter 1000, 999 per 15 præmissam: Jam igitur excessus ipsius 500, 499 super ipsam 1000, 999, superat ipsam 1000, 999, quare in sectione proportionum, quæ proximè antecedunt proportionem inter 500 & 499 per unam superabundantem, bisectionem, vel etiam quadrisectionem, redigendum est elementum, emergens ad propinquitatem elementi prioris in propositione præfinitam. At in iis quæ sequuntur proportionem inter 501 & 500 sectione porro non est opus. Nam quia proportio inter 500, 499 eadem est quæ inter 1000, 998, idcirco si fuerit notificata 1000, 998 & 1000, 500, noscetur etiam composita ex utraque 1000, 499 sine sectione laboriosa. Igitur inquisitio proportionalium definit in proportionem dupla, scilicet inter 1000 & 500.

XVIII. PROPOSITIO.

Cognitâ proportionē numeri cujuscunque ad primum 1000: simul cognoscitur etiam numerorum reliquorum continuæ ejusdem proportionis, ad eundem primum 1000 proportio.



Nota sit mensura proportionis inter A & B. Et sit ut A ad B, sic B ad C, & C ad D, & D ad E. Erunt igitur æquales mensuræ proportionum harum singularum ei, quæ est primò notæ A ad B per 1 postul. Jam verò proportio A ad C componitur ex proportionibus A ad B, & B ad C, per 1 ax. quare & mensura proportionis AC componetur ex duarum proportionum AB & BC mensuris. Id est, mensura ipsius AB duplicata, dat mensuram ipsius AC, triplicata ipsius AD, quadruplicata ipsius AE.

Hoc pacto cognitâ proportionē inter 1000, 900, cognoscitur etiam ipsius 1000 ad 810, & ad 729.

Et ex 1000 ad 800, etiam 1000 ad 640 & ad 512.

Et ex 1000 ad 700, etiam 1000 ad 490 & ad 343.

Et ex 1000 ad 600, etiam 1000 ad 360 & ad 216.

Et ex 1000 ad 500, etiam 1000 ad 250 & ad 125.

COROLLARIUM.

Hinc oritur præceptum quadrandi, cubicè multiplicandi, &c. & vicissim radicem quadratam, cubicam, &c. extrahendi in primis numerorum figuris. Est enim ut maximus chiliadis tanquam denominator ad numerum propositum tanquam numeratorem, sic hic ad fractionis quadratum, & hoc ad cubum.

XIX. PROPOSITIO.

Cognitâ proportionē numeri ad primum 1000; si duo alii in eâdem inter se proportionē fuerint; eorum unius proportionē ad 1000 cognitâ, noscetur etiam reliqui proportio ad eundem 1000.

Sit A 1000, & nota mensura proportionis A ad B. Sit verò ut A ad B, sic C ad D, & sit nota mensura proportionis A ad C. Dico etiam innotescere mensuram proportionis A ad D. Quia enim nota est mensura ipsius AB proportionis, nota etiam erit ipsius CD proportionis, ut quæ illi ponitur æqualis, per 1 postul. Nota verò est etiam AC, & AD est composita ex AC & CD, per 1 ax. quare etiam mensura ipsius AD componetur ex mensurâ ipsius AC, ut ex mensurâ ipsius CD, id est, ipsius AB.

COROL-

COROLLARIUM I.

Hoc pacto, ex notificatis proportionibus quindécim, prop. 18 præmissis, noscentur aliæ centum viginti numerorum intra millenarium, ad ipsum millenarium.

Quoties enim datæ fuerint proportionēs ad 1000 duorum numerorum talium, in quibus vel ambobus, duos ultimos locos habuerint cyphræ, ut 900, 800, vel in eorum altero quidem duos, in reliquo verò unum, ut 700, 810, velut 700, 10. Vel si unum solum, eumque ultimum in utroque numero locum ex tribus obtinuerit cyphra : quæ tamen hanc cyphram antecedit figura in altero numero, ex paribus in reliquo quinaris fuerit, ut 620, 950 sic 620, 50 sic 20, 950, vel si alter quidem cyphrâ in ultimo loco caruerit, ex paribus tamen fuerit ut 512, 12 vel 2 reliquus fuerit 500. Omnibus hisce casibus institutâ multiplicatione numerorum, proveniunt in fine tres cyphræ, quibus abjectis formatur numerus, unus ex mille ordinis naturalis, seu progressionis arithmeticæ.

COROLLARIUM II.

Hinc oritur præceptum tractandæ regulæ trium, quando uno loco occurrit rotundus 1000.

Nam si ille occurrit primo loco in tali situ :

A 1000 dat B, quid C ?

Tunc additur mensura proportionis AB ad mensuram proportionis AC, ita fit mensura proportionis AD.

Sin autem 1000 occurrat loco secundo vel tertio, in tali situ B dat A 1000, quid C ? vel tali B dat C, quid A 1000 ?

Tunc aufertur mensura proportionis AB à mensura proportionis AC, vel ejus multiplicis proximè majoris, ita relinquitur mensura vel ipsius proportionis AD, vel ejus æquè multiplicis.

XX. PROPOSITIO.

Quando fuerint ut primus ad secundum, sic tertius ad quartum, notæ verò fuerint proportionēs ipsius 1000 ad tres priores, innotescet etiam proportio ejusdem 1000 ad quartum.

Sit enim A 1000, & sit ut B ad C, sic D ad E. Notæ verò sint proportionēs AB, AC, AD ; Dico innotescere etiam proportionem AE. Nam quia ut B ad C, sic D ad E, æqualis est igitur mensura proportionis BC, mensuræ proportionis DE. Sed BE est composita ex BD, DE per 1 axiom. Æqualis igitur erit proportio BE, proportionibus BD, BC simul sumptis. Sed & AE est composita ex AB, BE : sic AD ex AB, BD, sic AC ex AB, BC per 1 axiom. Si ergò pro BD, BC, sumantur proportionēs notæ AD, AC, tunc AB bis accessit. Vicissim si pro BE sumatur AE proportio quæsitæ, tunc AB semel tantum accessit. Si ergò à junctis AD, AC notis, abstuleris AB notam semel, relinquitur AE proportio quæsitæ.

COROLLARIUM I.

Hâc methodo præterquam quod superiorum chiliadis multæ iteratò exquiruntur, accedunt illis insuper aliquot aliæ.

COROLLARIUM II.

Hinc oritur præceptum tractandi regulam trium, quando nuspiam occurrit rotundus 1000: ut si sic collocentur tres, B dat c, quid D?

Nam additur mensura proportionis AC, mensuræ proportionis AD, & à summâ aufertur mensura proportionis AB, vel ejus aliqua pars aliquota; ita relinquitur mensura proportionis AE, vel ejus æquè multiplex.

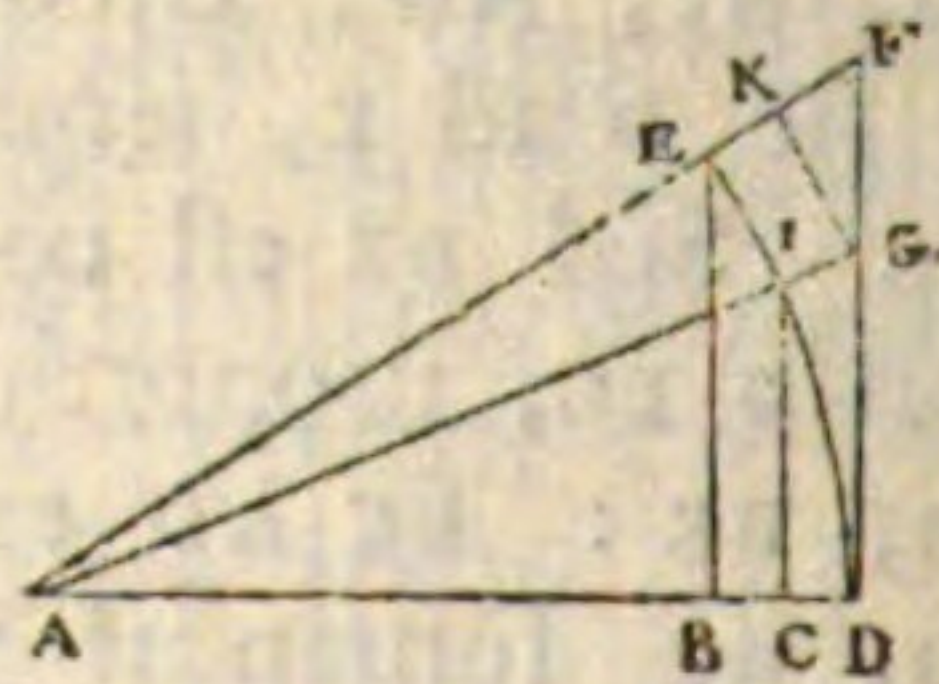
DEFINITIO.

Mensura cujuslibet proportionis inter 1000 & numerum eo minorem, ut est definita in superioribus, expressa numero, apponatur ad hunc numerum minorem in Chiliade, dicaturque LOGARITHMUS ejus, hoc est, numerus ($\alpha\rho\iota\theta\mu\omicron\varsigma$) indicans proportionem ($\lambda\omicron\gamma\omicron\nu$) quam habet ad 1000 numerus ille cui logarithmus apponitur.

XXI. PROPOSITIO.

Si primus numerus fit semidiameter circuli seu finus totus: omnis numerus minor, ut finus complementi alicujus arcus, logarithmum habet majorem sagittâ arcus, minorem verò excessu secantis arcus supra radium seu semidiametrum, excepto unico proximo post semidiametrum, quia illius logarithmus ex hypothesi est æqualis sagittæ.

Sit A centrum circuli, AD semidiameter, DI, DE arcus, eorumque finus IC, EB, finus verò complementorum sint CA, BA, sagittæ CD, BD. Sit autem ut AD ad AC, sic AC ad AB. Amplius sint eorundem arcuum secantes AG, AF, per terminos I, E, in tangentes DG, DF educti, & ipsi AG æqualis abscindatur ab AF, quæ sit AK; denique sit CD mensura proportionis CA, AD, ut minimi elementi arbitrarii. Dico mensuram proportionis BA ad AD, hoc est, logarithmum ipsius BA, majorem esse quàm BD, minorem verò quàm EF. Quod major sit quàm BD, demonstratum est suprâ propositione duodecima. Quod verò minores sint mensuræ proportionum harum IG, CF sic probatur. Primum de CA, cum ipsa CD sagitta, utpote



in

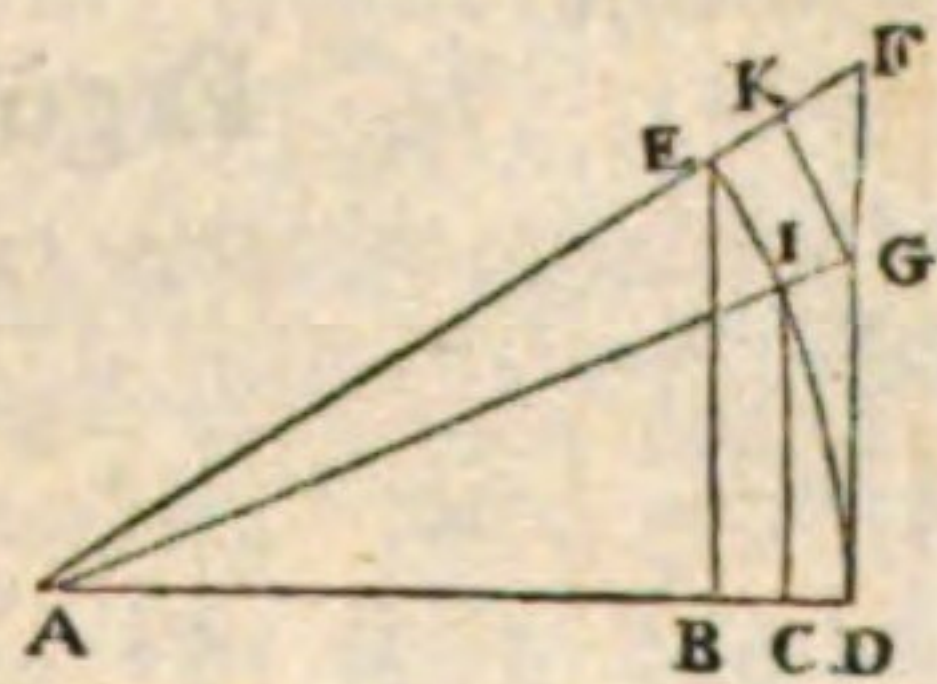
in minimo proportionum elemento ponatur esse logarithmus ipsius CA : CD fanè est minor quàm IG . Ut enim CA ad AD , sic IA , hoc est DA , ad AG , quia DG & CI parallelæ. Est igitur AD media proportionalis inter CA & AG . Ut igitur CA ad AD , sic differentia CA , AD , hoc est, CD , ad differentiam sequentem DA , AG , hoc est, ad IG . Sed CA est minor quàm AD , ergò & CD est minor quàm IG . Sed CD est logarithmus ipsius CA sinus complementi arcus ID , & IG est excessus secantis ejusdem arcus. Ergò logarithmus minor est hoc excessu.

Transeamus ad BA : cujus logarithmus major est quàm BD , demonstrandum est, illum non esse tantò majorem ipsâ BD , quin interim maneat minor ipso EF . Rursum igitur AD est media proportionalis inter BA & AF . Et quia posita est ut BA ad AC , sic CA ad AD : quare etiam ut EA ad AG , vel AK , sic KA vel GA ad AF . Sunt igitur continuè proportionales istæ AB , AC , AD , AK vel AG & AF . In eâdem igitur proportionem sunt etiam BC , CD , IG , vel EK & KF . Minor verò est CD quàm IG , ut prius ostensum, minor igitur erit etiam IG vel EK quàm KE . Tota igitur EF major est quàm dupla ipsius IG , multò magis igitur EF major erit quàm dupla ipsius CD minoris. At proportionis inter BA , AD , ut quæ dupla est ipsius BA , AC per 1 propositionem, mensura seu logarithmus ipsius BA , est præcisè duplus ipsius CD per 1 postul. Minor est ergo logarithmus ipsius BA excessu secantis EF . Erat autem major sagittâ BD . Patet ergo propositum.

XXII. PROPOSITIO.

Iisdem positis, sagitta arcus cum excessu secantis superat duplum
logarithmi, ad sinum complementi apponendi.

Sit enim primò Sinus complementi longissimus, aut longissima mediarum proportionalium, quibus aliqua proportio dividitur in partes arbitrario numero multas; sic ut ejus, verbi causa, AC residuum CD, seu sagitta arcus ID sit ipsissima mensura arbitraria proportionis CD, logarithmus igitur ipsius CA est CD. At IG est major ipsâ CD. Juncti igitur excessus secantis IG & sagitta CD, plus efficiunt quàm duplum ipsius CD.



Sit deinde alia quaecunque minor linea proportionis
continuae, ut AB, & educta ex B perpendiculari in cir-
cumferentiam E, connexisque AE, & DG continuatis in F, sit EF excessus se-
cantis, & BD sagitta ejusdem scilicet arcus ED. Dico junctos EF & BD, facere
plus quam duplum logarithmi ad BA apponendi, seu mensurae ipsius BA,
AD proportionis.

Quia est ut BA ad AD, sic DA ad AF, & CA media proportionalis inter BA, AD, ut igitur BA ad AC, sic GA ad AF, quare per 25 quinti Eucl. BA, AF junctæ sunt longiores junctis CA, AG, sive quia ut BA ad AD, sic DA ad AF, quare.

BA 2.

BA, AF junc̄tæ, superant DA, AD duas medias. Ut verò BA ad AC, sic etiam BC ad CD, & IG ad KF per 2 prop. quare etiam BC & KF junc̄tæ, superant CD & IG junc̄tas. At CD, IG plus sunt quàm duplum ipsius CD. Ergò BC & KF junc̄tæ, multò plus sunt quàm duplum ipsius CD. Et sic BD, EF plus sunt quàm quadruplum ipsius CD. Sed duplum ipsius CD est logarithmus seu mensura proportionis BA, AD per 1 postul. Ergò BD, EF junc̄tæ, plus sunt quàm duplum logarithmi ipsius BA, seu mensuræ proportionis BA, AD.

Sit tertio proportio BA, AC minor vel major quàm CA, AD, demonstrabitur nihilominus quod junc̄tæ BC, KF, superent duplum tantæ partitionis de CD, quanta portio est proportio BA, AC, proportionis CA, AD.

COROLLARIUM.

Logarithmus Sinus complementi est minor medio arithmetico inter sagittam & excessum secantis.

PRÆCEPTUM.

Sinus inventi in Canone Sinuum residuum ad totum adde excessui secantis complementi, summæ dimidium superat logarithmum, sagitta ipsa proximè minor est logarithmo.

Est Sinus 99970,1490 arcus
Ejus residuum ad fin. totum 29,8510 sagitta arcus complementi minor logarithmo
29,8599 excessus secantis
59,7109 summa
29,8555 dimidium majus logarithmo.

Ergò logarithmus est inter $\left\{ \begin{array}{l} 29,8510 \\ 29,8555. \end{array} \right.$

PRÆCEPTUM ALIUD.

Invento sinus logarithmo, invenies etiam proximè logarithmum numeri rotundi, qui sinu tuo scrupulato proximè minor est, si sinus scrupulosi excessum supra numerum rotundum adjeceris logarithmo sinus invento.

Ut quia sinus 99970,149, logarithmus inventus est 29,854 circiter, si jam vis scire logarithmum rotundi 99970,000, vides excessum tui sinus
6 scru-

scrupulosi esse 149, hunc adde ad inventum sinus logarithmum, observato puncto. Sic

29,854
149
30,003

Hic est logarithmus rotundi numeri 99970,000 proximè.

XXIII. PROPOSITIO.

Si tres quantitates invicem successerint, æqualibus excessibus differentes, mensura proportionis inter maximam & mediam, cum mensurâ alterius inter mediam & minimam constituet proportionem, majorem quidem proportionem majorum, minorem verò proportionem minorum.

A	H	B	C	D	L
	η	β	γ	δ	λ

Sint tres quantitates AD, AC, AH, æqualibus excessibus DC, CH; fit autem mensura proportionis DA, AC linea $\delta\gamma$, mensura verò proportionis CA, AH, linea $\gamma\eta$. Dico proportionem ipsius $\delta\gamma$ ad $\gamma\eta$ majorem quidem esse proportionem ipsius CA ad AD, minorem verò proportionem ipsius HA ad AC. Fiat enim ut DA ad AC, sic CA ad AB. Ut igitur DA ad AC, sic DC ad CB per 2 prop. Sed longior est DA quàm AC, longior igitur DC quàm CB, longior igitur & CH quàm CB, differentia BH.

Cum igitur æquales sint proportionem DA, AC & CA, AB, per 1 prop. mensura verò ipsius DA, AC sit linea $\delta\gamma$, habebit & CA, AB mensuram æqualem ipsi $\delta\gamma$, per 1 postul. Et cum proportio CA, AH sit composita ex proportionem CA, AB & proportionem BA, AH. Major igitur erit proportio CA, AH, quàm proportio DA, AC, æqualis ipsi CA, AB. Major igitur etiam mensura ejus $\gamma\eta$ quàm $\gamma\delta$. Abscindatur à $\gamma\eta$ æqualis ipsi $\gamma\delta$, quæ sit $\gamma\beta$, residua igitur $\beta\eta$, mensura erit residuæ proportionis BA, AH. Erit autem proportio $\gamma\beta$ ad CB minor quàm $\beta\eta$ ad BH per 12 coroll. id est, major est $\beta\eta$ respectu $\beta\gamma$ vel $\gamma\delta$, quàm HB respectu BC. Compositis igitur terminis, illic $\gamma\beta$ & $\beta\eta$ in $\gamma\eta$, hic CB & BH in CH, major erit $\gamma\eta$ respectu $\gamma\beta$, vel ejus æqualis $\gamma\delta$, quàm CH respectu CB. Major igitur est proportio inter $\delta\gamma$, $\gamma\eta$, quàm inter BC, CH, id est, CD.

Ut verò BC ad CD, sic CA ad AD per 2 prop. Major igitur proportio inter $\delta\gamma$, $\gamma\eta$, quàm inter CA, AD. Sed $\delta\gamma$ & $\gamma\eta$ sunt mensuræ, illa quidem proportionis inter DA, AC majores, hæc verò proportionis inter CA, AH minores. Ergò proportio mensurarum major est proportionem terminorum minorum.

Rursum

Rursum fiat ut HA ad AC , sic CA ad AL . Ut igitur HA ad AC , sic HC ad CL , per 2. Sed brevior est HA quam AC , brevior igitur HC , hoc est DC , quam CL , differentia DL . Cum igitur æquales sint proportionēs HA , AC , & CA , AL , per 1 prop. Mensura verò ipsius HA , AC , sic linea $\eta\gamma$, habebit & CA , AL mensuram æqualem ipsi $\eta\gamma$, per 1 postul. Esto $\gamma\lambda$. Minor verò erat proportionis CA , AD mensura, puta $\gamma\delta$, minor igitur est $\gamma\delta$ quam $\gamma\lambda$. Excessus igitur $\delta\lambda$ erit mensura proportionis inter DA , AL , appositæ ad proportionem inter CA , AD . Et quia termini DA , AL , sunt longiores quam CA , AD , quare per 12 coroll. minor est $\delta\lambda$ respectu $\lambda\gamma$, vel $\gamma\eta$, quam DL respectu LC . Major igitur residua $\gamma\delta$ respectu $\gamma\lambda$, vel $\gamma\eta$, quam CD vel HC respectu CL . Et quia proportio minuitur cuncto minori termino, per ax. 1. Minor igitur est proportio inter $\delta\gamma$, $\gamma\eta$, quam inter HC , CL . Ut vero HC ad CL , sic termini trium minores HA ad AC , per 2 conversam. Minor est igitur proportio inter $\delta\gamma$, $\gamma\eta$ mensuras proportionum, quarum unam facit major terminus cum medio, alteram medius cum minimo, quam inter terminos minores.

XXIV. PROPOSITIO.

Dicta proportio inter duas mensuras, est minor dimidiâ proportionē inter terminos extremos.

A	H	V	C	D
·	·	·	·	·
	η	ν	γ	δ
	·	·	·	·

Sint enim termini extremi, ut prius AH , AD , media duo, arithmeticum AC , geometricum AV , sit proportionis DA , AC mensura $\delta\gamma$, vel æqualis ipsi DC , vel etiam major per postul. 3. & applicetur ipsi $\delta\gamma$, alia $\gamma\nu$, quæ sit mensura iusta proportionis CA , AV , ut sic tota $\delta\nu$ mensuret proportionem DA , AV , residuæ igitur proportionis VA , AH mensura, erit priori $\delta\nu$ æqualis, per 1 postul. & 1 prop. sit ea $\nu\eta$: Quia igitur $\delta\gamma$ est vel æqualis, vel major quam DC , sequentis $\gamma\nu$ proportio ad CV , erit major quàm proportio ipsius $\gamma\delta$ ad CD , & tertiæ $\nu\eta$ proportio ad VH rursus erit major quàm secundæ $\gamma\nu$ ad secundam CV . Et per compositionem totius $\eta\delta$ ad totam HD major erit proportio quàm partis $\nu\gamma$ ad partem VC . Permutatim igitur major erit proportio totius $\eta\delta$ termini majoris in priori, ad $\nu\gamma$ maiorem, in posteriori, quàm totius HD termini minoris, in priori proportionē ad VC terminum minorem in posteriori. Sed tota $\eta\delta$ constat ex terminis $\eta\gamma$, $\gamma\delta$, quorum differentia $\gamma\nu$, similiter DH constat ex terminis DV , VH , quorum differentia VC : Major igitur est proportio terminorum $\eta\gamma$, $\gamma\delta$, junctarum, ad suam differentiam $\gamma\nu$, quàm terminorum DV , VH , ad differentiam VC . At verò auctâ proportionē summæ terminorum, ad suam differentiam, minuitur ipsorum inter se terminorum seorsim positorum proportio, per coroll. 4, 13 & communem notitiam, quod proportionalium eadem sit ratio: Minor est itaque proportio inter $\eta\gamma$, $\gamma\delta$, quàm inter DV , VH .
Sed

Sed proportio DV ad VH est æqualis proportioni DA ad AV per 2. Hæc verò DA ad AV proportio est dimidia ipsius DA ad AH per 1 prop. Minor est ergò proportio inter $\eta\gamma$, $\gamma\delta$, mensuras proportionum HA, AC & CA, AD, quàm dimidia inter terminos HA, AD.

EXEMPLUM HIC EST.

Numeri.	Numerorum differentiæ.	Logarithmi.	Logarithmorum differentia, seu mensuræ pro- portio.
1000		00	
750	---250	28768,21	28768,21
500	---250	69314,72	40546,51

Si verò fiat ut 1000 ad medium proportionale inter terminos extremos 1000 & 500 (id est, ad 70710,68), sic mensura proportionis prioris, quæ est 28768,21 ad aliquem; is prodibit 40684,40, major quàm mensura proportionis posterioris.

Si verò fiat ut medium proportionale inter 1000 & 360 (id est, ut 600) ad 1000, sic mensura proportionis posterioris 63598,86 ad aliquem, is prodibit 38159,32, minor quàm mensura proportionis prioris inter 1000 & 600.

ALIUD EXEMPLUM.

1000		0	
680	320	38566,25	38566,25
360	320	102165,11	63598,86

COROLLARIUM.

Cum igitur medium arithmeticum dispescat proportionem in partes inæquales, quarum una major est semisse totius, altera minor, si quærat, quæ ergò sit ipsarum proportionum proportio inter se, respondetur, quod ea sit paulò minor semisse dicto.

E

Exemplum

Exemplum inquirendi proxime majus, & proximè minus
aliquid, mensurâ proportionis propositæ.

Nota sit mensura proportionis inter 1000, 900, quæ sit 10536,05, quæritur mensura proportionis 900, 800, ut sint æquales differentiæ inter 1000, 900 & inter 900, 800.

Est igitur ut 9 ad 8 sic 10536,05 ad 11706,72. Mensura verò proportionis 9, 8 est major. Rursum ut medium proportionale inter 8, 10 (quod est 89,442719) ad 10, sic 10536,05, ad 11779,66. At mensura proportionis inter 9 & 8 est minor, scilicet 11778,30.

NOTITIA COMMUNIS.

Omnis Numerus quantitatem exprimit effabilem.

XXV. PROPOSITIO.

Si numeri mille succedant invicem, ordine naturali, bini differentes unitate; suscipiantur verò bini quicunque, deinceps ordinati, tanquam termini proportionis alicujus; erit hujus proportionis mensura, ad mensuram proportionis inter duos maximos Chiliadis, in proportione majore quidem quàm quantam habet maximus ipse 1000 ad majorem ex terminis susceptis; minore verò quàm quantam habet idem 1000 ad minorem ex susceptis. Minore etiam, quàm quantam habet 1000 ad medium proportionale inter susceptos.

Suscipiantur enim ex mille duo quicunque deinceps, puta 501 & 500. Et sit logarithmus prioris 69114,92, posterioris 69314,72. Horum logarithmorum differentia est 199,80. Quare per definitionem proportionis inter 501 & 500 mensura est 199,80. Eodem modo, quia maximi 1000 logarithmus est 0; proximi verò 999 logarithmus est 100,05, & horum duorum logarithmorum differentia itidem 100,05, mensura igitur proportionis inter 1000, 900 (vel inter 100000,00 & 99900,00) est 100,05; copuletur jam maximus 1000 cum utroque susceptorum, scilicet & cum 501, & cum 500, copuletur etiam mensura 199,80 cum mensurâ 100,05, dico proportionem inter 1000 & 501 majorem esse proportionem inter 199,80 & 100,05, minorem verò eandem proportionem inter 1000, 500.

Demonstratum est enim in prop. 23, mensuras duarum proportionum deinceps ordinarum, comprehendere proportionem majorem minore earum, quæ sint deinceps. Ut si sint tres termini deinceps 1000, 999, 998 quorum priores quidem 1000, 999 faciant proportionem præcedentem, posteriores 999, 998 sequentem, præcedentis quidem proportionis mensura, ut prius erit 100,05, sequentis verò mensura erit 100,15. Hæ duæ mensuræ 100,15 & 100,05, constituunt proportionem majorem, quàm termini 1000 & 999.

Jam verò duæ mensuræ, altera quidem proportionis inter 1000 & 999 hoc est, 100,05 altera verò proportionis inter 501 & 500, hoc est, 199,80; hæ, inquam,

inquam, duæ mensuræ, ut termini consideratæ proportionem inter se constituunt compositam ex proportionibus omnibus, omnium binarum, & binarum deinceps mensurarum, intercedentium per axiom. 1. Similiter verò etiam termini ipsi 1000 & 501 proportionem inter se constituunt compositam ex totidem, hoc est, omnibus proportionibus omnium binorum, & binorum deinceps numerorum inter 1000 & 501 intercedentem per idem ax. 1.

Compositorum verò ex partibus æquè multiplicibus proportionalibus, eadem est proportio, quæ partium inter se singularum, hinc & inde combinatorum per 1. quinti Eucl. Ergò etiam hæ non deinceps sitæ, sed longè distantes mensuræ 100,05 & 199,80 proportionem facient majorem quàm termini non deinceps siti, sed distantes longè, scilicet 1000 & 501. Eodem tenore rursum incipiendo syllogismum eundem inferemus duas mensuras deinceps 00,15, & 100,05, constituere proportionem minorem quàm duos terminos deinceps 999 & 998. Mensurarum verò 100,05 & 199,80 proportionem componi ex omnibus interjectis, & similiter terminorum 999 & 500 proportionem componi ex totidem interjectis, quare etiam proportionem inter 100 & 999,80 minorem esse quàm inter 999 & 500. Multò igitur minorem quàm inter 1000 & 500, ut quæ ad proportionem 999, 500, addit proportionem 1000, 999.

Tertio per eadem syllogismi vestigia euntes, sic colligemus ex prop. 24 præmissâ.

Mensurarum deinceps sitarum, scilicet 1000,15 & 1000,05, proportio est minor quàm ea quæ est inter 1000 & medium proportionale terminorum 1000, 998, vel quàm ea quæ est inter hoc medium proportionale terminorum 1000, 998 & terminum 998. Mensurarum vero non deinceps, ut 100,05 & 199,80, proportio componitur ex interjectis omnium binarum, & binarum deinceps proportionalibus; & terminorum, quorum unus est medium proportionale inter 1000 & 998, id est, $\sqrt{998000}$ alter 500: vel quod idem est unus 1000 & alter medium proportionale inter 501 & 500, seu $\sqrt{250500}$; hæc, inquam, proportio componitur ex proportionibus, quas constituunt totidem interjectæ binorum, & binorum numerorum mediæ proportionales lineæ, accensito termino 1000. Quare etiam mensuræ non deinceps sitæ, scilicet 100,05 & 199,80 proportionem, exhibent minorem quàm 1000, cum medio proportionali inter susceptos 501 & 500.

COROLLARIUM I.

Proposito numero quocunque infra 1000, ejusque logarithmo, quæcunque differentię logarithmorum antecedunt propositum, versus initium chiliadis, sunt ad primum logarithmum (qui scilicet ad 999 apponitur) in proportionem majore, quàm 1000 ad propositum quæcunque sequuntur versus ultimum logarithmum, sunt ad eum in proportionem minore.

COROLLARIUM II.

Hoc adjumento facilè implentur loca chiliadis, quæ per superiores propositiones nondum sunt sortita suos logarithmos.

XXVI. PROPOSITIO.

Differentia binorum logarithmorum, qui sunt adscripti ad numeros deinceps, est ad eorundem numerorum differentiam, in proportionem majori quidem quàm est 1000 ad numerorum majorem; minore verò quàm idem 1000 est ad numerorum minorem.

Facile demonstratur per antecedentem ejusque corollarium. Nam ex unâ parte differentia duorum deinceps numerorum perpetuò est unitas (seu in prolongatis perpetuò 100,00.) Jam verò ultimus logarithmus est 100,05, idem, qui & mensuræ ultimæ & minimæ proportionis. Nulla igitur differentia numerorum deinceps differt à mensurâ ultimæ proportionis plus quàm quinque unitatibus numerorum prolongatorum. Ex alterâ verò parte mensura proportionis duorum deinceps numerorum nihil est aliud, quàm differentia duorum ad illos numeros positorum logarithmorum, per definitionem. Si igitur inter mensuram susceptæ proportionis, & inter mensuram proportionis ultimæ est major proportio, quàm inter 1000 & terminum susceptæ majorem, erit etiam inter differentiam logarithmorum deinceps, & inter differentiam numerorum, ad quos sunt logarithmi, proportio major quàm inter 1000 & terminum susceptæ majorem. Nam quod majori majus est, ipsius quoque multo est majus. Sed proportio inter differentiam logarithmorum, deinceps (verbi causa) inter 199,80 & 1000,00 differentiam perpetuam numerorum deinceps, est major quàm proportio inter 199,80 & 100,05; excessus enim est proportio inter 100,05, & 100,00. Et inter 199,80 & 100,05 proportio major fuit demonstrata in præcedenti. Ergò multò est major proportio inter 199,80 & 100,00 quàm inter 1000 & (verbi causa) 501 terminum susceptæ majorem. Est verò eadem proportio inter 199,80 differentiam logarithmorum, & 100,00 differentiam numerorum, etiam minor quàm proportio inter 1000 & 500, terminum susceptæ minorem, quod sic probo.

Proportio inter 199,80 & 100,05, est minor quàm proportio inter 999 & 500, per demonstrata 25 præcedente; similiter verò etiam proportio inter 100,05 & 100,00, seu inter 20,01 & 20,00, est minor quàm proportio inter 1000 & 999 per coroll. ad 13, quia scilicet eadem est differentia numerorum majorum 2001 & 2000, quæ minorum 1000 & 999, utrobique scilicet unitas.

Componitur verò proportio inter 199,80 & 100,00 ex utrâque suæ societatis minore, scilicet ex prop. 199,80 ad 100,05 & ex prop. 100,05, ad 100,00. Sic etiam proportio 1000 ad 500 componitur ex utrâque suæ societatis majore, scilicet ex prop. 999 ad 500, & ex prop. 1000 ad 999. Ergò etiam ipsa composita prior erit minor, & composita posterior erit major.

(si placet labor) medio geometrico inter secantes majores, exque differentiis continuè additis, accumulari logarithmi.

COROLLARIUM III. PRÆCEPTUM.

Divide finum totum per utrumque proportionis susceptæ terminum, quotientis utriusque medium arithmeticum est quæsitum incrementum, hoc adde ad logarithmum termini majoris, prodit logarithmus termini minoris.

EXEMPLUM.

Propius & omnino proximum vero est medium geometricum, at non est pretium operæ tam operosæ, quando unus solus vel alter logarithmus est constituendus.

Datus esto logarithmus ad 700, scilicet 35667,4948, quæritur logarithmus ad 699. Divide ergò radium per 700, prodit 142857,142857,142857,142857, &c. divide etiam per 699, prodit 1430672, &c.

Medium arithmeticum est 142,962
Adde hoc ad 35667,4948

Prodit logarithmus ad 699.....35810,4568.

COROLLARIUM IV. PRÆCEPTUM DE LOGARITHMIS SINUUM.

Incrementum logarithmorum inter duos sinus sic inquire: inter secantes complementorum constituatur medium geometricum, dividaturque per differentiam sinuum, prodit differentia logarithmorum.

EXEMPLUM.

Sit sin. gr. 0 1 29 09, sec. compl. 343774682
0 2 58 18, sec. compl. 171887348

Diff. 29 09, medium geom. 2428, circiter
2909.

Quotiens 80000, est major quæsito incremento logarithmorum, quia secantes admodum magni sunt.

APPENDIX.

Eodem ferè modo posset etiam demonstrari, differentias secundas esse in duplâ proportionem primarum, tertias in duplâ secundarum*. Verbi causâ, cum in ipso principio logarithmorum differentia prima sit 100, 00000, æqualis scilicet ipsi differentiæ numerorum 100000,00000, & 99900,00000, secunda seu differentiarum differentia 10000; tertia

* Hæc falsa est. Vide Huttoni Mathematical Tables, pa. 58. EDIT.

20. Postquam ad numerum 50000,00000 ventum fuerit, logarithmi quidem proximi differentiam faciunt 200,00000 quæ sic habet ad differentiam primam, sicut numerus 50000,00000, ad maximum 100000,00000. Secunda verò differentia est 40000, in qua 10000 continetur 4ter. Tertia 328, in qua 20 continetur 16ies. At cum in re insolitâ laboremus penuriâ vocabulorum: quare ne nimium obscura proponamus, demonstratio dimittatur intentata.

XXVIII. PROPOSITIO.

Nullus numerus exactè exprimit mensuram proportionis inter binos unius millenarii numeros, methodo superiori constitutam.

Nam quia termini uniuscujusque proportionis extremi non sunt ab invicem, ut duo numeri ejusdem speciei figurativæ; tam multorum graduum, quot vices arbitrariæ sunt assumptæ ad secundam proportionem in minima elementa arbitraria: mediæ ergò proportionales elementa constituentes sunt ineffabiles per 9 prop. Differentia igitur inter mediarum proportionalium maximam, & terminum per 1000 significatum est & ipsa ineffabilis. Sed mensura proportionis inter 1000 & terminum minorem effabilem in chiliade est multiplex hujus differentiolæ, id est, & commensurabilis est illi. Ergò mensura hæc est terminis incommensurabilis, hoc est, ineffabilis. At nullus ergò numerus, & sic neque logarithmus, exactè exprimit hanc mensuram.

ADMONITIO.

Interest igitur observare, quousque sese proferat hoc vitium. Nam si proportio 999, 1000 secatur in particulas 1677216 per 24tas proportionales medias, & in particulæ unius mensurâ numero expressâ peccetur semisse unitatis: multiplicatus hic error cum ipsâ mensurâ elementi in numerum elementorum proportionis, efficiet 8000000 unitates.

XXIX. PROPOSITIO.

Si mensuræ proportionum omnium exprimantur numeris, seu logarithmis: non omnes proportiones sortientur legitimam suam mensuræ portionem ad omnem minutiarum scrupulositatem.

Nam per 11 prop. proportiones numerorum chiliadis inter se sunt incommensurabiles; omnes verò eorum logarithmi sunt effabiles per ax. 3, & sic indicant mensuras inter se commensurabiles: injustâ igitur partitione, si ad minima veniatur.

NOTA.

Locum idem habet etiam in sinibus, inque proportionem circuli ad circumferentiam, & passim; & interest animadvertere, à quâ figurâ numeri, vitium incipiat, ne in numeris eâ posterioribus frustra impendamus operam.

XXX. PROPOSITIO.

Si ad numerum 1000 chiliadis maximum referantur aliqui majores, ipsi verò 1000, fit applicatus logarithmus 0, logarithmi majoribus competentes erunt privativi.

Referatur ad 1000 major 1024, fiatque ut 1024 ad 1000, sic hic ad 97656,25; fit etiam mensura proportionis inter 100000,00 & 97656,25, logarithmus 2371,6526. Cùm ergò proportio inter 1024 & 1000 sit æqualis proportioni inter 100000,00 & 97656,25. Erit eadem mensura ejus. Et si ad 1024 apponeretur logarithmus 0, tunc ad 1000 apponendus esset logarithmus 2371,6526, ad numerum verò 97656,25 duplum hujus logarithmi, quia proportio 10240000 ad 97656,25 est dupla proportionis 1024 ad 1000; sed quia ad 1000 in chiliade apponitur log. 0, deterfio logarithmo, & 2371,6526, & à duplo hujus etiam apud 97656,25, simplum est deterfsum in canone chiliadis; ergò etiam à logarithmo ipsius 1024, cui applicaveramus logarithmum 0, detergendum est tantundem. Si verò à 0 auferas 2371,6526, relinquitur 2371,6526 privativum cum signo Cossico.

METHODUS COMPENDIOSISSIMA

CONSTRUENDI

CHILIADA

LOGARITHMORUM.

PRINCIPIO inquiratur logarithmus, qui metitur proportionem inter 100000,00 & 97656,25 quæsitâ mediâ proportionali, maximâ vicefimarum quartarum inter hos terminos, ejusque & numeri totalis prolongati differentiâ toties duplicatâ: emerget autem logarithmus 2371,6526, qui idem est etiam numeri 1024 defectivus per 30. Secundò idem fiat etiam cum proportionem inter 1000 & 500: emerget autem logarithmus ad 500, iste 69314,7193, qui idem etiam duplicationis logarithmus dicitur.

Jam quia ut 1000 ad 500 sic 1024 ad 512, & hic ad 256, & hic ad 128, & hic ad 64, & hic ad 32, & hic ad 16, & hic ad 8, & hic ad 4, & hic ad 2, et hic ad 1.

Decupla est igitur proportio 1024 ad 1, proportionis 1000 ad 500. Quare logarithmus ad 1, tunc quidem erit decuplus logarithmi ad 500, cum numerus 1024 acceperit logarithmum 0, sed ubi ei privativus 2371,6526, fuerit applicatus, etiam ipsius 1 logarithmus erit diminuendus tanto. Diminuatur decuplum duplicantis.

693147,1928

2371,6526

Hic est logarithmus unitatis in
chiliade nostrâ seu 100,00

690775,5422

Et igitur 10,00.

921034,0563.

Et quia ut 1 ad 10, sic hic ad 100, & hic ad 1000, tripla est itaque proportio 1000 ad 1 proportionis 1000 ad 100, tertia igitur pars logarithmi ad 1 est apponenda pro logarithmo ad 100, puta 230258,5141, & hic etiam est logarithmus decuplicationis, duæ verò tertiæ sunt logarithmus ad 10, scilicet 460517,0282. Quod si duplicantem abstuleris a logarithmo ad 1, emergit logarithmus ad 2, si ab hoc abstuleris logarithmum ad 10 restat logarithmus quinduplicationis.

Logar. ad 1, 690775,5422
 Duplicans 69314,7193

Logar. ad 2, 621460,8229
 Logar. ad 10, 460517,0281

Quinduplicans 160943,7948.

Hiscæ sic præparatis, jam construantur logarithmi centum maximorum, initio facto à 999, hâc methodo. Totalis 1000 prolongatus 7, cyphris dividatur per singulos ordine; quotientes referantur in tabellam, sunt enim secantes illorum arcuum, quorum complementa habent divisores istos pro sinubus; ut si divisor seu sinus complementi 999, quotiens seu secans erit 100,10010, sin dividat 901, quotiens erit 110,98779; sin 900 quotiens erit 111,11111. Divisio autem continuatur propterea usque ad octavam figuram, ut constet nobis quantum differat medium arithmeticum à geometrico, ubi maximè. Ut si multiplicatis in se duobus ultimis quotientibus, radix quæretur, ea erit 111,04942. At medium arithmeticum inter duos quotientes est 111,04945.

In structurâ igitur centum minimorum logarithmorum media hæc duo inter se sunt æqualia, usque ad septimam figuram inclusivè: in octavâ oritur differentia ternarii. Pone in omnibus centum mediis esse tantam. Si ergo centum media arithmetica ordine accumules, peccarent illa excessu non majore quàm 00300, ternarii scilicet in tertiâ figurâ post punctum. At verò non est in omnibus centum tanta differentia: in initialibus enim penitus evanescit, ut inter 100,00000 & 100,10010. Hic enim utrumque medium est 100,05005 & differentia occultatur in figuris ulterioribus, si quis illas erueret. Quare centum minimos logarithmos tutissimè constituimus per acumulationem mediarum proportionalium inter quotientes per 27 prop. Semper enim additis duobus secantibus ad logarithmi prioris duplum conflatur duplum logarithmi posterioris.

Constructis his logarithmis; eligatur logar. ad 960, qui erit 4082,2001 ferè. Huic si continuè adjeceris duplicantem, emergent logarithmi ad 480, 240, 120, 60, 30, 15. Quia proportio 960, ad 15 est sextupla proportionis 1000 ad 500. Ita emerget logarithmus ad 15, qui est 419970,5159,

& ad 30 log. 350655,7965.

Hunc si abstuleris à logar. ad 10 scilicet 460517,0281

Restabit 109861,2316 triplicans logarithmus.

Hunc ergo aufer à logar. ad 1,690775,5422

Restat logar. ad 3,580914,3106.

Idem etiam ex log. ad 900 elici potest, probationis causa. Nam hujus logar. est 10536,0535. Et si jam hic est nimius. ob causam dictam. Nam ut 1000 ad 900, sic 9000 ad 8100, & ut 1000 ad 8100 sic hic ad 6561. Quadrumplum ergo logarithmi ad 900 respondet numero 6561 (seu 90000,00 numero 65610,00) scilicet 42144,2140. At qui numerus 65610,00 est ternarii de 1000 con-

1000 continuè multiplex. Ecce 6561, 2187, 729, 243, 81, 27, 9, 3, 1.
Ergò proportio 6561 ad 1, est octupla proportionis 3 ad 1.

Ergò logarithmo ad 1000 scilicet 921034,0563
Aufer log. ad 65610,00 scilicet 42144,2140

Residui 878889,8423
Pars octava 109861,2303 et paulò minor quàm prius.

Scilicet, quia logarithmus ad 900, & sic etiam ejus quadruplum, justo majus fuit, id verò subtractum à justo logar. ad 1, relinquit justo minus. Et sanè etiam bisectio continua proportionis inter 10 & 9 ostendit ultimas hujus logar. ad 900 figuras, non 0535 sed 0513 per 22 minus, cujus quadruplum est 88 peccatum residui, & hujus pars octava 11 ad 2303 addita facit 2314, planè ut in priori processu invenimus 2316.

Idem etiam ex priori logar. ad 960, aliâ viâ, quia 1000 ad 960 est ut 9600 ad 9216, & cum logarithmus 960 sit 4082,2001, erit logar. ad 9216 duplus prioris scilicet 8164,4002. At verò proportio 9216 ad 9 est decupla proportionis 1000 ad 500. Ecce: 9216, 4608, 2304, 1152, 576, 288, 144, 72, 36, 18, 9, & proportio 9 ad 1 est dupla proportionis 3 ad 1. Ergò ad decuplum duplicationis

693147,1928 adde logar. ad numerum 92160,00
scilicet 8164,4002

701311,5930 aufer à logar. ad 10,00 summam
quæ est logar. ad 90,00 scilicet 921034,0563

Restat 219722,4633
Hujus dimid. 109861,2316 est triplicans ut prius.

Sic ex log. ad 990 (vel 99000,00) qui est 1005,0331, vel sine decuplicante, vel per eum perveniamus ad logarithmum ad 11 (vel 1100,00). Nam ad 98010,00 erit logarithmus duplus, scilicet 2010,0675. Hic additur ad quadruplum triplicantis,

439444,9256.

Hoc aufer à log.
ad 10,00

facit 441454,9931 log. ad 1210,00

921034,0563

Restat
Hujus dimid.
à logar. ad 1.

479579,0632
239789,5316 Undecuplat. aufer
690775,5422

Restat

450986,0106 log. ad 1100,00.

Sic quia ut 1000 ad 980, sic hic est ad 9604. Duplus igitur logarithmi ad 980 est logarithmus ad 9604, scil. 4040,5422. Et verò proportio 9604 ad 2401 est dupla proportionis 1000 ad 500, & proportio 24010,00 ad 10,00 est

quadrupla proportionis 70,00 ad 10,00. Adde igitur duplum duplicantis,

	scilicet	138629,4386
	ad	4040,5422
Summam	142669,9808 log. ad 2401	
Aufer à log. 10,00000	921034,0563	
Residui	778364,0775	
Pars quarta	194591,0194 septupl.	
Hoc igitur aufer à log. ad 100	230258,5141	
Restat logar. ad 700	35667,4947	
vel 70000,00		

Atque hic idem ad 700 per continuam bisectionem proportionis 10,7, per maximam tricesimarum proportionalium exactissimi tantus prodiit. Vide hanc suprà in tabellâ. Sic quia ut 1000 ad 950 sic 9500 est ad 9025. Ergò logarithmus ad 9025 est illius duplus, scilicet 10258,6606,

Adde duplum quinduplicantis sc. 321887,5896

	Confurgit	332146,2502 log. ad 3610,00
Hunc aufer à log. ad 10,00		921034,0563

	Residui	588887,8061
	Dimidium	294443,9030 novemdecuplat.
Aufer id à logar. ad 1		690775,5422
vel ad 100,00		

Restat 396331,6392 log. ad 1900,00.

Eundem derivabimus etiam ex 912, cujus logar. 9211,5306, & quia proportio 912 ad 57 est quadrupla proportionis 1000 ad 500. Duplicantis adde quadruplum, scilicet 277258,8771. Confurgit 286470,4077. Huic adde tripl-

109861,2316

Restat 396331,6393 log. ad 1900,00.

Eundem ex 950 derivabimus aliâ viâ :

Logar. ad 95000,00	5129,3303
Decuplans	230258,5141

Logar. ad 9500,00	235387,8444
Quinduplicans	160943,7948

Logar. ad 1900,00	396331,6392.
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Sic

Sic ad logar. ad 988	1207,2583	
Adde duplicantis duplum	138629,4386	
Venit logar. ad 247-----	139836,6969	
Novemdecuplicans addatur	294443,9030	
Prodit logar. ad 13 -----	434280,5999	
Auferatur logarith. ad 1	690775,5422	
Restat	256494,9423	tridecuplans.
Sic ad logar. ad 969	3149,0672	
Adde triplicantem	109861,2316	
Venit log. ad 323 -----	113010,2988	
Adde novemdecuplum	294443,9030	
Venit log. ad 17 -----	407454,2018	aufer hunc à logarith.
ad 1	690775,5422	
Restat septemdecuplans	283321,3404	
Sic ad logar. 986	1409,8927	
Adde duplicantem	69314,7193	
Et septemdecuplum	283321,3404	
Venit logar. ad 29-----	354045,9524	auferatur
à logar. 1	690775,5422	
Restat	336729,5898	undetriginduplans.
Et ad log. ad 966	3459,1450	
Adde duplicantem	69314,7193	
Et triplicantem	109861,2316	
Et septuplicantem	194591,0194	
Venit log. ad 23 -----	377226,1153	qui ablatus
à logar. ad 10,00	690775,5422	
Relinquit	313549,4269	viginticuplans.
Eundem ex log. 920. Quia ut 1000 ad 920 sic 9200 ad 8464, ergò		
Logar. ad 920 duplum	16676,3247	
Adde ad quadruplum duplicantis	277258,8771	
Confurgit	293935,2018	aufer
à logar. 1 de 100,00	921034,0563	
Restat	627098,8545	
Dimidium	313549,4272	
Eundem ex logar. ad 920 jam abundanti sic extruo:		
Ad logar. ad 920	8338,1624	
Adde duplicantis duplum	138629,4386	
Et decuplicantem	230258,5141	
Venit logar. ad 23	377226,1151	

Sic ad logar. ad 930	7257,0706 majusculum
Adde triplicantem	109861,2316
Et decuplicantem	230258,5141
Venit logar. ad 31 - - -	347376,8163 majusculus autem
	8149 verior
à logar. ad 1	690775,5422
	343398,7259 undetrigintuplicans mi-
	nusculus.

Eundem ex log. ad 961 elicio sic. Nam quia ut 961 ad 31 sic hoc ad 1. Ergò,

Aufer logarithmum ad 961	3978,0876
à logar. ad 1	69077,5422
Restat	686797,4546
Hujus dimid.	343398,7273 undetrigintupla hic est
	justior.

Jam quia intra millenarium quadratus major non est quàm 961, non igitur opus nobis erit multiplicationibus aliis, secundum primos majores ipso 31. Nam omnis primi majoris multiplex infra 1000 secundum aliquem numerum minorem, quàm 31 est multiplex: ut 979 est undecuplus primi 89.

Ubi notandum, quod logarithmus alicujus proportionis multiplicis sit differentia inter logarithmum unitatis in chiliade, & inter logarithmum numeri, qui multiplicitate proportionis prodit. Is igitur logarithmus multiplicator additus logarithmo cujusque numeri, constituit logarithmum partis, abstractus logarithmum multiplicis. Idem verum est de logarithmis proportionum non multiplicium, qui sunt nihil aliud quàm differentiae, vel multiplicantium logarithmorum, vel appositorum ad numeros multiplicium denominatores: ut eadem est differentia inter septuplicantem & triplicantem, quæ est inter logar. ad 7 & logar. ad 3. Hæc igitur addita logarithmo numeri tertii constituit logarithmum numeri proportionalis tertio minoris, ablata constituit logar. numeri proportionalis tertio majoris.

Hoc modo plerorumque primorum infra 500 logarithmi eruuntur ex uno centum positorum: nam in primo centenariio nullus superest, in secundo soli quinque 127, 149, 167, 173, 179, in tertio undecim 211, 223, 251, 257, 263, 269, 271, 277, 281, 283, 293, in quarto undecim 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397, in quinto novem 401, 409, 419, 421, 431, 433, 439, 443, 449. Summa 36.

Restant ii qui sunt inter 500 & 900, primi 59 numero, prætereaque etiam multiplices jam expressorum triginta sex totidem scilicet dupli; ex iis minorum sedecim tripli, primorum septem quadrupli, & ex his priorum quinque quintupli, duorum 127 & 149 sextupli, unius 127 septuplus.

Ut igitur etiam ad hos, & ad primos supra 500 numero 59 logarithmi habeantur, considerandæ sunt differentiae logarithmorum positorum ad interspersos hisce: & quâ methodo differentiae prius sunt constitutæ logarithmorum 100 minorum serie continua, eadem nunc etiam per prop. 27. coroll. 3, & maximè per ejus appendicem, si quis ea dextrè utatur, interrupta serie, & quidem longè facilius hæ differentiae sunt supplendæ, quia plerunque vel unus solus, vel bini deinceps, rarò tres deinceps logarithmis suis carebunt, ita ut in accumulatione differentiarum crebrò reversio fiat, ad logarithmum jam antea certum.

C H I L I A S
L O G A R I T H M O R U M

JOH. KEPLERI, MATHEM. CÆSAREI.

CHITIN

FOR THE MONTH

JOE. REED. MATH. CHARTER

Arcus Circuli cum diffe- rentiis.	Sinus seu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
	1		1611809,60	
			69314,72	
	2		1542494,88	
			40546,51	
	3		1501948,37 —	
			28768,21	
	4		1473180,16	
			22314,35	
	5		1450865,80 +	
			18232,16	
	6		1432633,65	
			15415,07	
	7		1417218,58	
			13353,14	
	8		1403865,44	
			11778,30	
	9		1392087,14 —	
			10536,05	
	10		1381551,08 +	
			69314,72	
	20		1312236,37	
			40546,51	
	30		1271689,85 +	
			28768,21	
	40		1242921,65 —	
			22314,35	
	50		1220607,29	
			18232,16	
	60		1202375,13	
			15415,07	
	70		1186960,07 —	
			13353,14	
	80		1173606,93 —	
			11778,30	
	90		1161828,62	
			10536,05	
	1,00		1151292,57	
			69314,72	
	2,00		1081977,85	
			40546,51	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicefimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
	3,00		1041431,34 28768,21	
	4,00		1012663,13 22314,35	
	5,00		990348,78 — 18232,16	
	6,00		972116,62 15415,08	
	7,00		956701,55 13353,14	
	8,00		943348,41 + 11778,30	
	9,00		931570,11 — 10536,05	
	10,00		921034,06 — 69314,72	
	20,00		851719,34 — 40546,51	
	30,00		811172,82 + 28768,21	
	40,00		782404,62 22314,35	
	50,00		760090,26 18232,16	
	60,00		741858,11 — 15415,07	
	70,00		726443,04 — 13353,14	
	80,00		713089,90 11778,30	
	90,00		701311,59 — 10536,05	
o 3 26	100,00	o 1 26	690775,54	o 4
o 3 27			69314,72	
o 6 53	200,00	o 2 53	621460,82 +	o 7
o 3 26			40546,51	
o 10 19	300,00	o 4 19	580914,31	o 11
o 3 26			28768,21	
o 13 45	400,00	o 5 46	552146,10 +	o 14
o 3 27			22314,35	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicefimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
3 27 o 17 12 3 26	500,00	o 7 12	22314,35 529831,75 - 18232,16	o 18
o 20 38 3 26	600,00	o 8 38	511599,59 15415,07	o 22
o 24 4 3 26	700,00	o 10 5	496184,52 + 13353,14	o 25
o 27 30 3 26	800,00	o 11 31	482831,38 + 11778,30	o 29
o 30 56 3 27	900,00	o 12 58	471053,08 10536,05	o 32
o 34 23 3 26	1000,00	o 14 24	460517,03 9531,02	o 36
o 37 49 3 26	1100,00	o 15 50	450986,01 8701,14	o 40
o 41 15 3 27	1200,00	o 17 17	442284,87 8004,27	o 43
o 44 42 3 26	1300,00	o 18 43	434280,60 7410,80	o 47
o 48 8 3 26	1400,00	o 20 10	426869,80 + 6899,27	o 50
o 51 34 3 26	1500,00	o 21 36	419970,52 - 6453,86	o 54
o 55 0 3 27	1600,00	o 23 2	413516,67 - 6062,47	o 58
o 58 27 3 26	1700,00	o 24 29	407454,20 5715,84	I 1
I 1 53 3 27	1800,00	o 25 55	401738,36 5406,72	I 5
I 5 20 3 26	1900,00	o 27 22	396331,64 5129,33	I 8
I 8 46 3 26	2000,00	o 28 48	391202,31 4879,02	I 12
I 12 12 3 26	2100,00	o 30 14	386323,29 4652,00	I 16
I 15 38 3 27	2200,00	o 31 41	381679,29 4445,17	I 19
I 19 5 3 26	2300,00	o 33 7	377226,12 - 4255,97	I 23
I 22 31 3 26	2400,00	o 34 34	372970,15 + 4082,20	I 26

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicefimæ quartæ.	Logarithmi cum differentiis.	Partes fexa- genariæ.
3 26			4082,20	
I 25 57	2500,00	0 36 0	368887,95 +	I 30
3 27			3922,07	
I 29 24	2600,00	0 37 26	364965,88	I 34
3 26			3774,03	
I 32 50	2700,00	0 38 53	361191,85 -	I 37
3 26			3636,77	
I 36 16	2800,00	0 40 19	357555,08 +	I 41
3 27			3509,13	
I 39 43	2900,00	0 41 46	354045,95	I 44
3 26			3390,15	
I 43 9	3000,00	0 43 12	350655,80 -	I 48
2 26			3278,99	
I 46 35	3100,00	0 44 38	347376,81 +	I 52
3 27			3174,86	
I 50 2	3200,00	0 46 5	344201,95 -	I 55
3 26			3077,17	
I 53 28	3300,00	0 47 31	341124,78	I 59
3 27			2985,29	
I 56 55	3400,00	0 48 58	338139,49 +	2 2
3 26			2898,76	
2 0 21	3500,00	0 50 24	335240,73	2 6
3 26			2817,08	
2 3 47	3600,00	0 51 50	332423,65 -	2 10
3 26			2739,90	
2 7 13	3700,00	0 53 17	329683,75 -	2 13
3 27			2666,83	
2 10 40	3800,00	0 54 43	327016,92	2 17
3 27			2597,55	
2 14 7	3900,00	0 56 10	324419,37 +	2 20
3 26			2531,78	
2 17 33	4000,00	0 57 36	321887,59	2 24
3 26			2469,26	
2 20 59	4100,00	0 59 2	319418,33	2 28
3 27			2409,75	
2 24 26	4200,00	I 0 29	317008,58 -	2 31
3 26			2353,05	
2 27 52	4300,00	I 1 55	314655,53 -	2 35
3 27			2298,95	
2 31 19	4400,00	I 3 22	312356,58 -	2 38
3 26			2247,29	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
3 26			2247,29	
2 34 45	4500,00	I 4 48	310109,29 —	2 42
3 27			2197,89	
2 38 12	4600,00	I 6 14	307911,40 —	2 46
3 26			2150,62	
2 41 38	4700,00	I 7 41	305760,78 —	2 49
3 26			2105,34	
2 45 4	4800,00	I 9 7	303655,44 —	2 53
3 27			2061,93	
2 48 31	4900,00	I 10 34	301593,51 —	2 56
3 27			2020,27	
2 51 58	5000,00	I 12 0	299573,24 —	3 0
3 27			1980,27	
2 55 25	5100,00	I 13 26	297592,97 +	3 4
3 26			1941,81	
2 58 51	5200,00	I 14 53	295651,16 +	3 7
3 27			1904,81	
3 2 18	5300,00	I 16 19	293746,35 —	3 11
3 27			1869,22	
3 5 45	5400,00	I 17 46	291877,13 +	3 14
3 26			1834,91	
3 9 11	5500,00	I 19 12	290042,22 —	3 18
3 27			1801,85	
3 12 38	5600,00	I 20 38	288240,37	3 22
3 26			1769,96	
3 16 4	5700,00	I 22 5	286470,41	3 25
3 27			1739,17	
3 19 31	5800,00	I 23 31	284731,24 —	3 29
3 27			1709,45	
3 22 58	5900,00	I 24 58	283021,79	3 32
3 26			1680,71	
3 26 24	6000,00	I 26 24	281341,08	3 36
3 27			1652,93	
3 29 51	6100,00	I 27 50	279688,15	3 40
3 26			1626,05	
3 33 17	6200,00	I 30 17	278062,10 —	3 43
3 27			1600,04	
3 36 44	6300,00	I 39 43	276462,06 +	3 47
3 27			1574,83	
3 40 11	6400,00	I 32 10	274887,23	3 50
3 26			1550,42	

Arcus Circuli cum differ- entiis.	Sinus feu Nu- meri absoluti.	Partes vicefimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
3 26			1550,42	
3 43 37	6500,00	I 33 36	273336,81 -	3 54
3 27			1526,75	
3 47 4	6600,00	I 35 2	271810,06	3 58
3 26			1503,79	
3 50 30	6700,00	I 36 29	270306,27 +	4 1
3 27			1481,50	
3 53 57	6800,00	I 37 55	268824,77 +	4 5
3 27			1459,88	
3 57 24	6900,00	I 39 22	267364,89	4 8
3 26			1438,88	
4 0 50	7000,00	I 40 48	265926,01	4 12
3 27			1418,46	
4 4 17	7100,00	I 42 14	264507,55	4 16
3 27			1398,62	
4 7 44	7200,00	I 43 41	263108,93 -	4 19
3 26			1379,33	
4 11 10	7300,00	I 45 7	261729,60 -	4 23
3 27			1360,57	
4 14 37	7400,00	I 46 34	260369,03 -	4 26
3 27			1342,31	
4 18 4	7500,00	I 48 0	259026,72 +	4 30
3 27			1324,52	
4 21 31	7600,00	I 49 26	257702,20	4 34
3 26			1307,21	
4 24 57	7700,00	I 50 53	256394,99 +	4 37
3 27			1290,34	
4 28 24	7800,00	I 52 19	255104,65 +	4 41
3 27			1273,90	
4 31 51	7900,00	I 53 46	253830,75	4 44
3 27			1257,88	
4 35 18	8000,00	I 55 12	252572,87	4 48
3 27			1242,25	
4 38 45	8100,00	I 56 38	251330,62	4 52
3 27			1227,01	
4 42 12	8200,00	I 58 5	250103,61	4 55
3 26			1212,14	
4 45 38	8300,00	I 59 31	248891,47 +	4 59
3 27			1197,61	
4 49 5	8400,00	2 0 58	247693,86 -	5 2
3 27			1183,45	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
3 27			1183,45	
4 52 32	8500,00	2 2 24	246510,41	5 6
3 27			1169,60	
4 55 59	8600,00	2 3 50	245340,81 -	5 10
3 27			1156,09	
4 59 26	8700,00	2 5 17	244184,72 +	5 13
3 27			1142,86	
5 2 53	8800,00	2 6 43	243041,86 -	5 17
3 27			1129,96	
5 6 20	8900,00	2 8 10	241911,90	5 20
3 28			1117,33	
5 9 48	9000,00	2 9 36	240794,57 -	5 24
2 27			1104,99	
5 13 15	9100,00	2 11 2	239689,58 +	5 28
3 27			1092,90	
5 16 42	9200,00	2 12 29	238596,68	5 31
3 27			1081,09	
5 20 9	9300,00	2 13 55	237515,59	5 35
3 27			1069,53	
5 23 36	9400,00	2 15 22	236446,06 -	5 38
3 27			1058,21	
5 27 3	9500,00	2 16 48	235387,85 -	5 42
3 28			1047,13	
5 30 31	9600,00	2 18 14	234340,72 -	5 46
3 27			1036,28	
5 33 58	9700,00	2 19 41	233304,44 -	5 49
3 27			1025,65	
5 37 25	9800,00	2 21 7	232278,79 -	5 53
3 27			1015,24	
5 40 52	9900,00	2 22 34	231263,55 -	5 56
3 28			1005,03	
5 44 20	10000,00	2 24 0	230258,52 -	6 0
3 27			995,04	
5 47 47	10100,00	2 25 26	229263,48 +	6 4
3 27			985,23	
5 51 14	10200,00	2 26 53	228278,25 +	6 7
3 28			976,61	
5 54 55	10300,00	2 28 19	227302,64	6 11
3 27			966,20	
5 58 22	10400,00	2 29 46	226336,44 +	6 14
3 28			956,94	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicefimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
3 28			956,94	
6 1 50	10500,00	2 31 12	225379,50	6 18
3 27			947,87	
6 5 17	10600,00	2 32 38	224431,63 —	6 22
3 28			938,98	
6 8 45	10700,00	2 34 5	223492,65 +	6 25
3 27			930,24	
6 12 12	10800,00	2 35 31	222562,41 +	6 29
3 28			921,66	
6 15 40	10900,00	2 36 58	221640,75	6 32
3 28			913,25	
6 18 55	11000,00	2 38 24	220727,50 —	6 36
3 27			904,99	
6 22 22	11100,00	2 39 50	219822,51 +	6 40
3 28			896,86	
6 25 50	11200,00	2 41 17	218925,65	6 43
3 28			888,90	
6 29 18	11300,00	2 42 43	218036,75	6 47
3 27			881,06	
6 32 45	11400,00	2 44 10	217155,69	6 50
3 28			873,37	
6 36 13	11500,00	2 45 36	216282,32	6 54
3 28			865,80	
6 39 41	11600,00	2 47 2	215416,52 —	6 58
3 27			858,38	
6 43 8	11700,00	2 48 29	214558,14	7 1
3 28			851,07	
6 46 36	11800,00	2 49 55	213707,07	7 5
3 28			843,88	
6 50 4	11900,00	2 51 22	212863,19 —	7 8
3 27			836,83	
6 53 31	12000,00	2 52 48	212026,36	7 12
3 28			829,88	
6 56 59	12100,00	2 54 14	211196,48	7 16
3 28			823,05	
7 0 27	12200,00	2 55 41	210373,43	7 19
3 28			816,33	
7 3 55	12300,00	2 57 7	209557,10	7 23
3 27			809,72	
7 7 22	12400,00	2 58 34	208747,38 —	7 26
3 28			803,22	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
3 28			803,22	
7 10 50	12500,00	3 0 0	207944,16	7 30
3 28			796,82	
7 14 18	12600,00	3 1 26	207147,34 +	7 34
3 28			790,51	
7 17 46	12700,00	3 2 53	206356,83 -	7 37
3 28			784,32	
7 21 14	12800,00	3 4 19	205572,51	7 41
3 28			778,22	
7 24 42	12900,00	3 5 46	204794,29 +	7 44
3 28			772,20	
7 28 10	13000,00	3 7 12	204022,09 -	7 48
3 29			766,29	
7 31 39	13100,00	3 8 38	203255,80	7 52
3 28			760,46	
7 35 7	13200,00	3 10 5	202495,34	7 55
3 28			754,72	
7 38 35	13300,00	3 11 31	201740,62	7 59
3 28			749,07	
7 42 3	13400,00	3 12 58	200991,55 +	8 2
3 28			743,50	
7 45 31	13500,00	3 14 24	200248,05 +	8 6
3 28			738,00	
7 48 59	13600,00	3 15 50	199510,05 +	8 10
3 29			732,61	
7 22 28	13700,00	3 17 17	198777,44	8 13
3 29			727,27	
7 55 56	13800,00	3 18 43	198050,17	8 17
3 28			722,03	
7 59 24	13900,00	3 20 10	197328,14	8 20
3 28			716,85	
8 2 52	14000,00	3 21 36	196611,29	8 24
3 28			711,74	
8 6 20	14100,00	3 23 2	195899,55 -	8 28
3 29			706,72	
8 9 49	14200,00	3 24 29	195192,83	8 31
3 28			701,76	
8 13 17	14300,00	3 25 55	194491,07 +	8 35
3 28			696,86	
8 16 45	14400,00	3 27 22	193794,21 -	8 38
3 29			692,05	

Arcus Circuli cum diffe- rentiis.	Sinus seu Nu- meri absoluti.	Partes vicefimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
3 29			692,05	
8 20 14	14500,00	3 28 48	193102,16	8 42
3 28			687,28	
8 23 42	14600,00	3 30 14	192414,88 —	8 46
3 28			682,61	
8 27 10	14700,00	3 31 41	191732,27 +	8 49
3 29			677,96	
8 30 39	14800,00	3 33 7	191054,31 —	8 53
3 28			673,41	
8 34 7	14900,00	3 34 34	190380,90 +	8 56
3 29			668,90	
8 37 36	15000,00	3 36 0	189712,00 +	9 0
3 29			664,45	
8 41 5	15100,00	3 37 26	189047,55	9 4
3 28			660,07	
8 44 33	15200,00	3 38 53	188387,48	9 7
3 29			655,74	
8 48 2	15300,00	3 40 19	187731,74	9 11
3 29			651,47	
8 51 31	15400,00	3 41 46	187080,27 +	9 14
3 28			647,25	
8 54 59	15500,00	3 43 12	186433,02 +	9 18
3 29			643,09	
8 58 28	15600,00	3 44 38	185789,93 +	9 22
3 29			638,98	
9 1 57	15700,00	3 46 5	185150,95 +	9 25
3 29			634,92	
9 5 26	15800,00	3 47 31	184516,03	9 29
3 29			630,92	
9 8 55	15900,00	3 48 58	183885,11 +	9 32
3 30			626,96	
9 12 25	16000,00	3 50 24	183258,15	9 36
3 29			623,06	
9 15 54	16100,00	3 51 50	182635,09 +	9 40
3 29			619,19	
9 19 23	16200,00	3 53 17	182015,90	9 43
3 29			615,38	
9 22 52	16300,00	3 54 43	181400,52	9 47
3 29			611,63	
9 26 21	16400,00	3 56 10	180788,89	9 50
3 30			607,90	

Arcus Circuli cum differ- entiis.	Sinus feu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
3 30			607,90	
9 29 51	16500,00	3 57 36	180180,99 —	9 54
3 29			604,24	
9 33 20	16600,00	3 58 2	179576,75 +	9 58
3 29			600,60	
9 36 49	16700,00	4 0 29	178976,15 +	10 1
3 29			597,01	
9 40 18	16800,00	4 1 55	178379,14 —	10 5
3 29			593,48	
9 43 47	16900,00	4 3 22	177785,66	10 8
3 29			589,97	
9 47 16	17000,00	4 4 48	177195,69	10 12
3 30			586,51	
9 50 46	17100,00	4 6 14	176609,18 —	10 16
3 29			583,09	
9 54 15	17200,00	4 7 41	176026,09 —	10 19
3 29			579,72	
9 57 44	17300,00	4 9 7	175446,37	10 23
3 29			576,37	
10 1 13	17400,00	4 10 34	174870,00 +	10 26
3 30			573,07	
10 4 43	17500,00	4 12 0	174296,93 +	10 30
3 29			569,79	
10 8 12	17600,00	4 13 26	173727,14 —	10 34
3 30			566,58	
10 11 42	17700,00	4 14 53	173160,56	10 37
3 30			563,38	
10 15 12	17800,00	4 16 19	172597,18	10 41
3 30			560,23	
10 18 42	17900,00	4 17 46	172036,95	10 44
3 29			557,10	
10 22 11	18000,00	4 19 12	171479,85 —	10 48
3 30			554,02	
10 25 41	18100,00	4 20 38	170925,83	10 52
3 30			550,97	
10 29 11	18200,00	4 22 5	170374,86 +	10 55
3 30			547,94	
10 32 41	18300,00	4 23 31	169826,92 —	10 59
3 30			544,96	
10 36 11	18400,00	4 24 58	169281,96 —	11 2
3 29			542,01	

Arcus Circuli cum diffe- rentiis.	Sinus seu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
3 29			542,01	
IO 39 40	18500,00	4 26 24	168739,95	II 6
3 30			539,08	
IO 43 10	18600,00	4 27 50	168200,87	II 10
3 30			536,20	
IO 46 40	18700,00	4 29 17	167664,67	II 13
3 30			533,33	
IO 50 10	18800,00	4 30 43	167131,34	II 17
3 30			530,51	
IO 53 40	18900,00	4 32 10	166600,83	II 20
3 30			527,70	
IO 57 10	19000,00	4 33 36	166073,13 —	II 24
3 31			524,94	
II 0 41	19100,00	4 35 2	165548,19	II 28
3 30			522,19	
II 4 11	19200,00	4 36 29	165026,00 —	II 31
3 30			519,49	
II 7 41	19300,00	4 37 55	164506,51 +	II 35
3 30			516,79	
II 11 11	19400,00	4 39 22	163989,72 —	II 38
3 30			514,15	
II 14 41	19500,00	4 40 48	163475,57 +	II 42
3 31			511,50	
II 18 12	19600,00	4 42 14	162964,07 —	II 46
3 30			508,91	
II 21 42	19700,00	4 43 41	162455,16	II 49
3 30			506,33	
II 25 12	19800,00	4 45 7	161948,83 —	II 53
3 30			503,78	
II 28 42	19900,00	4 46 34	161445,05	II 56
3 31			501,25	
II 32 13	20000,00	4 48 0	160943,80 —	12 0
3 30			498,76	
II 35 43	20100,00	4 49 26	160445,04 +	12 4
3 31			496,28	
II 39 14	20200,00	4 50 53	159948,76 +	12 7
3 31			493,83	
II 42 45	20300,00	4 52 19	159454,93 +	12 11
3 30			491,40	
II 46 15	20400,00	4 53 46	158963,53 +	12 14
3 31			489,00	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
3 31			489,00	
11 49 46	20500,00	4 55 12	158474,53 +	12 18
3 31			486,61	
11 53 17	20600,00	4 56 38	157987,92	12 22
3 31			484,26	
11 56 48	20700,00	4 58 5	157503,66 -	12 25
3 31			481,94	
12 0 19	20800,00	4 59 31	157021,72 +	12 29
3 30			479,61	
12 3 49	20900,00	5 0 58	156542,11 -	12 32
3 31			477,33	
12 7 20	21000,00	5 2 24	156064,78	12 36
3 31			475,06	
12 10 51	21100,00	5 3 50	155589,72 -	12 40
3 31			472,81	
12 14 22	21200,00	5 5 17	155116,91 -	12 43
3 31			470,59	
12 17 53	21300,00	5 6 43	154646,32 -	12 47
3 31			468,39	
12 21 24	21400,00	5 8 10	154177,93 +	12 50
3 31			466,20	
12 24 55	21500,00	5 9 36	153711,73	12 54
3 32			464,04	
12 28 27	21600,00	5 11 2	153247,69 +	12 58
3 31			461,89	
12 31 58	21700,00	5 12 29	152785,80	13 1
3 31			459,77	
12 35 29	21800,00	5 13 55	152326,03	13 5
3 31			457,67	
12 39 0	21900,00	5 15 22	151868,36	13 8
3 32			455,58	
12 42 32	22000,00	5 16 48	151412,78 -	13 12
3 31			453,52	
12 46 3	22100,00	5 18 14	150959,26	13 16
3 31			451,47	
12 49 34	22200,00	5 19 41	150507,79 +	13 19
3 32			449,43	
12 53 6	22300,00	5 21 7	150058,36 -	13 23
3 31			446,43	
12 56 37	22400,00	5 22 34	149610,93	13 26
3 32			445,44	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicefimæ quartæ.	Logarithmi cum differentiis.	Partes fexa- genariæ.
13 3 32			445,44	
13 0 9	22500,00	5 24 0	149165,49	13 30
13 3 32			443,46	
13 3 41	22600,00	5 25 26	148722,03	13 34
13 3 32			441,50	
13 7 13	22700,00	5 26 53	148280,53	13 37
13 3 32			439,56	
13 10 45	22800,00	5 28 19	147840,97	13 41
13 3 32			437,64	
13 14 17	22900,00	5 29 46	147403,33 +	13 44
13 3 33			435,73	
13 17 50	23000,00	5 31 12	146967,60	13 48
13 3 32			433,84	
13 21 22	23100,00	5 32 38	146533,76	13 52
13 3 32			431,96	
13 24 54	23200,00	5 34 5	146101,80 —	13 55
13 3 32			430,11	
13 28 26	23300,00	5 35 31	145671,69	13 59
13 3 32			428,27	
13 31 58	23400,00	5 36 58	145243,42	14 2
13 3 33			426,44	
13 35 31	23500,00	5 38 24	144816,98	14 6
13 3 32			424,63	
13 39 3	23600,00	5 39 50	144392,35	14 10
13 3 32			422,83	
13 42 35	23700,00	5 41 17	143969,52	14 13
13 3 32			421,05	
13 46 7	23800,00	5 42 43	143548,47 —	14 17
13 3 33			419,29	
13 49 40	23900,00	5 44 10	143129,18 —	14 20
13 3 32			418,54	
13 53 12	24000,00	5 45 36	142711,64	14 24
13 3 32			415,80	
13 56 44	24100,00	5 47 2	142295,84	14 28
14 3 33			414,08	
14 0 17	24200,00	5 48 29	141881,76	14 31
14 3 33			412,37	
14 3 50	24300,00	5 49 55	141469,39	14 35
14 3 32			410,68	
14 7 22	24400,00	5 51 22	141058,71	14 38
14 3 33			409,00	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
3 33			409,00	
14 10 55	24500,00	5 52 48	140649,71	14 42
3 33			407,33	
14 14 28	24600,00	5 54 14	140242,38	14 46
3 33			405,68	
14 18 1	24700,00	5 55 41	139836,70	14 49
3 33			404,04	
14 21 34	24800,00	5 57 7	139432,66 —	14 53
3 32			402,42	
14 25 6	24900,00	5 58 34	139030,24	14 56
3 33			400,80	
14 28 39	25000,00	6 0 0	138629,44	15 0
3 33			399,20	
14 32 12	25100,00	6 1 26	138230,24 —	15 4
3 33			397,62	
14 35 45	25200,00	6 2 53	137832,62 +	15 7
3 33			396,04	
14 39 18	25300,00	6 4 19	137436,58 +	15 11
3 33			394,47	
14 42 51	25400,00	6 5 46	137042,11 —	15 14
3 34			392,93	
14 46 25	25500,00	6 7 12	136649,18 —	15 18
3 33			391,39	
14 49 58	25600,00	6 8 38	136257,79	15 22
3 33			389,87	
14 53 31	25700,00	6 10 5	135867,92 +	15 25
3 34			388,35	
14 57 5	25800,00	6 11 31	135479,57 +	15 29
3 33			386,84	
15 0 38	25900,00	6 12 58	135092,73 —	15 32
3 34			385,36	
15 4 12	26000,00	6 14 24	134707,37 —	15 36
3 33			383,88	
15 7 45	26100,00	6 15 50	134323,49	15 40
3 34			382,41	
15 11 19	26200,00	6 17 17	133941,08	15 43
3 34			380,95	
15 14 53	26300,00	6 18 43	133560,13	15 47
3 33			379,51	
15 18 26	26400,00	6 20 10	133180,62	15 50
3 34			378,07	

Arcus Circuli cum differ- entiis.	Sinus seu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
3 34			378,07	
15 22 0	26500,00	6 21 36	132802,55	15 54
3 34			376,65	
15 25 34	26600,00	6 23 2	132425,90	15 58
3 35			375,23	
15 29 9	26700,00	6 24 29	132050,67 —	16 1
3 34			373,84	
15 32 43	26800,00	6 25 55	131676,83 +	16 5
3 34			372,43	
15 36 17	26900,00	6 27 22	131304,40	16 8
3 35			371,07	
15 39 52	27000,00	6 28 48	130933,33 +	16 12
3 34			369,67	
15 43 26	27100,00	6 30 14	130563,66 —	16 16
3 34			368,33	
15 47 0	27200,00	6 31 41	130195,33 —	16 19
3 35			366,98	
15 50 35	27300,00	6 33 7	129828,55	16 23
3 34			365,63	
15 54 9	27400,00	6 34 34	129462,72	16 26
3 35			364,30	
15 57 44	27500,00	6 36 0	129098,42 +	16 30
3 34			362,97	
16 1 18	27600,00	6 37 26	128735,45 —	16 34
3 35			361,67	
16 4 53	27700,00	6 38 53	128373,78	16 37
3 35			360,36	
16 8 28	27800,00	6 40 19	128013,42	16 41
3 34			359,07	
16 12 2	27900,00	6 41 46	127654,35	16 44
3 35			357,78	
16 15 37	28000,00	6 43 12	127296,57	16 48
3 35			356,51	
16 19 12	28100,00	6 44 38	126940,06 +	16 52
3 35			355,23	
16 22 47	28200,00	6 46 5	126584,83 —	16 55
3 35			353,98	
16 26 22	28300,00	6 47 31	126230,85 +	16 59
3 35			352,74	
16 29 57	28400,00	6 48 58	125878,11	17 2
3 36			351,50	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
16 3 36	28500,00	6 50 24	351,50 125526,61 +	17 6
16 3 35	28600,00	6 51 50	350,26 125176,35 +	17 10
16 3 35	28700,00	6 53 17	349,04 124827,31	17 13
16 3 35	28800,00	6 54 43	347,82 124479,49 —	17 17
16 3 36	28900,00	6 56 10	346,63 124132,86	17 20
16 3 35	29000,00	6 57 36	345,42 123787,44	17 24
16 3 35	29100,00	6 59 2	344,24 123443,20 +	17 28
16 3 36	29200,00	7 0 29	343,04 123100,16 —	17 31
17 3 35	29300,00	7 1 55	341,89 122758,27	17 35
17 3 36	29400,00	7 3 22	340,72 122417,55 +	17 38
17 3 36	29500,00	7 4 48	339,55 122078,00 —	17 42
17 3 35	29600,00	7 6 14	338,41 121739,59 —	17 46
17 3 36	29700,00	7 7 41	337,27 121402,32	17 49
17 3 36	29800,00	7 9 7	336,14 121066,18 +	17 53
17 3 36	29900,00	7 10 34	335,01 120731,17 +	17 56
17 3 37	30000,00	7 12 0	333,89 120397,28 +	18 0
17 3 36	30100,00	7 13 26	332,78 120064,50 +	18 4
17 3 36	30200,00	7 14 53	331,67 119732,83	18 7
17 3 37	30300,00	7 16 19	330,58 119402,25	18 11
17 3 36	30400,00	7 17 46	329,49 119072,76	18 14
17 3 37			328,41	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicefimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
3 37			328,41	
17 45 29	30500,00	7 19 12	118744,35 +	18 18
3 37			327,33	
17 49 6	30600,00	7 20 38	118417,02	18 22
3 36			326,26	
17 52 42	30700,00	7 22 5	118090,76 -	18 25
3 37			325,21	
17 56 19	30800,00	7 23 31	117765,55 +	18 29
3 37			324,15	
17 59 56	30900,00	7 24 58	117441,40 +	18 32
3 37			323,10	
18 3 33	31000,00	7 26 24	117118,30 +	18 36
3 37			322,06	
18 7 10	31100,00	7 27 50	116796,24	18 40
3 37			321,03	
18 10 47	31200,00	7 29 17	116475,21 +	18 43
3 37			320,00	
18 14 24	31300,00	7 30 43	116155,21 +	18 47
3 38			318,98	
18 18 2	31400,00	7 32 10	115836,23 +	18 50
3 37			317,96	
18 21 39	31500,00	7 33 36	115518,27	18 54
3 37			316,96	
18 25 16	31600,00	7 35 2	115201,31	18 58
3 37			315,96	
18 28 53	31700,00	7 36 29	114885,35 +	19 1
3 38			314,96	
18 32 31	31800,00	7 37 55	114570,39 +	19 5
3 37			313,97	
18 36 8	31900,00	7 39 22	114256,42 +	19 8
3 38			312,99	
18 39 46	32000,00	7 40 48	113943,43	19 12
3 38			312,01	
18 43 24	32100,00	7 42 14	113631,42	19 16
3 38			311,05	
18 47 2	32200,00	7 43 41	113320,37 +	19 19
3 38			310,07	
18 50 40	32300,00	7 45 7	113010,30	19 23
3 39			309,12	
18 54 19	32400,00	7 46 34	112701,18	19 26
3 38			308,17	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
18 3 38			308,17	
18 57 57	32500,00	7 48 0	112393,01	19 30
19 3 38			307,21	
19 1 35	32600,00	7 49 26	112085,80	19 34
19 3 38			306,29	
19 5 13	32700,00	7 50 53	111779,51 +	19 37
19 3 39			305,34	
19 8 52	32800,00	7 52 19	111474,17	19 41
19 3 38			304,41	
19 12 30	32900,00	7 53 46	111169,76 -	19 44
19 3 38			303,49	
19 16 8	33000,00	7 55 12	110866,27 -	19 48
19 3 39			302,58	
19 19 47	33100,00	7 56 38	110563,69 +	19 52
19 3 38			301,66	
19 23 25	33200,00	7 58 5	110262,03 +	19 55
19 3 39			300,75	
19 22 4	33300,00	7 59 31	109961,28	19 59
19 3 38			299,85	
19 30 42	33400,00	8 0 58	109661,43 +	20 2
19 3 39			298,95	
19 34 21	33500,00	8 2 24	109362,48	20 6
19 3 39			298,06	
19 38 0	33600,00	8 3 50	109064,42 -	20 10
19 3 39			297,18	
19 41 39	33700,00	8 5 17	108767,24 -	20 13
19 3 39			296,30	
19 45 18	33800,00	8 6 43	108470,94	20 17
19 3 40			295,42	
19 48 58	33900,00	8 8 10	108175,52	20 20
19 3 39			294,55	
19 52 37	34000,00	8 9 36	107880,97	20 24
19 3 39			293,69	
19 56 16	34100,00	8 11 2	107587,28 +	20 28
19 3 40			292,82	
19 59 56	34200,00	8 12 29	107294,46 -	20 31
20 3 39			291,97	
20 3 35	34300,00	8 13 55	107002,49 -	20 35
20 3 39			291,12	
20 7 14	34400,00	8 15 22	106711,37 -	20 38
20 3 40			290,28	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicefimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
3 40			290,28	
20 10 54	34500,00	8 10 48	106421,09	20 42
3 40			289,44	
20 14 34	34600,00	8 18 14	106131,65	20 46
3 39			288,60	
20 18 13	34700,00	8 19 41	105843,05 +	20 49
3 40			287,77	
20 21 53	34800,00	8 21 7	105555,28 +	20 53
3 40			286,94	
20 25 33	34900,00	8 22 34	105268,34	20 56
3 41			286,13	
20 29 14	35000,00	8 24 0	104982,21 +	21 0
3 40			285,30	
20 32 54	35100,00	8 25 26	104696,91	21 4
3 40			284,49	
20 36 34	35200,00	8 26 53	104412,42 -	21 7
3 40			283,70	
20 40 14	35300,00	8 28 19	104128,72 +	21 11
3 41			282,88	
20 43 55	35400,00	8 29 46	103845,84	21 14
3 40			282,09	
20 47 35	35500,00	8 31 12	103563,75	21 18
3 41			281,29	
20 51 16	35600,00	8 32 38	103282,46 -	21 22
3 41			280,50	
20 54 57	35700,00	8 34 5	103001,96 -	21 25
3 41			279,73	
20 58 38	35800,00	8 35 31	102722,23	21 29
3 42			278,94	
21 2 20	35900,00	8 36 58	102443,29	21 32
3 41			278,16	
21 6 1	36000,00	8 38 24	102165,13 -	21 36
3 41			277,39	
21 9 42	36100,00	8 39 50	101887,74 -	21 40
3 41			276,63	
21 13 23	36200,00	8 41 17	101611,11	21 43
3 42			275,86	
21 17 5	36300,00	8 42 43	101335,25	21 47
3 41			275,11	
21 20 46	36400,00	8 44 10	101060,14 +	21 50
3 42			274,34	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
3 42			274,34	
21 24 28	36500,00	8 45 36	100785,80 —	21 54
3 41			273,60	
21 28 9	36600,00	8 47 2	100512,20 —	21 58
3 42			272,86	
21 31 51	36700,00	8 48 29	100239,34	22 1
3 41			272,10	
21 35 32	36800,00	8 49 55	99967,24	22 5
3 42			271,38	
21 39 14	36900,00	8 51 22	99695,86 +	22 8
3 42			270,63	
21 42 56	37000,00	8 52 48	99425,23	22 12
3 41			269,90	
21 46 37	37100,00	8 54 14	99155,33 —	22 16
3 42			269,18	
21 50 19	37200,00	8 55 41	98886,15	22 19
3 42			268,46	
21 54 1	37300,00	8 57 7	98617,69	22 23
3 42			267,74	
21 57 43	37400,00	8 58 34	98349,95	22 26
3 43			267,02	
22 1 26	37500,00	9 0 0	98082,93 —	22 30
3 42			266,31	
22 5 8	37600,00	9 1 26	97816,62	22 34
3 43			265,61	
22 8 51	37700,00	9 2 53	97551,01	22 37
3 43			264,90	
22 12 34	37800,00	9 4 19	97286,11	22 41
3 43			264,20	
22 16 17	37900,00	9 5 46	97021,91	22 44
3 44			263,50	
22 20 1	38000,00	9 7 12	96758,41 —	22 48
3 43			262,82	
22 23 44	38100,00	9 8 38	96495,59 +	22 52
3 43			262,12	
22 27 27	38200,00	9 10 5	96233,47	22 55
3 44			261,44	
22 31 11	38300,00	9 11 31	95972,03	22 59
3 43			260,75	
22 34 54	38400,00	9 12 58	95711,28 —	23 2
3 44			260,08	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
22 38 38	38500,00	9 14 24	260,08 95451,20	23 6
22 42 21	38600,00	9 15 50	259,41 95191,79 +	23 10
22 46 5	38700,00	9 17 17	258,73 94933,06	23 13
22 49 49	38800,00	9 18 43	258,06 94675,00 -	23 17
22 53 32	38900,00	9 20 10	257,40 94417,60	23 20
22 57 16	39000,00	9 21 36	256,75 94160,85 +	23 24
23 1 0	39100,00	9 23 2	256,07 93904,78 -	23 28
23 4 44	39200,00	9 24 29	255,43 93649,35 -	23 31
23 8 29	39300,00	9 25 55	254,78 93394,57	23 35
23 12 13	39400,00	9 27 22	254,13 93140,44	23 38
23 15 58	39500,00	9 28 48	253,49 92886,95 +	23 42
23 19 43	39600,00	9 30 14	252,84 92634,11 -	23 46
23 23 28	39700,00	9 31 41	252,21 92381,90	23 49
23 27 12	39800,00	9 33 7	251,57 92130,33	23 53
23 30 57	39900,00	9 34 34	250,94 91879,39	23 56
23 34 42	40000,00	9 36 0	250,31 91629,08 -	24 0
23 38 27	40100,00	9 37 26	249,69 91379,39	24 4
23 42 12	40200,00	9 38 53	249,07 91130,32 +	24 7
23 45 58	40300,00	9 40 19	248,45 90881,87	24 11
23 49 43	40400,00	9 41 46	247,83 90634,04 +	24 14
			247,22	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
3 46			247,22	
23 53 29	40500,00	9 43 12	90386,82 +	24 18
3 46			246,61	
23 57 15	40600,00	9 44 38	90140,21 +	24 22
3 46			246,00	
24 1 1	40700,00	9 46 5	89894,21 +	24 25
3 45			245,40	
24 4 46	40800,00	9 47 31	89648,81 +	24 29
3 46			244,79	
24 8 32	40900,00	9 48 58	89404,02 -	24 32
3 46			244,21	
24 12 18	41000,00	9 50 24	89159,81 +	24 36
3 46			243,60	
24 16 4	41100,00	9 51 50	88916,21	24 40
3 47			243,01	
24 19 51	41200,00	9 53 17	88673,20	24 43
3 47			242,43	
24 23 38	41300,00	9 54 43	88430,77	24 47
3 46			241,83	
24 27 24	41400,00	9 56 10	88188,94 -	24 50
3 47			241,26	
24 31 11	41500,00	9 57 36	87947,68	24 54
3 47			240,68	
24 34 58	41600,00	9 59 2	87707,00 +	24 58
3 47			240,09	
24 38 45	41700,00	10 0 29	87466,91	25 1
3 46			239,52	
24 42 31	41800,00	10 1 55	87227,39 -	25 5
3 47			238,95	
24 46 18	41900,00	10 3 22	86988,44 -	25 8
3 47			238,38	
24 50 5	42000,00	10 4 48	86750,06	25 12
3 47			237,81	
24 53 52	42100,00	10 6 14	86512,25 -	25 16
3 48			237,25	
24 57 40	42200,00	10 7 41	86275,00 -	25 19
3 47			236,69	
25 1 27	42300,00	10 9 7	86038,31 +	25 23
3 48			236,12	
25 5 15	42400,00	10 10 34	85802,19 -	25 26
3 48			235,57	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicefimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
25 3 48	42500,00	10 12 0	235,57 85566,62 —	25 30
25 8 3	42600,00	10 13 26	235,02 85331,60 —	25 34
25 12 51	42700,00	10 14 53	234,47 85097,13	25 37
25 16 40	42800,00	10 16 19	233,92 84863,21 +	25 41
25 20 28	42900,00	10 17 46	233,37 84629,84	25 44
25 24 16	43000,00	10 19 12	232,83 84397,01	25 48
25 28 4	43100,00	10 20 38	232,27 84164,74 —	25 52
25 31 52	43200,00	10 22 5	231,77 83932,97 +	25 55
25 35 41	43300,00	10 23 31	231,21 83701,76 —	25 59
25 39 29	43400,00	10 24 58	230,68 83471,08	26 2
25 43 18	43500,00	10 26 24	230,15 83240,93 —	26 6
25 47 7	43600,00	10 27 50	229,62 83011,31	26 10
25 50 56	43700,00	10 29 17	229,10 82782,21 +	26 13
25 54 46	43800,00	10 30 43	228,57 82553,64	26 17
25 58 35	43900,00	10 32 10	228,05 82325,59	26 20
26 2 24	44000,00	10 33 36	227,53 82098,06 —	26 24
26 6 14	44100,00	10 35 2	227,02 81871,04	26 28
26 10 3	44200,00	10 36 29	226,50 81644,54	26 31
26 13 53	44300,00	10 37 55	225,99 81418,55 +	26 35
26 17 43	44400,00	10 39 22	225,48 81193,07 +	26 38
26 21 33			224,97	
26 25 21				

Arcus circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicefimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
26 3 51			224,97	
26 25 24	44500,00	10 40 48	80968,10	26 42
3 50			224,46	
26 29 14	44600,00	10 42 14	80743,64 —	26 46
3 51			223,97	
26 33 5	44700,00	10 43 41	80519,67	26 49
3 51			223,46	
26 36 56	44800,00	10 45 7	80296,21	26 53
3 50			222,97	
26 40 46	44900,00	10 46 34	80073,24	26 56
3 51			222,47	
26 44 37	45000,00	10 48 0	79850,77	27 0
3 51			221,97	
26 48 28	45100,00	10 49 26	79628,80 —	27 4
3 51			221,49	
26 52 19	45200,00	10 50 53	79407,31	27 7
3 51			220,99	
26 56 10	45300,00	10 52 19	79186,32	27 11
3 52			220,51	
27 0 2	45400,00	10 53 46	78965,81	27 14
3 51			220,02	
27 3 53	45500,00	10 55 12	78745,79	27 18
3 52			219,54	
27 7 45	45600,00	10 56 38	78526,25	27 22
3 52			219,06	
27 11 37	45700,00	10 58 5	78307,19	27 25
3 52			218,58	
27 14 29	45800,00	10 59 31	78088,61 +	27 29
3 52			218,10	
27 18 21	45900,00	11 0 58	77870,51	27 32
3 53			217,63	
27 23 14	46000,00	11 2 24	77652,88	27 36
3 52			217,15	
27 27 6	46100,00	11 3 50	77435,73 —	27 40
3 53			216,69	
27 30 59	46200,00	11 5 17	77219,04	27 43
3 52			216,21	
27 34 51	46300,00	11 6 43	77002,83 —	27 47
3 53			215,75	
27 38 44	46400,00	11 8 10	76787,08 —	27 50
3 53			215,29	

Arcus circuli cum diffe- rentiis.	Sinus seu Nu- meri absoluti.	Partes vicefimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
3 53			215,29	
27 42 37	46500,00	II 9 36	76571,79	27 54
3 53			214,82	
27 46 30	46600,00	II II 2	76356,97	27 58
3 54			214,36	
27 50 24	46700,00	II 12 29	76142,61 —	28 1
3 53			213,91	
27 54 17	46800,00	II 13 55	75928,70	28 5
3 54			213,45	
27 58 11	46900,00	II 15 22	75715,25 +	28 8
3 53			212,99	
28 2 4	47000,00	II 16 48	75502,26	28 12
3 54			212,54	
28 5 58	47100,00	II 18 14	75289,72	28 16
3 54			212,09	
28 9 52	47200,00	II 19 41	75077,63	28 19
3 54			211,64	
28 13 46	47300,00	II 21 7	74865,99	28 23
3 54			211,19	
28 17 40	47400,00	II 22 34	74654,80	28 26
3 55			210,75	
28 21 35	47500,00	II 24 0	74444,05	28 30
3 54			210,30	
28 25 29	47600,00	II 25 26	74233,75 —	28 34
3 55			209,87	
28 29 24	47700,00	II 26 53	74023,88	28 37
3 54			209,42	
28 33 18	47800,00	II 28 19	73814,46 —	28 41
3 55			208,99	
28 37 13	47900,00	II 29 46	73605,47	28 44
3 55			208,55	
28 41 8	48000,00	II 31 12	73396,92	28 48
3 55			208,12	
28 45 3	48100,00	II 32 38	73188,80 +	28 52
3 56			207,68	
28 48 59	48200,00	II 34 5	72981,12	28 55
3 55			207,25	
28 52 54	48300,00	II 35 31	72773,87 —	28 59
3 56			206,83	
28 56 50	48400,00	II 36 58	72567,04	29 2
3 55			206,40	

Arcus Circuli cum diffe- rentiis.	Sinus seu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
29 3 55			206,40	
29 0 45	48500,00	11 38 24	72360,64	29 6
3 56			205,97	
29 4 41	48600,00	11 39 50	72154,67	29 10
3 56			205,55	
29 8 37	48700,00	11 41 17	71949,12	29 13
3 56			205,13	
29 12 33	48800,00	11 42 43	71743,99	29 17
3 57			204,71	
29 16 30	48900,00	11 44 10	71539,28	29 20
3 56			204,29	
29 20 26	49000,00	11 45 36	71334,99	29 24
3 57			203,87	
29 24 23	49100,00	11 47 2	71131,12 —	29 28
3 57			203,46	
29 28 20	49200,00	11 48 29	70927,66	29 31
3 57			203,05	
29 32 17	49300,00	11 49 55	70724,61 +	29 35
3 58			202,63	
29 36 15	49400,00	11 51 22	70521,98	29 38
3 57			202,23	
29 40 12	49500,00	11 52 48	70319,75 +	29 42
3 57			201,81	
29 44 9	49600,00	11 54 14	70117,94 —	29 46
3 58			201,41	
29 48 7	49700,00	11 55 41	69916,53 —	29 49
3 57			201,01	
29 52 4	49800,00	11 57 7	69715,52	29 53
3 58			200,60	
29 56 2	49900,00	11 58 34	69514,92	29 56
3 58			200,20	
30 0 0	50000,00	12 0 0	69314,72	30 0
3 58			199,80	
30 3 58	50100,00	12 1 26	69114,92	30 4
3 59			199,40	
30 7 57	50200,00	12 2 53	68915,52 —	30 7
3 58			199,01	
30 11 55	50300,00	12 4 19	68716,51 +	30 11
3 59			198,61	
30 15 54	50400,00	12 5 46	68517,90 +	30 14
3 59			198,21	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicefimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
3 59			198,21	
30 19 53	50500,00	12 7 12	68319,69	30 18
3 59			197,83	
30 23 52	50600,00	12 8 38	68121,86 +	30 22
4 0			197,43	
30 27 52	50700,00	12 10 5	67924,43	30 25
3 59			197,04	
30 31 51	50800,00	12 11 31	67727,39 —	30 29
4 0			196,66	
30 35 51	50900,00	12 12 58	67530,73 —	30 32
4 0			196,27	
30 39 51	51000,00	12 14 24	67334,46 —	30 36
4 0			195,89	
30 43 51	51100,00	12 15 50	67138,57	30 40
4 1			195,50	
30 47 52	51200,00	12 17 17	66943,07	30 43
4 0			195,12	
30 51 52	51300,00	12 18 43	66747,95	30 47
4 0			194,75	
30 55 52	51400,00	12 20 10	66553,20 +	30 50
4 1			194,36	
30 59 53	51500,00	12 21 36	66358,84	30 54
4 0			193,99	
31 3 53	51600,00	12 23 2	66164,85 +	30 58
4 1			193,61	
31 7 54	51700,00	12 24 29	65971,24 +	31 1
4 1			193,23	
31 11 55	51800,00	12 25 55	65778,01 —	31 5
4 1			192,87	
31 15 56	51900,00	12 27 22	65585,14	31 8
4 1			192,49	
31 19 57	52000,00	12 28 48	65392,65 —	31 12
4 2			192,12	
31 23 59	52100,00	12 30 14	65200,53 —	31 16
4 1			191,76	
31 28 0	52200,00	12 31 41	65008,77	31 19
4 2			191,39	
31 32 2	52300,00	12 33 7	64817,38	31 23
4 2			191,02	
31 36 4	52400,00	12 34 34	64626,36	31 26
4 2			190,65	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicefimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
4 2 31 40 6	52500,00	12 36 0	190,65 64435,71 -	31 30
4 3 31 44 9	52600,00	12 37 26	190,30 64245,41	31 34
4 2 31 48 11	52700,00	12 38 53	189,94 64055,47 +	31 37
4 3 31 52 14	52800,00	12 40 19	189,57 63865,90	31 41
4 3 31 56 17	52900,00	12 41 46	189,21 63676,69 -	31 44
4 3 32 0 20	53000,00	12 43 12	188,86 63487,83	31 48
4 4 32 4 24	53100,00	12 44 38	188,50 63299,33	31 52
4 3 32 8 27	53200,00	12 46 5	188,15 63111,18	31 55
4 4 32 12 31	53300,00	12 47 31	187,79 62923,39	31 59
4 4 32 16 35	53400,00	12 48 58	187,44 62735,95 -	32 2
4 4 32 20 39	53500,00	12 50 24	187,10 62548,85 +	32 6
4 4 32 24 43	53600,00	12 51 50	186,74 62362,11 +	32 10
4 5 32 28 48	53700,00	12 53 17	186,38 62175,73 +	32 13
4 4 32 32 52	53800,00	12 54 43	186,05 61989,68	32 17
4 5 32 36 57	53900,00	12 56 10	185,71 61803,97	32 20
4 5 32 41 2	54000,00	12 57 36	185,36 61618,61 +	32 24
4 5 32 45 7	54100,00	12 59 2	185,00 61433,61	32 28
4 5 32 49 12	54200,00	13 0 29	184,67 61248,94 -	32 31
4 6 32 53 18	54300,00	13 1 55	184,34 61064,60	32 35
4 5 32 57 23	54400,00	13 3 22	183,99 60880,61 -	32 38
4 6			183,66	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
33 4 6	54500,00	13 4 48	183,66	
33 1 29			60696,95	32 42
33 4 6	54600,00	13 6 14	183,32	
33 5 35			60513,63	32 46
33 4 6	54700,00	13 7 41	182,98	
33 9 41			60330,65	32 49
33 4 7	54800,00	13 9 7	182,65	
33 13 48			60148,00	32 53
33 4 6	54900,00	13 10 34	182,31	
33 17 54			59965,69	32 56
33 4 7	55000,00	13 12 0	181,99	
33 22 2			59783,70 +	33 0
33 4 7	55100,00	13 13 26	181,65	
33 26 8			59602,05 +	33 4
33 4 7	55200,00	13 14 53	181,32	
33 30 15			59420,73	33 7
33 4 8	55300,00	13 16 19	181,00	
33 34 23			59239,73	33 11
33 4 7	55400,00	13 17 46	180,67	
33 38 30			59059,06	33 14
33 4 8	55500,00	13 19 12	180,34	
33 42 38			58878,72	33 18
33 14 8	55600,00	13 20 38	180,02	
33 46 46			58698,70	33 22
33 4 8	55700,00	13 22 5	179,70	
33 50 54			58519,00	33 25
33 4 9	55800,00	13 23 31	179,37	
33 55 3			58339,63	33 29
33 4 8	55900,00	13 24 58	179,05	
33 59 11			58160,58	33 32
34 4 9	56000,00	13 26 24	178,73	
34 3 20			57981,85	33 36
34 4 9	56100,00	13 27 50	178,41	
34 7 29			57803,44	33 40
34 4 9	56200,00	13 29 17	178,10	
34 11 38			57625,34 +	33 43
34 4 10	56300,00	13 30 43	177,77	
34 15 48			57447,57	33 47
34 4 9	56400,00	13 32 10	177,46	
34 19 57			57270,11 -	33 50
34 4 10			177,16	

Arcus Circuli cum differ- entiis.	Sinus feu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
4 10			177,16	
34 24 7	56500,00	13 33 36	57092,95 +	33 54
4 10			176,82	
34 28 17	56600,00	13 35 2	56916,13 +	33 58
4 11			176,53	
34 32 28	56700,00	13 36 29	56739,60 +	34 1
4 10			176,21	
34 36 38	56800,00	13 37 55	56563,39	34 5
4 11			175,90	
34 40 49	56900,00	13 39 22	56387,49 -	34 8
4 11			175,60	
34 45 0	57000,00	13 40 48	56211,89 +	34 12
4 11			175,28	
34 49 11	57100,00	13 42 14	56036,61	34 16
4 12			174,98	
34 53 23	57200,00	13 43 41	55861,63 +	34 19
4 11			174,67	
34 57 34	57300,00	13 45 7	55686,96 -	34 23
4 12			174,37	
35 1 46	57400,00	13 46 34	55512,59	34 26
4 12			174,06	
35 5 58	57500,00	13 48 0	55338,53 -	34 30
4 12			173,76	
35 10 11	57600,00	13 49 26	55164,77 -	34 34
4 12			173,47	
35 14 23	57700,00	13 50 53	54991,30 +	34 37
4 13			173,16	
35 18 36	57800,00	13 52 19	54818,14	34 41
4 13			172,86	
35 22 49	57900,00	13 53 46	54645,28	34 44
4 13			172,56	
35 27 2	58000,00	13 55 12	54472,72	34 48
4 14			172,27	
35 31 16	58100,00	13 56 38	54300,45 +	34 52
4 13			171,97	
35 35 29	58200,00	13 58 5	54128,48 +	34 55
4 14			171,67	
35 39 43	58300,00	13 59 31	53956,81 +	34 59
4 14			171,37	
35 43 57	58400,00	14 0 58	53785,44 -	35 2
4 14			171,09	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicefimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
4 14			171,09	
35 48 11	58500,00	14 2 24	53614,35 —	35 6
4 15			170,80	
35 52 26	58600,00	14 3 50	53443,55	35 10
4 14			170,50	
35 56 40	58700,00	14 5 17	53273,05 —	35 13
4 15			170,22	
36 0 55	58800,00	14 6 43	53102,83 +	35 17
4 15			169,92	
36 5 10	58900,00	14 8 10	52932,91	35 20
4 16			169,63	
36 9 26	59000,00	14 9 36	52763,28 —	35 24
4 15			169,35	
36 13 41	59100,00	14 11 2	52593,93 —	35 28
4 16			169,06	
36 17 57	59200,00	14 12 29	52424,87 —	35 31
4 16			168,78	
36 22 13	59300,00	14 13 55	52256,09	35 35
4 17			168,49	
36 26 30	59400,00	14 15 22	52087,60	35 38
4 16			168,21	
36 30 46	59500,00	14 16 48	51919,39	35 42
4 17			167,93	
36 35 3	59600,00	14 18 14	51751,46 +	35 46
4 17			167,64	
36 39 20	59700,00	14 19 41	51583,82	35 49
4 18			167,37	
36 43 38	59800,00	14 21 7	51416,45 +	35 53
4 17			167,08	
36 47 55	59900,00	14 22 34	51249,37	35 56
4 18			166,81	
36 52 13	60000,00	14 24 0	51082,56 +	36 0
4 18			166,52	
36 56 31	60100,00	14 25 26	50916,04 —	36 4
4 19			166,20	
37 0 50	60200,00	14 26 53	50749,78 +	36 7
4 19			165,97	
37 5 9	60300,00	14 28 19	50583,81	36 11
4 18			165,70	
37 9 27	60400,00	14 29 46	50418,11	36 14
4 19			165,43	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
4 19			165,43	
37 13 46	60500,00	14 31 12	50252,68 +	36 18
4 19			165,15	
37 18 5	60600,00	14 32 38	50087,53	36 22
4 20			164,88	
37 22 25	60700,00	14 34 5	49922,65	36 25
4 19			164,61	
37 26 44	60800,00	14 35 31	49758,04	36 29
4 20			164,34	
37 31 4	60900,00	14 36 58	49593,70 +	36 32
4 20			164,07	
37 35 24	61000,00	14 38 24	49429,63 +	36 36
4 21			163,80	
37 39 45	61100,00	14 39 50	49265,83 +	36 40
4 20			163,53	
37 44 5	61200,00	14 41 17	49102,30	36 43
4 21			163,26	
37 48 26	61300,00	14 42 43	48939,04 -	36 47
4 21			163,00	
37 52 47	61400,00	14 44 10	48776,04 -	36 50
4 22			162,74	
37 57 9	61500,00	14 45 36	48613,30 +	36 54
4 21			162,47	
38 1 30	61600,00	14 47 2	48450,83 +	36 58
4 22			162,20	
38 5 52	61700,00	14 48 29	48288,63	37 1
4 22			161,95	
38 10 4	61800,00	14 49 55	48126,68 +	37 5
4 23			161,68	
38 14 37	61900,00	14 51 22	47965,00 +	37 8
4 22			161,42	
38 18 59	62000,00	14 52 48	47803,58 +	37 12
4 23			161,16	
38 23 22	62100,00	14 54 14	47642,42	37 16
4 24			160,90	
38 27 46	62200,00	14 55 41	47481,52	37 19
4 23			160,64	
38 32 9	62300,00	14 57 7	47320,88	37 23
4 24			160,39	
38 36 33	62400,00	14 58 34	47160,49 +	37 26
4 24			160,12	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicefimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
4 24			160,12	
38 40 57	62500,00	15 0 0	47000,37 —	37 30
4 25			159,88	
38 45 22	62600,00	15 1 26	46840,49 +	37 34
4 25			159,61	
38 49 47	62700,00	15 2 53	46680,88 —	37 37
4 24			159,37	
38 54 11	62800,00	15 4 19	46521,51 +	37 41
4 25			159,11	
38 58 36	62900,00	15 5 46	46362,40 +	37 44
4 25			158,85	
39 3 1	63000,00	15 7 12	46203,55	37 48
4 26			158,61	
39 7 27	63100,00	15 8 38	46044,94 +	37 52
4 27			158,35	
39 11 54	63200,00	15 10 5	45886,59	37 55
4 27			158,10	
39 16 21	63300,00	15 11 31	45728,49	37 59
4 27			157,86	
39 20 48	63400,00	15 12 58	45570,63 +	38 2
4 26			157,60	
39 25 14	63500,00	15 14 24	45413,03	38 6
4 27			157,36	
39 29 41	63600,00	15 15 50	45255,67 +	38 10
4 27			157,10	
39 34 8	63700,00	15 17 17	45098,57 —	38 13
4 28			156,87	
39 38 36	63800,00	15 18 43	44941,70 +	38 17
4 27			156,62	
39 43 3	63900,00	15 20 10	44785,08 +	38 20
4 28			156,37	
39 47 31	64000,00	15 21 36	44628,71	38 24
4 28			156,12	
39 51 59	64100,00	15 23 2	44472,59 —	38 28
4 29			155,89	
39 56 28	64200,00	15 24 29	44316,70	38 31
4 29			155,65	
40 0 57	64300,00	15 25 55	44161,05 +	38 35
4 30			155,40	
40 5 27	64400,00	15 27 22	44005,65 +	38 38
4 29			155,15	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicefimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
40 4 29	64500,00	15 28 48	155,15 43850,50 —	38 42
40 4 30	64600,00	15 30 14	154,92 43695,58	38 46
40 4 30	64700,00	15 31 41	154,68 43540,90	38 49
40 4 31	64800,00	15 33 7	154,44 43386,46	38 53
40 4 31	64900,00	15 34 34	154,20 43232,26	38 56
40 4 32	65000,00	15 36 0	153,97 43078,29	39 0
40 4 32	65100,00	15 37 26	153,73 42924,56 +	39 4
40 4 33	65200,00	15 38 53	153,48 42771,08	39 7
40 4 32	65300,00	15 40 19	153,26 42617,82	39 11
40 4 33	65400,00	15 41 46	153,03 42464,79 +	39 14
40 4 33	65500,00	15 43 12	152,79 42312,00 +	39 18
40 4 33	65600,00	15 44 38	152,55 42159,45	39 22
40 4 33	65700,00	15 46 5	152,32 42007,13 —	39 25
40 4 34	65800,00	15 47 31	152,09 41855,04 —	39 29
40 4 33	65900,00	15 48 58	151,87 41703,17 +	39 32
40 4 34	66000,00	15 50 24	151,62 41551,55 —	39 36
40 4 35	66100,00	15 51 50	151,40 41400,15 —	39 40
40 4 35	66200,00	15 53 17	151,18 41248,97 +	39 43
40 4 36	66300,00	15 54 43	150,94 41098,03	39 47
40 4 35	66400,00	15 56 10	150,72 40947,31 +	39 50
40 4 36			150,49	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicefimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
4 36			150,49	
41 40 57	66500,00	15 57 36	40796,82	39 54
4 36			150,26	
41 45 33	66600,00	15 59 2	40646,56	39 58
4 37			150,03	
41 50 10	66700,00	16 0 29	40496,53 —	40 1
4 37			149,82	
41 54 47	66800,00	16 1 55	40346,71 +	40 5
4 38			149,58	
41 59 25	66900,00	16 3 22	40197,13 —	40 8
4 37			149,37	
42 4 2	67000,00	16 4 48	40047,76	40 12
4 38			149,14	
42 8 40	67100,00	16 6 14	39898,62	40 16
4 38			148,92	
42 13 18	67200,00	16 7 41	39749,70 —	40 19
4 39			148,70	
42 17 57	67300,00	16 9 7	39601,00 —	40 23
4 39			148,48	
42 22 36	67400,00	16 10 34	39452,52 —	40 26
4 39			148,26	
42 27 15	67500,00	16 12 0	39304,26	40 30
4 40			148,04	
42 31 55	67600,00	16 13 26	39156,22	40 34
4 40			147,76	
42 36 35	67700,00	16 14 53	39008,46	40 37
4 40			147,66	
42 41 15	67800,00	16 16 19	38860,80	40 41
4 41			147,38	
42 45 56	67900,00	16 17 46	38713,42 —	40 44
4 41			147,17	
42 50 37	68000,00	16 19 12	38566,25	40 48
4 42			146,95	
42 55 19	68100,00	16 20 38	38419,30	40 52
4 42			146,74	
43 0 1	68200,00	16 22 5	38272,56 +	40 55
4 43			146,52	
43 4 44	68300,00	16 23 31	38126,64 +	40 59
4 42			146,30	
43 9 26	68400,00	16 24 58	37979,74 —	41 2
4 43			146,09	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
4 43			146,09	
43 14 9	68500,00	16 26 24	37833,65 —	41 6
4 43			145,88	
43 18 52	68600,00	16 27 50	37687,77 —	41 10
4 43			145,67	
43 23 35	68700,00	16 29 17	37542,10	41 13
4 44			145,45	
43 28 19	68800,00	16 30 43	37396,65 —	41 17
4 44			145,25	
43 33 3	68900,00	16 32 10	37251,40 +	41 20
4 45			145,03	
43 37 48	69000,00	16 33 36	37106,37	41 24
4 45			144,83	
43 42 33	69100,00	16 35 2	36961,54 +	41 28
4 45			144,61	
43 47 18	69200,00	16 36 29	36816,93	41 31
4 46			144,40	
43 52 4	69300,00	16 37 55	36672,53 —	41 35
4 46			144,20	
43 56 50	69400,00	16 39 22	36528,33 +	41 38
4 47			143,99	
44 1 37	69500,00	16 40 48	36384,34 +	41 42
4 47			143,78	
44 6 24	69600,00	16 42 14	36240,56 +	41 46
4 48			143,57	
44 11 12	69700,00	16 43 41	36096,99	41 49
4 48			143,37	
44 16 0	69800,00	16 45 7	35953,62	41 53
4 48			143,17	
44 20 48	69900,00	16 46 34	35810,45 +	41 56
4 49			142,96	
44 25 37	70000,00	16 48 0	35667,49 +	42 0
4 49			142,75	
44 30 26	70100,00	16 49 26	35524,74	42 4
4 50			142,55	
44 35 15	70200,00	16 50 53	35382,19	42 7
4 50			142,35	
44 40 5	70300,00	16 52 19	35239,84 +	42 11
4 50			142,14	
44 44 55	70400,00	16 53 46	35097,70 —	42 14
4 51			141,95	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
4 51			141,95	
44 49 46	70500,00	16 55 12	34955,75 —	42 18
4 51			141,75	
44 54 37	70600,00	16 56 38	34814,00 +	42 22
4 52			141,54	
44 59 29	70700,00	16 58 5	34672,46	42 25
4 52			141,34	
45 4 21	70800,00	16 59 31	34531,12	42 29
4 52			141,14	
45 9 13	70900,00	17 0 58	34389,98 —	42 32
4 53			140,95	
45 14 6	71000,00	17 2 24	34249,03	42 36
4 53			140,74	
45 18 59	71100,00	17 3 50	34108,29 —	42 40
4 54			140,55	
45 23 53	71200,00	17 5 17	33967,74 —	42 43
4 54			140,35	
45 28 47	71300,00	17 6 43	33827,39	42 47
4 54			140,15	
45 33 41	71400,00	17 8 10	33687,24 —	42 50
4 55			139,96	
45 38 36	71500,00	17 9 36	33547,28 —	42 54
4 55			139,77	
45 43 31	71600,00	17 11 2	33407,51	42 58
4 56			139,57	
45 48 27	71700,00	17 12 29	33267,94 +	43 1
4 56			139,37	
45 53 23	71800,00	17 13 55	33128,57	43 5
4 57			139,18	
45 58 20	71900,00	17 15 22	32989,39	43 8
4 57			138,98	
46 3 17	72000,00	17 16 48	32850,41 —	43 12
4 58			138,79	
46 8 15	72100,00	17 18 14	32711,62 —	43 16
4 58			138,60	
46 13 13	72200,00	17 19 41	32573,02 —	43 19
4 59			138,41	
46 18 12	72300,00	17 21 7	32434,61 —	43 23
4 59			138,22	
46 23 11	72400,00	17 22 34	32296,39	43 26
4 59			138,03	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
46 4 59			138,03	
46 28 10	72500,00	17 24 0	32158,36 +	43 30
5 0			137,83	
46 33 10	72600,00	17 25 26	32020,53	43 34
5 0			137,65	
46 38 10	72700,00	17 26 53	31882,88	43 37
5 1			137,46	
46 43 11	72800,00	17 28 19	31745,42 +	43 41
5 1			137,27	
46 48 12	72900,00	17 29 46	31608,15 +	43 44
5 1			137,07	
46 53 13	73000,00	17 31 12	31471,08 -	43 48
5 2			136,90	
46 58 15	73100,00	17 32 38	31334,18	43 52
5 2			136,70	
47 3 17	73200,00	17 34 5	31197,48 -	43 55
5 3			136,53	
47 8 20	73300,00	17 35 31	31060,95	43 59
5 3			136,33	
47 13 23	73400,00	17 36 58	30924,62	44 2
5 4			136,14	
47 18 27	73500,00	17 38 24	30788,48	44 6
5 5			135,96	
47 23 32	73600,00	17 39 50	30652,52	44 10
5 5			135,78	
47 18 37	73700,00	17 41 17	30516,74	44 13
5 6			135,60	
47 33 43	73800,00	17 42 43	30381,14 +	44 17
5 6			135,40	
47 38 49	73900,00	17 44 10	30245,74 -	44 20
5 7			135,23	
47 43 56	74000,00	17 45 36	30110,51	44 24
5 7			135,04	
47 49 3	74100,00	17 47 2	29975,47	44 28
5 7			134,86	
47 54 10	74200,00	17 48 29	29840,61 -	44 31
5 8			134,68	
47 59 18	74300,00	17 49 55	29705,93	44 35
5 8			134,50	
48 4 26	74400,00	17 51 22	29571,43	44 38
5 9			134,32	

Arcus Circuli cum diffe- rentiis.	Sinus seu Nu- meri absoluti.	Partes vicefinæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
48 5 9			134,32	
48 9 35	74500,00	17 52 48	29437,11	44 42
48 5 9			134,14	
48 14 44	74600,00	17 54 14	29302,97	44 46
48 5 10			133,96	
48 19 54	74700,00	17 55 41	29169,01	44 49
48 5 10			133,78	
48 25 4	74800,00	17 57 7	29035,23	44 53
48 5 11			133,60	
48 30 15	74900,00	17 58 34	28901,63	44 56
48 5 11			133,42	
48 35 26	75000,00	18 0 0	28768,21 —	45 0
48 5 12			133,25	
48 40 38	75100,00	18 1 26	28634,96 +	45 4
48 5 13			133,06	
48 45 51	75200,00	18 2 53	28501,90	45 7
48 5 13			132,90	
48 51 4	75300,00	18 4 19	28369,00 +	45 11
48 5 14			132,71	
48 56 18	75400,00	18 5 46	28236,29	45 14
48 5 14			132,54	
49 1 32	75500,00	18 7 12	28103,75 +	45 18
49 5 15			132,36	
49 6 47	75600,00	18 8 38	27971,39	45 22
49 5 15			132,19	
49 12 2	75700,00	18 10 5	27839,20 +	45 25
49 5 16			132,01	
49 17 18	75800,00	18 11 31	27707,19	45 29
49 5 16			131,84	
49 22 34	75900,00	18 12 58	27575,35 +	45 32
49 5 17			131,66	
49 27 51	76000,00	18 14 24	27443,69 —	45 36
49 5 18			131,50	
49 33 9	76100,00	18 15 50	27312,19	45 40
49 5 18			131,32	
49 38 27	76200,00	18 17 17	27180,87 +	45 43
49 5 19			131,14	
49 43 46	76300,00	18 18 43	27049,73 —	45 47
49 5 19			130,98	
49 49 5	76400,00	18 20 10	26918,75	45 50
49 5 20			130,81	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicefimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
5 20			130,81	
49 54 25	76500,00	18 21 36	26787,94 +	45 54
5 21			130,63	
49 59 46	76600,00	18 23 2	26657,31	45 58
5 21			130,46	
50 5 7	76700,00	18 24 29	26526,85	46 1
5 22			130,29	
50 10 29	76800,00	18 25 55	26396,56 —	46 5
5 22			130,13	
50 15 51	76900,00	18 27 22	26266,43 +	46 8
5 23			129,95	
50 21 14	77000,00	18 28 48	26136,48	46 12
5 24			129,79	
50 26 38	77100,00	18 30 14	26006,69 +	46 16
5 24			129,62	
50 32 2	77200,00	18 31 41	25877,07 +	46 19
5 25			129,45	
50 37 27	77300,00	18 33 7	25747,62 +	46 23
5 26			129,28	
50 42 53	77400,00	18 34 34	25618,34	46 26
5 26			129,11	
50 48 19	77500,00	18 36 0	25489,23 —	46 30
5 27			128,95	
50 53 46	77600,00	18 37 26	25360,28 —	46 34
5 27			128,79	
50 59 13	77700,00	18 38 35	25231,49 +	46 37
5 28			128,61	
51 4 41	77800,00	18 40 19	25102,88	46 41
5 28			128,46	
51 10 9	77900,00	18 41 46	24974,42 +	46 44
5 29			128,29	
51 15 38	78000,00	18 43 12	24846,13 +	46 48
5 30			128,12	
51 21 8	78100,00	18 44 38	24718,01 +	46 52
5 31			127,95	
51 26 39	78200,00	18 46 5	24590,06 —	46 55
5 31			127,80	
51 32 10	78300,00	18 47 31	24462,26 —	46 59
5 32			127,63	
51 37 42	78400,00	18 48 58	24334,63 —	47 2
5 33			127,47	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
5 33			127,47	
51 43 15	78500,00	18 50 24	24207,16	47 6
5 33			127,31	
51 48 48	78600,00	18 51 50	24079,85	47 10
5 34			127,15	
51 54 22	78700,00	18 53 17	23952,70 +	47 13
5 34			126,98	
51 59 56	78800,00	18 54 43	23825,72	47 17
5 35			126,82	
52 5 31	78900,00	18 56 10	23698,90 -	47 20
5 36			126,67	
52 11 7	79000,00	18 57 36	23572,23 +	47 24
5 37			126,50	
52 16 44	79100,00	18 59 2	23445,73	47 28
5 38			126,34	
52 22 22	79200,00	19 0 29	23319,39 -	47 31
5 38			126,19	
52 28 0	79300,00	19 1 55	23193,20 +	47 35
5 39			126,02	
52 33 39	79400,00	19 3 22	23067,18	47 38
5 40			125,86	
52 39 19	79500,00	19 4 48	22941,32	47 42
5 41			125,71	
52 45 0	79600,00	19 6 14	22815,61	47 46
5 41			125,55	
52 50 41	79700,00	19 7 41	22690,06	47 49
5 42			125,36	
52 56 23	79800,00	19 9 7	22564,67	47 53
5 42			125,24	
53 2 5	79900,00	19 10 34	22439,43 +	47 56
5 43			125,07	
53 7 48	80000,00	19 12 0	22314,36 -	48 0
5 44			124,93	
53 13 32	80100,00	19 13 26	22189,43 +	48 4
5 45			124,76	
53 19 17	80200,00	19 14 53	22064,67	48 7
5 46			124,61	
53 25 3	80300,00	19 16 19	21940,06	48 11
5 46			124,46	
53 30 49	80400,00	19 17 46	21815,60 +	48 14
5 47			124,30	

Arcus Circuli cum diffe- rentiis.	Sinus seu Nu- meri absoluti.	Partes vicefimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
53 5 47	80500,00	19 19 12	124,30 21691,30	48 18
53 5 48	80600,00	19 20 38	124,15 21567,15	48 22
53 5 49	80700,00	19 22 5	123,98 21443,17 —	48 25
53 5 50	80800,00	19 23 31	123,85 21319,32 +	48 29
53 5 51	80900,00	19 24 58	123,68 21195,64 —	48 32
53 5 52	81000,00	19 26 24	123,54 21072,10 +	48 36
54 5 52	81100,00	19 27 50	123,38 20948,72 +	48 40
54 5 53	81200,00	19 29 17	123,23 20825,49 +	48 43
54 5 54	81300,00	19 30 43	123,07 20702,42	48 47
54 5 55	81400,00	19 32 10	122,93 20579,49 +	48 50
54 5 56	81500,00	19 33 36	122,77 20456,72 —	48 54
54 5 57	81600,00	19 35 2	122,63 20334,09 +	48 58
54 5 58	81700,00	19 36 29	122,47 20211,62 —	49 1
54 5 58	81800,00	19 37 55	122,32 20089,30 —	49 5
54 5 59	81900,00	19 39 22	122,18 19967,12	49 8
54 6 0	82000,00	19 40 48	122,03 19845,09 +	49 12
55 6 1	82100,00	19 42 14	121,87 19723,22 —	49 16
55 6 2	82200,00	19 43 41	121,73 19601,49	49 19
55 6 3	82300,00	19 45 7	121,58 19479,91	49 23
55 6 3	82400,00	19 46 34	121,43 19358,48	49 26
55 6 4			121,29	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicefimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
6 4			121,29	
55 35 21	82500,00	19 48 0	19237,19	49 30
6 5			121,14	
55 41 26	82600,00	19 49 26	19116,05	49 34
6 6			120,99	
55 47 32	82700,00	19 50 53	18995,06	49 37
6 7			120,84	
55 53 39	82800,00	19 52 19	18874,22 —	49 41
6 8			120,71	
55 59 47	82900,00	19 53 46	18753,51 +	49 44
6 9			120,55	
56 5 56	83000,00	19 55 12	18632,96	49 48
6 10			120,41	
56 12 6	83100,00	19 56 38	18512,55	49 52
6 11			120,27	
56 18 17	83200,00	19 58 5	18392,28 +	49 55
6 12			120,12	
56 24 29	83300,00	19 59 31	18272,16 +	49 59
6 13			119,97	
56 30 42	83400,00	20 0 58	18152,19	50 2
6 14			119,83	
56 36 56	83500,00	20 2 24	18032,36	50 6
6 15			119,69	
56 43 11	83600,00	20 3 50	17912,67 —	50 10
6 16			119,55	
56 49 27	83700,00	20 5 17	17793,12	50 13
6 18			119,40	
56 55 45	83800,00	20 6 43	17673,72 —	50 17
6 19			119,26	
57 2 4	83900,00	20 8 10	17554,46 —	50 20
6 20			119,12	
57 8 24	84000,00	20 9 36	17435,34	50 24
6 21			118,98	
57 14 45	84100,00	20 11 2	17316,36 +	50 28
6 22			118,83	
57 21 7	84200,00	20 12 29	17197,53 —	50 31
6 23			118,70	
57 27 30	84300,00	20 13 55	17078,83	50 35
6 24			118,55	
57 33 54	84400,00	20 15 22	16960,28 —	50 38
6 25			118,41	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
6 25			118,41	
57 40 19	84500,00	20 16 48	16841,87 —	50 42
6 26			118,28	
57 46 45	84600,00	20 18 14	16723,59 +	50 46
6 27			118,13	
57 53 12	84700,00	20 19 41	16605,46	50 49
6 28			117,99	
57 59 40	84800,00	20 21 7	16487,47 —	50 53
6 30			117,86	
58 6 10	84900,00	20 22 34	16369,61 +	50 56
6 31			117,71	
58 12 41	85000,00	20 24 0	16251,90 —	51 0
6 32			117,58	
58 19 13	85100,00	20 25 26	16134,32	51 4
6 33			117,44	
58 25 46	85200,00	20 26 53	16016,88 —	51 7
6 35			117,31	
58 32 21	85300,00	20 28 19	15899,57 +	51 11
6 36			117,16	
58 38 57	85400,00	20 29 46	15782,41	51 14
6 37			117,03	
58 45 34	85500,00	20 31 12	15665,38	51 18
6 39			116,89	
58 52 13	85600,00	20 32 38	15548,49 +	51 22
6 40			116,75	
58 58 53	85700,00	20 34 5	15431,74	51 25
6 41			116,62	
59 5 34	85800,00	20 35 31	15315,12	51 29
6 42			116,48	
59 12 16	85900,00	20 36 58	15198,64 —	51 32
6 44			116,35	
59 19 0	86000,00	20 38 24	15082,29	51 36
6 45			116,21	
59 25 45	86100,00	20 39 50	14966,08 —	51 40
6 46			116,06	
59 32 31	86200,00	20 41 17	14850,02 —	51 43
6 47			115,95	
59 39 18	86300,00	20 42 43	14734,07 +	51 47
6 49			115,82	
59 46 7	86400,00	20 44 10	14618,25 +	51 50
6 51			115,68	

Arcus Circuli cum diffe- rentiis.	Sinus seu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
6 51	86500,00	20 45 36	115,68	
59 52 58	86500,00		14502,57	51 54
6 52	86600,00	20 47 2	115,53	
59 59 50	86600,00		14387,04 —	51 58
6 53	86700,00	20 48 29	115,41	
60 6 43	86700,00		14271,63	52 1
6 54	86800,00	20 49 55	115,27	
60 13 37	86800,00		14156,36	52 5
6 56	86900,00	20 51 22	115,14	
60 20 33	86900,00		14041,22	52 8
6 57	87000,00	20 52 48	115,01	
60 27 30	87000,00		13926,21 —	52 12
6 59	87100,00	20 54 14	114,88	
60 34 29	87100,00		13811,33	52 16
7 1	87200,00	20 55 41	114,74	
60 41 30	87200,00		13696,59	52 19
7 3	87300,00	20 57 7	114,72	
60 48 33	87300,00		13581,87	52 23
7 4	87400,00	20 58 34	114,38	
60 55 37	87400,00		13467,49 +	52 26
7 6	87500,00	21 0 0	114,35	
61 2 43	87500,00		13353,14	52 30
7 8	87600,00	21 1 26	114,22	
61 9 51	87600,00		13238,92	52 34
7 9	87700,00	21 2 53	114,09	
61 17 0	87700,00		13124,83 +	52 37
7 10	87800,00	21 4 19	113,96	
61 24 10	87800,00		13010,87	52 41
7 12	87900,00	21 5 46	113,83	
61 31 22	87900,00		12897,04	52 44
7 13	88000,00	21 7 12	113,70	
61 38 35	88000,00		12783,34 —	52 48
7 15	88100,00	21 8 38	113,57	
61 45 50	88100,00		12669,77 —	52 52
7 16	88200,00	21 10 5	113,45	
61 53 6	88200,00		12556,32	52 55
7 18	88300,00	21 11 31	113,31	
62 0 24	88300,00		12443,01 —	52 59
7 20	88400,00	21 12 58	113,19	
62 7 44	88400,00		12329,82	53 2
7 22			113,06	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
7 22			113,06	
62 15 6	88500,00	21 14 24	12216,76 +	53 6
7 24			112,93	
62 22 30	88600,00	21 15 50	12103,83 +	53 10
7 26			112,80	
62 29 56	88700,00	21 17 17	11991,03	53 13
7 27			112,68	
62 37 23	88800,00	21 18 43	11878,35 +	53 17
7 29			112,55	
62 44 52	88900,00	21 20 10	11765,80 +	53 20
7 31			112,42	
62 53 23	89000,00	21 21 36	11653,38	53 24
7 33			112,29	
62 59 56	89100,00	21 23 2	11541,09 —	53 28
7 35			112,17	
63 7 31	89200,00	21 24 29	11428,92 —	53 31
7 37			112,05	
63 15 8	89300,00	21 25 55	11316,87	53 35
7 40			111,92	
63 22 48	89400,00	21 27 22	11204,95	53 38
7 42			111,79	
63 30 30	89500,00	21 28 48	11093,16 —	53 42
7 44			111,67	
63 38 14	89600,00	21 30 14	10981,49	53 46
7 46			111,55	
63 46 0	89700,00	21 31 41	10869,94 +	53 49
7 48			111,42	
63 53 48	89800,00	21 33 7	10758,52	53 53
7 50			111,29	
64 1 38	89900,00	21 34 34	10647,23 —	53 56
7 52			111,18	
64 9 30	90000,00	21 36 0	10536,05	54 0
7 54			111,05	
64 17 24	90100,00	21 37 26	10425,00	54 4
7 56			110,92	
64 25 20	90200,00	21 38 53	10314,08 —	54 7
7 58			110,81	
64 33 18	90300,00	21 40 19	10203,27 +	54 11
8 1			110,68	
64 41 19	90400,00	21 41 46	10092,59	54 14
8 3			110,56	

Arcus Circuli cum diffe- rentiis.	Sinus seu Nu- meri absoluti.	Partes vicefimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
8 3			110,56	
64 49 22	90500,00	21 43 12	9982,03 +	54 18
8 6			110,43	
64 57 28	90600,00	21 44 38	9871,60	54 22
8 9			110,32	
65 5 37	90700,00	21 46 5	9761,28 +	54 25
8 12			110,19	
65 13 49	90800,00	21 47 31	9651,09	54 29
8 14			110,07	
65 22 3	90900,00	21 48 58	9541,02	54 32
8 17			109,95	
65 30 20	91000,00	21 50 24	9431,07	54 36
8 19			109,83	
65 38 39	91100,00	21 51 50	9321,24	54 40
8 21			109,71	
65 47 0	91200,00	21 53 17	9211,53	54 43
8 24			109,59	
65 55 24	91300,00	21 54 43	9101,94	54 47
8 26			109,47	
66 3 50	91400,00	21 56 10	8992,47	54 50
8 29			109,35	
66 12 19	91500,00	21 57 36	8883,12 +	54 54
8 32			109,23	
66 29 51	91600,00	21 59 2	8773,89 +	54 58
8 35			109,11	
66 29 26	91700,00	22 0 29	8664,78	55 1
8 39			108,99	
66 38 5	91800,00	22 1 55	8555,79	55 5
8 42			108,87	
66 46 47	91900,00	22 3 22	8446,92 -	55 8
8 46			108,76	
66 55 33	92000,00	22 4 48	8338,16	55 12
8 49			108,63	
67 4 22	92100,00	22 6 14	8229,53 -	55 16
8 51			108,52	
67 13 13	92200,00	22 7 41	8121,01 -	55 19
8 54			108,40	
67 22 7	92300,00	22 9 7	8012,61 -	55 23
8 58			108,29	
67 31 5	92400,00	22 10 34	7904,32	55 26
9 1			108,16	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
9 1			108,16	
67 40 6	92500,00	22 12 0	7796,16 —	55 30
9 4			108,05	
67 49 10	92600,00	22 13 26	7688,11 —	55 34
9 8			107,94	
67 58 18	92700,00	22 14 53	7580,17 +	55 37
9 13			107,81	
68 7 31	92800,00	22 16 19	7472,36 —	55 41
9 15			107,70	
68 16 46	92900,00	22 17 46	7364,66 —	55 44
9 20			107,59	
68 26 6	93000,00	22 19 12	7257,07	55 48
9 23			107,47	
68 35 29	93100,00	22 20 38	7149,60	55 52
9 27			107,35	
68 44 56	93200,00	22 22 5	7042,25	55 55
9 31			107,24	
68 54 27	93300,00	22 23 31	6935,01	55 59
9 35			107,12	
69 4 2	93400,00	22 24 58	6827,89 —	56 2
9 39			107,01	
69 13 41	93500,00	22 26 24	6720,88 —	56 6
9 44			106,90	
69 23 25	93600,00	22 27 50	6613,98	56 10
9 48			106,78	
69 33 13	93700,00	22 29 17	6507,20	56 13
9 53			106,67	
69 43 6	93800,00	22 30 43	6400,53 +	56 17
9 57			106,55	
69 53 3	93900,00	22 32 10	6293,98	56 20
10 3			106,44	
70 3 6	94000,00	22 33 36	6187,54	56 24
10 7			106,32	
70 13 13	94100,00	22 35 2	6081,22 —	56 28
10 12			106,22	
70 23 25	94200,00	22 36 29	5975,00	56 31
10 17			106,10	
70 33 42	94300,00	22 37 55	5868,90	56 35
10 22			105,99	
70 44 4	94400,00	22 39 22	5762,91	56 38
10 27			105,87	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
10 27			105,87	
70 54 31	94500,00	22 40 48	5657,04 —	56 42
10 34			105,77	
71 5 5	94600,00	22 42 14	5551,27	56 46
10 39			105,65	
71 15 44	94700,00	22 43 41	5445,62	56 49
10 46			105,54	
71 26 30	94800,00	22 45 7	5340,08	56 53
10 51			105,43	
71 37 21	94900,00	22 46 34	5234,65	56 56
10 57			105,32	
71 48 18	95000,00	22 48 0	5129,33	57 0
11 4			105,21	
71 59 22	95100,00	22 49 26	5024,12	57 4
11 12			105,09	
72 10 34	95200,00	22 50 53	4919,03 —	57 7
11 18			104,99	
72 21 52	95300,00	22 52 19	4814,04	57 11
11 24			104,88	
72 33 16	95400,00	22 53 46	4709,16	57 14
11 31			104,77	
72 44 47	95500,00	22 55 12	4604,39 +	57 18
11 39			104,65	
72 56 26	95600,00	22 56 38	4499,74 —	57 22
11 47			104,55	
73 8 13	95700,00	22 58 5	4395,19	57 25
11 54			104,44	
73 20 7	95800,00	22 59 31	4290,75	57 29
12 3			104,33	
73 32 10	95900,00	23 0 58	4186,42	57 32
12 12			104,22	
73 44 22	96000,00	23 2 24	4082,20	57 36
12 23			104,11	
73 56 45	96100,00	23 3 50	3978,09	57 40
12 30			104,01	
74 9 15	96200,00	23 5 17	3874,08 +	57 43
12 40			103,89	
74 21 55	96300,00	23 6 43	3770,19 —	57 47
12 50			103,79	
74 34 45	96400,00	23 8 10	3666,40	57 50
13 0			103,68	

Arcus Circuli cum differ- entiis.	Sinus seu Nu- meri absoluti.	Partes vicesimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
13 0			103,68	
74 47 45	96500,00	23 9 36	3562,72	57 54
13 12			103,57	
75 0 57	96600,00	23 11 2	3459,15 —	57 58
13 25			103,47	
75 14 22	96700,00	23 12 29	3355,68	58 1
13 38			103,36	
75 28 0	96800,00	23 13 55	3252,32	58 5
13 50			103,25	
75 41 50	96900,00	23 15 22	3149,07 —	58 8
14 3			103,15	
75 55 53	97000,00	23 16 48	3045,92	58 12
14 16			103,04	
76 10 9	97100,00	23 18 14	2942,88	58 16
14 30			102,93	
76 24 39	97200,00	23 19 41	2839,95	58 19
14 45			102,83	
76 39 24	97300,00	23 21 7	2737,12	58 23
15 1			102,72	
76 54 25	97400,00	23 22 34	2634,40	58 26
15 18			102,62	
77 9 43	97500,00	23 24 0	2531,78	58 30
15 37			102,51	
77 25 20	97600,00	23 25 26	2429,27	58 34
15 55			102,41	
77 41 15	97700,00	23 26 53	2326,86 +	58 37
16 17			102,30	
77 57 32	97800,00	23 28 19	2224,56	58 41
16 40			102,20	
78 14 12	97900,00	23 29 46	2122,36 +	58 44
17 5			102,09	
78 31 17	98000,00	23 31 12	2020,27	58 48
17 31			101,99	
78 48 48	98100,00	23 32 38	1918,28	58 52
17 59			101,88	
79 6 47	98200,00	23 34 5	1816,40 —	58 55
18 27			101,78	
79 25 14	98300,00	23 35 31	1714,62 —	58 59
18 58			101,68	
79 44 12	98400,00	23 36 58	1612,94	59 2
19 34			101,58	

Arcus Circuli cum diffe- rentiis.	Sinus feu Nu- meri abfoluti.	Partes vicefimæ quartæ.	Logarithmi cum differentiis.	Partes sexa- genariæ.
19 34			101,58	
80 3 46	98500,00	23 38 24	1511,36 +	59 6
20 12			101,47	
80 23 58	98600,00	23 39 50	1409,89 +	59 10
20 53			101,37	
80 44 51	98700,00	23 41 17	1308,52 +	59 13
21 42			101,26	
81 6 33	98800,00	23 42 43	1207,26	59 17
22 53			101,17	
81 29 26	98900,00	23 44 10	1106,09 +	59 20
24 6			101,06	
81 53 32	99000,00	23 45 36	1005,03 +	59 24
25 6			100,96	
82 18 38	99100,00	23 47 2	904,07 +	59 28
26 28			100,85	
82 45 6	99200,00	23 48 29	803,22 -	59 31
27 54			100,76	
83 13 0	99300,00	23 49 55	702,46	59 35
30 20			100,65	
83 43 20	99400,00	23 51 22	601,81	59 38
32 40			100,56	
84 16 0	99500,00	23 52 48	501,25 +	59 42
36 30			100,45	
84 52 30	99600,00	23 54 14	400,80	59 46
41 9			100,35	
85 33 39	99700,00	23 55 41	300,45	59 49
48 54			100,25	
86 22 33	99800,00	23 57 7	200,20	59 53
1 3 42			100,15	
87 26 15	99900,00	23 58 34	100,05	59 56
2 33 45			100,05	
90 0 0	100000,00	24 0 0	000000,00	60 0

JOANNIS KEPLERI,
IMP. CÆS. FERDINANDI II. MATHEMATICI,
SUPPLEMENTUM
CHILIADIS
LOGARITHMORUM,

CONTINENS

PRÆCEPTA DE EORUM USU.

AD

ILLUSTRISS. PRINCIPEM ET DOMINUM,
DN. PHILIPPUM, LANDGRAVIUM HASSIÆ, &c.

Prima hujus tractatûs editio impressa fuit Marpurgi, et excusa typis Casparis Chemlini, Anno
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JOANNIS KEPLERI
 SUPPLEMENTUM
 CHILIADIS
 LOGARITHMORUM,
 CONTINENS
 PRÆCEPTA DE EORUM USU.

LECTORI S.

CUM anno 1621, venissem in Germaniam superiorem, passimque cum peritis rerum Mathematicarum de Logarithmis Neperianis contulissem; deprehendi eos, quibus ætas prudentiam addebat, promptitudinem minuebat, super hoc genere numerorum, loco Canonis Sinuum in usum recipiendo, cunctari: quod dicerent turpe esse Professori Mathematico, super compendio aliquo calculi pueriliter exultare, interimque sine demonstratione legitimâ formam calculi in usum recipere, quæ olim, cum minimè metueres, in erroris insidias te pertrahere posset. Querebantur Neperi demonstrationem niti figmento motus cujusdam Geometrici; cujus lubricitas & fluxibilitas inepta esset, in quâ solidus ille stilus Rationis Demonstrationumque firmum poneret vestigium. Hæc mihi causa fuit, statim tunc concipiendi rudimentum aliquod Demonstrationis legitimæ; quod posterius, ut primùm Lincium reversus sum, excolui diligentius: maximè postquàm initio anni 1622, supra ea re cum Illustri & Generoso D. D. Petro Henrico à Stralendorff, S. Cæs. Majest. in consistorio Imperiali Aulico, vice Præside, &c. contulissem; qui ut est harum artium avidissimus, rogando plurimum me rerum admonuit, quàm liber aliquis benè crassus præcipiendo admonere posset. Per illam igitur hyemem solennem rei demonstrationem sum aggressus; subiectum hujus speculationis, quodnam esset genuinum, descripsi; quodque id non esset verè sub genere vel linearum, vel motus fluxusque, aut cujusquam alterius quantitatis sensilis, ut sic dicam, sed sub genere Relationum, quantitatisque mentalis, evidentibus enunciationibus constitui: deinde, cum,

ut omnes cæteræ, sic etiam mentalis ista quantitas ($\lambda\acute{o}\gamma o s$ Græcis dicta, quod Latini *Rationem* minùs usitatè, crebriùs *Proportionem* transferunt) divisionem in infinitum recipiat: metas etiam hujus divisionis congruas, ex eodem, scilicet, genere rerum, constitui: Lineæ enim punctis, motus articulis temporum dividi solet; at Proportionem discescunt termini inter extremos magnitudine medii. Hisce sic constitutis feliciter, procedebat demonstratio: vera proportionum omnium communicantium communisque mensura, facta est & ipsa de genere proportionum: locus est factus Arbitrio, in eligendo proportionis Elemento, quodnam deberet haberi pro minimo, proque mensura: demonstratum etiam est plerunque esse proportionem inter se incommunicabiles; eoque minimum unius Elementum quod placuisset, non posse esse mensuram genuinam proportionis cujuscunque alterius: semper enim peccari vel excessu, vel defectu. Ea de causa factus jam est hic alter locus Arbitrio, ut quinam defectus, quæ minimorum Elementorum incommunicabilium differentia dissimulari posset, inque profundum insensibilitatis demergi, constitueretur: ut sic tandem in usum ipsum numerorum redundaret, absurda quidem demonstranti, sed utilissima computanti, *incommunicabilium proportionum communis mensura, Logarithmus dicta*.

Simul autem fuit ipso opere monendus Lector meus, Logarithmos non primum nasci cum Sinubus, seu rectis in Circulo, quod Neperiana descriptio incautius inspecta præ se ferre videtur, sed foris extra Geometriam Circuli constitui, tanquam intra metas libri Quinti Euclidis: inde verò transsumptos, applicari Sinubus; eoque concipiendam esse animo (quam ipse dudum descripseram in Tabula) Matricem quandam Logarithmorum, ex qua omnis per numeros expressa quantitas, etiam Sinus ipsi, suos Logarithmos peterent, ea ipsa numerorum fide, quâ Sinus ipsi sunt in Geometria Circuli definiti: ut si qui Sinus non legitimo numero sit expressus, idem etiam ex hac Matrice Logarithmum imperfectum hausurus sit, nullo Matricis, sed suo proprio vitio. Hoc igitur proposito & computata est à me Chilias Logarithmorum, & adornata etiam demonstratio; quæ ordinem præcipuè numerorum Chiliadis est complexa, causasque ejus in lucem profert. De usu Chiliadis populari cogitatio æqualis quidem tempore, naturâ tamen fuit posterior. Chiliada cum demonstratione, ut primùm perfecta fuit, transmissi Lincio tanquam ad PHILIPPUM LANDGRAVIUM HASSIÆ, sed per ambages itinerum, in quibus libellus hæsit in tertium annum usque. Spes interdum affulsit mihi, libellum excusum iri curâ peritorum: sed ea subinde iterum extincta fuit. Itaque refrixit etiam apud me studium libelli exornandi. Tandem in manus Patroni sui Illustrissimi perlatus, me penitus ignaro, nec quicquam tale opinante, typis mandatus fuit; ut prius ex Catalogo Nundinarum

Nundinarum Francofurtensium, quàm ex Literis Cels. suæ (quas adhucdum in itinere esse nuper admodum rescivi) quid ageretur perceperim. Non dubito, quin Cels. sua in literis me moneat, si quid de usu Chiliadis conceperim, ut id subsidio mittam, quo commendatior & utilior libellus exeat. Nam illi jam superius commemorati fines scriptionis, reconditiorem sensum habent, adque paucos pertinent. Et habet sanè ipsa Chalias tres columnas superadditas, quæ totæ se à subtilitate illa demonstrationum subducunt, adque usum conferunt. Ut igitur, quod in hac editione, temporum et locorum culpâ fuit omissum, suppleatur, ut scilicet Chalias ista, è penetralibus illis speculationum & demonstrationum, quarum præcipuè causâ composita fuit, etiam ad populares usus educatur, prius sunt aliqua mihi præfanda de Titulo. Is profitetur Demonstrationem legitimam ortus Logarithmorum. Et spero fastuosos esse Geometras, quod in chartis excusis hunc scopum sim consecutus. Profitetur verò etiam Demonstrationem usus Logarithmorum. Hic subsiste, Lector. Dum enim rei demonstrationem polliceor, rem ipsam præsuppono jam antea notam, vulgoque tritam. Quænam verò res ista? quis usus Logarithmorum? Nimirum is ipse, qui jam ante decem annos à primo authore Nepero fuit traditus, quique tribus verbis potest concipi: *Ubicunque occurrunt in Arithmetica communi, inque Regula Trium duo numeri inter se multiplicandi, ibi sunt eorum Logarithmi in unam summam addendi; ubi verò Numerus absolutus jubetur factum dividere, ibi Logarithmus illius est auferendus à Logarithmorum summâ: ut Logarithmus illic acervatus, hic residuus, monstret numerum in qualibet operatione quæsitum.* Hic, inquam, est usus Logarithmorum. Hujus usus demonstrationem habes in Propositionibus XVIII, XIX, XX, ut vides in earum Corollariis expressis verbis indicatum. Perpende rem, dices, scio, nihil promissum esse in tituli verbis allegatis, quod non sit præstitum in his tribus Propositionibus. Quod verò pulli Arithmeticorum implumes, facilitatum avidi, rostra ad hanc usus mentionem in immensum pandunt, velut hausuri omnes præceptiuncularum particularium bolos, quibus exsatiarentur; id equidem præstare in chartis istis, quæ demonstrationi fundamentorum erant destinatæ, non potui. At dixeris, sequentia verba Tituli de novâ Arithmeticâ omninò plura polliceri? Video equidem irritamenta cupiditatum talium, & rideo; mirorque non incidisse mihi, cum subito tabellarii decessu ad properandum compulsus, illud Elogium inventi Neperiani, quo Typographus aliquis illiceretur, titulo libelli mei subjungerem; fore ut in majus acciperetur, veluti propria mea gloriatio. Sed utcunque quis hoc meum consilium accipiat, de intentione saltem, me excusat constructio Grammatica. Non de Chiliadis libello, non de mea demonstratione usus, dictum vides, sed de Logarithmis ipsis, invento scilicet Neperiano; quod iis nova tradatur

O

Arithmetica.

Arithmetica. Nec ego hæc sequentia præcepta de usu Chiliadis hujus in id submittenda libello jam excuso statui, ut hanc posteriorem tituli partem penitus ad ipsam meam Chiliada traducerem: segregetur ea pars tituli a labore meo, adque inventum ipsum Neperianum, & ad Canonem Logarithmorum Quadrantis referatur. Non titulus iste subito conceptus, sed utilitas ipsa emptoris me monet, ut quomodo singulæ Chiliadis columnæ possint ad usum popularem referri & quatenus, pauculis præceptis docerem. Ea verò referam in certa Capita. In primo paucula præmittam de Tabulâ Chiliadis. In secundum Caput referam comparisonem Columnæ arcuum, cum Columnâ Rotundorum. Tertio dicam de comparatione Columnæ secundæ Sinuum, cum tertia Quadrivicenaria, & quinta sexagesimorum. Quarto comparabo Quadrivicenariam cum Sexagenariâ. In quinto Capite erit usus Quadrivicenariæ, & Logarithmicæ columnarum junctarum. In sexto Sexagenariæ & Logarithmicæ. In septimo comparabitur Columna prima arcuum, cum quartâ Logarithmorum. In octavo jungentur Columna secunda Absolutorum, & Quarta Logarithmorum. In nono denique Columna Logarithorum jungetur duabus aliis columnis.

P R I M U M C A P U T.
D E T A B U L A C H I L I A D I S,
E J U S Q U E C O L U M N I S.

P R I M U M sciat Lector, exemplar meum in formâ folii scriptum, divisum fuisse in vestibulum Chiliadis, & in Chiliada ipsam. In vestibulum accensentur lineæ 36, ut Chiliadis ipsius principium habeatur ibi, ubi est numerus 100,00. Hac monitione opus est. Nam vestibuli non est eadem proportionum sequela, quæ Chiliadis ipsius, ut apparet ex ipsis Logarithmorum differentiis: quarum series quater ad idem revertitur initium. Quare nec potuit esse usus idem trium columnarum in vestibulo, qui in Chiliade, eaque de causa relictæ sunt vacantes: denique totum vestibulum abesse posset, nisi capite octavo tribuendum id esset analogiæ cum arithmetica communi, & integrationi numerorum omnium.

Jam columnas Chiliadis ipsius vides esse quinque, lineas mille. Prima est columna arcuum quadrantis, qui habent pro sinibus rotundos illos secundæ columnæ numeros. Arcus isti sunt expressi ternis membris, in primo sunt gradus, in secundo scrupula prima, in tertio scrupula secunda, quæ, cum suis tertiis & quartis careant, non possunt exacta esse, præterquam in unico arcu 30° inque fine quadrantis. Quanquam inepta est cura de parte unius secundi, cum ipsa applicatio arcuum horum ad suas lineas, rotundis numeris expressas, eoque effabiles, non sit secuta demonstrationis alicujus subtilitatem; sed solum popularem inquisitionem partis proportionalis ex canone sinuum. Itaque fieri potest, præsertim circa finem quadrantis, ut scrupula secunda paucula passim vel deficient, vel abundant. Inter lineas insertæ sunt differentiæ, quantæ arcubus per singulas millesimas totius semidiametri accrescunt; earum minima in principio quadrantis est $3' 26'' +$, maxima in fine quadrantis est $153' 45''$, seu $2^{\circ} 33' 45''$, quadragecupla quintupla minimæ.

Sequitur secunda columna numerorum: ipsorum rotundorum. Rotundos hic appello, qui continent exactè partes millesimas sui maximi in fine positi, scilicet 100000,00, hoc est, Rotundi habentur, qui terminantur in quatuor minimum cyphras rotundas, nullas dictas.

Vicissim numeri scrupulosi habentur respectu chiliadis hujus, qui desinunt in cyphras significativas, aut qui minus quàm quatuor nullas habent in fine. Ut 96300,00 est rotundus, sic etiam 90000,00. At 96350,00, & 96357,00, & 96300,68, & 90000,05, sunt scrupulosi. Atque hi rotundi nostræ Chiliadis, respectu quidem arcuum ad latus sinistrum positorum, appellantur Sinus, respectu verò columnarum cæterarum ad dextram appellantur Absoluti, quia in cæteris columnis intelliguntur numeri significatione aliquâ certâ revincti.

Hic venit observanda causa, cur, cùm Chiliada scribere proposuerim, numerus tamen maximus non sit 1000, sed potius 100000,00, ejusque unitas non 1, sed 100,00. Deinde cur semper ultimæ duæ numeri cujusque figuræ interposito puncto sint separatæ, & velut abscissæ à reliquis versus sinistram. Causa igitur utriusque rei est eadem, quia nimirum in vulgato canone sinuum, maximus seu semidiameter circuli constat figuris totidem, vel enim 100000, vel 10000000, solet usurpari. Et quia radius ille ad quinque cyphras continuatus plerunque sufficit ad calculum, alterius qui septem cyphris exprimitur, rarior est necessitas: hinc postremi canonis sinuum excultores cœperunt duas ultimas, tanquam minus necessarias, puncto separare. Atque hoc ego mihi ratus sum observandum in rotundis meis, ut qui sunt etiam loco sinuum. Si quis iis, ut absolutis, velit uti, poterit quatuor ultimas cyphras rejicere, quoties id usus aut præcepta sequentia exegerint.

Porro hæc causa attinet etiam columnam quartam, quæ præcipua est, Logarithmorum scilicet cum differentiis, de qua nunc plura dicam.

Primum operæ pretium fecerit computista, qui hanc Chiliade est usus, si subordinationem differentiarum: sub suos Logarithmos subtilibus lineolis adjuvet, ut singulæ figuræ illarum singulis horum respondeant.

Deinde notet calculator, numeros hos non esse ἀριθμοὺς absolutos, sed esse λογαριθμοὺς, numeros scilicet relatos singulos ad binos alios, numeros proportionem significantes, quæ est inter absolutum ad latus sinistrum positum, & inter maximum Chiliadis seu 100000,00.

Tertiò hi Logarithmi omnes sunt scrupulosi, hoc est, non rotundi. Non tamen exhausta est omnis eorum scrupulositas, continuatione hanc eorum usque ad duas figuras ultra punctum. Atque hoc indicatur + ibi per signa + plus & - minus passim apposita.

Nam quibus nullum tale adest signum, iis intellige non unam quadrantem unitatis neque deficere, neque abundare: qui verò habent signum +, ii habent insuper plus quàm quadrantem, minus quàm semissem unitatis, aut summum semissem: qui denique habent signum -, eos intellige minuendos aliquà particulâ unitatis, quæ sit inter quadrantem & semissem. Qui igitur vult accuratissimè computare; is si duos logarithmos est additurus cum signo +, summæ poterit addere unum insuper semissem: Sin duos cum signo -, de summâ detrahet semissem. Et in subtractione, si qui est cum signo - subtrahendus erit ab aliquo cum signo +, eis residuum faciet semisse auctius. Sin è contrario is qui habet +, subtrahi debet ab eo qui habet -. Residuum semisse minuendum erit.

$$\begin{array}{r|l} \text{Exempla additionis.} & \\ 12783,34 & - \\ 230258,52 & - \end{array} \quad \begin{array}{l} 59783,70 + \\ 61618,61 + \end{array}$$

$$\begin{array}{r|l} 243041,85\frac{1}{2} & \\ \text{five } 86 & - \end{array} \quad \begin{array}{l} 121402,31\frac{1}{2} \\ \text{five } 32 & - \end{array}$$

$$\begin{array}{r|l} \text{Exempla subtractionis.} & \\ 59783,70 & + \\ 12783,34 & - \end{array} \quad \begin{array}{l} 231263,55 - \\ 1005,03 + \end{array}$$

$$\begin{array}{r|l} 47000,36\frac{1}{2} & \\ \text{five } 37 & - \end{array} \quad \begin{array}{l} 230258,51\frac{1}{2} \\ \text{five } 52 & - \end{array}$$

Hæc

Hæc scrupulositas locum habet tantum in logarithmis, qui sunt expressi in Chiliade, in aliis intercidentibus præstari non facile potest: nusquam verò est absolutè necessaria. Immò plerumque contemni possunt duæ ultimæ post punctum. Denique quantò major logarithmus, tantò minus periculi in neglectu figurarum differentiæ, post duas ad sinistram primas omnium.

Restant columnæ tertia & quinta. Et in tertia quidem sunt numeri logistici non absoluti; maximus habet 24° quot scilicet sunt unius diei horæ, etsi possunt significare promiscuè vel horas, vel gradus.

In quinta rursus sunt numeri logistici, complectentes collectionem sexagenariam: itaque maximus est $60'$. Tertia quidem numeros exhibet suos tribus membris, in primo sunt integri gradus vel horæ, in secundo scrupula prima unius integri, in tertio scrupula secunda. Quinta duobus membris est contenta, scrupulis nimirum & secundis. Cum autem divisio millenaria non sit apta numeris 24° & $60'$, fit ut in his duabus columnis, ultima unitas scrupulorum secundorum non sit exacta: nam minutiae ejus, quæ minus semisse efficiunt, neglectæ sunt, quæ plus, ea completionem unitatis sunt confusæ & oblitteratæ. Pars enim millesima de 24° est $0^{\circ} 1' 26'' 24'''$, pro qua scripta sunt $0^{\circ} 1' 26''$. Duæ tales partes sunt $2' 52'' 48'''$, pro hoc scripta sunt $2' 53''$. Et in sexagenaria pars millesima est $0' 3'' 36'''$, pro qua scripta sunt $0' 4''$. Duæ millesimæ sunt $0' 7'' 12'''$, pro quo scripta sunt $0' 7''$. Itaque quaternis lineis semper excessu vel defectu peccantibus, solæ quintæ sunt exactæ. Propriam verò & accommodatam hæ columnæ subdivisionem fortientur in tabulis Rudolphi. In præsens Chiliadi dandæ fuerunt partes potiores: et electus est à me millenarius, tanquam propria collectio numerorum absolutorum: quia, ut initio dictum, scopus meus fuit, monere lectorem, quod proportionales, earumque mensuræ, logarithmi, accidant primò numeris absolutis. Nihil tamen impedit has duas columnas, tertiam & quintam etiam per hanc dispositionem minus exactam, & impropriam, millenariam scilicet, fieri utiles: ut in sequentibus dicetur.

C A P U T II.

De Associatione Columnæ Arcuum, & Columnæ Rotundorum.

DIXI cur in columnâ numerorum absolutorum statuerim progressionem arithmeticam continuam ab unitate ad mille, seu à 100,00 ad 100000,00, & qua ratione, cuilibet ex hisce rotundis in columnâ arcuum, associaverim suum arcum.

Cum autem arcus hi omnes sint scrupulosi, fit creberrimè, ut oblato arcu non scrupuloso, aut scrupuloso quidem, sed sic, ut non exactè reperiatur inter arcus Chiliadis, sed inter binos intercitat, ut his, inquam, arcubus jubeamur assignare suos sinus; ut ita Chilia nostra, quadamtenus etiam serviat loco canonis sinuum, ejusque vices suppleat. Id autem fieri poterit hac ratione.

Per

Per arcum Chiliadis proximè minorem proposito excerpe finum rotundum; eundem arcum aufer à proposito: residua scrupula, prima & secunda colloca in regula detri ad dextram, differentiam verò binorum arcuum, inter quos positus intercidit, colloca ad sinistram, in medio colloca numerum progressionis perpetuum scilicet 100,00, & operare per regulam detri prodibunt enim figuræ 4, ultimæ scrupulosæ adjiciendæ ex scripto sinui rotundo, seu loco deletarum ejus 4, cyphrarum finalium scribendæ.

Vicissim dato sinu scrupuloso, indagabimus etiam ejus arcum; si cum sinu rotundo Chiliadis proximè minore quàm est propositus, hoc est, cùm ejus tribus primis figuris ex septem, excerpamus arcum, deinde de differentia interjecta inter duos arcus proximos, sumamus partem proportionalem scrupulositati, quam 4, ultimæ sinus dati figuræ complectuntur.

Hic cùm misceantur numeri diversorum generum, facilè compendium aliquod invenies ex teipso, si bonus es arithmeticus, ingenioque polles. Si verò nescis compendium, utere nostro, quod huic rei servitutum in cap. IX num. VII & VIII rejectum est.

Non quidem erit exactissima vel una, vel altera operatio, præsertim in fine quadrantis; quia differentia tunc fit subitò magna. Itaque conducet, decem ultimas differentias in denas subdividere. Et interpositis sinubus usque ad quartam figuram scrupulosas, suos illis ex canone quadrantis assignare arcus, quod quilibet privatâ operâ poterit. Exceptis verò decem ultimis differentiis, per reliquum quadrantis perveniemus in notitiam sinuum ad 5, saltem figuras continuatorum, non multò incertius, facilius etiam, quàm per canonem sinuum, ad singula scrupula prima extensorum: quia laborem quærendæ partis, quæ sit proportionalis adhærentibus secundis, per logarithmos sublevamus, ut capite IX doceberis. Quomodo etiam secantes arcuum, quodammodo haberi ex hâc Chiliade possint, infra apparebit capite VIII.

Jam quorsum sint nobis utiles sinus arcuum; id non est hujus loci docere, ubi omnia per logarithmos perficimus. Adeantur libri qui tradunt doctrinam triangulorum, per rectos quadrantis.

C A P U T III.

De Comparatione Columnæ secundæ Sinuum, cum tertiâ Quadrivicenariâ, vel cum quintâ sexagesimorum.

FIT interdum, sic exigente parte arithmeticæ, quæ logistica dicitur; ut semidiameter circuli aliter divisa offeratur, quàm in partes denarias, centenarias, millenarias, &c. Verbi causâ diameter solis vel lunæ deficientis dividitur in partes 12, quas digitos appellamus. Quod si igitur detur aliqua portio diametri in partibus, qualium est semidiameter 100000. Ea portio quæ sita inter absolutos, statim monstrat è regione sub quadrivicenariâ quadruplum numerum digitorum, quos valet illa portio.

Ut si deficiat sexta pars de ipsâ circumferentiâ lunæ; ad sciendum quot digiti supersint in lumine, posito quod umbra terræ secet diametrum, ut recta linea ad rectos angulos: supersunt igitur in lumine 300 gradus. Horum dimidium est 150° , hoc est 90° , & 60° . Partes igitur diametri respondentes erunt 100000, & 86603. Illa in quadrivicenariâ ostendit 24° ista $20^{\circ} 47' 2''$ summa $44^{\circ} 47' 2''$. Hinc quarta pars est $11^{\circ} 11' 45''$. Et tot sunt digiti rescissi de diametro.

Vicissim si deficiat digitus, quid is valet in dimensione, qualium semidiameter, hoc est, 6 digiti, sunt 100000? Quia igitur 24, numerus hujus columnæ maximus est quadruplum de 6, sume etiam quadruplum digiti unius, id est, 4° . Hoc verò quæsitum in quadrivicenaria, offert inter absolutos 16675,00 circiter.

Eodem modo fit interdum, ut semidiameter valeat authoribus non 100000, sed 60', ut solet Ptolomæus, et plerique astronomorum in indicatione proportionis orbium. Ii authores, si exprimant in hac dimensione sinus arcuum, aut partes quascunque semidiametri; quæsitæ eæ partes in sexagenariâ, exhibent in absolutis, valorem illius partis in dimensione hodiernâ sinuum progressionis denariæ.

Exempli causa, quæritur $38' 47''$, quantum faciat sinum, qualium 60' est 100000, respondetur 64630 circiter.

Sic etiam vicissim quæritur 9265, eccentricitas Martis, quot faciat scrupula, qualium 100000 facit 60', respondetur $5' 33''$.

C A P U T IV.

De Comparatione Quadrivicenariæ cum Sexagenariâ.

SOLET esse pars tabularum astronomicarum conversio horarum in scrupula diei, & vicissim.

Hæc conversio habetur per has duas tabulas satis exactè, præsertim in quintis lineis, quas rotundas dicere possumus.

Verbi causâ horæ $6^{\circ} 28' 54''$, quot valeant scrupula diei? In columna tertia occurrunt $6^{\circ} 28' 48''$, quæ faciunt in quinta $16' 12''$ exactè, quia linea est rotunda & quinariï ultima. Residuum igitur $6''$ unius minuti horariï, quæsitæ ut horæ, dant $15'$, $0''$, quæ jam valent $15'''$, junctis scilicet apicibus residui horariï $6''$, et horum $15'$ horis 6° respondentium. Ergò totus numerus $6^{\circ} 28' 54''$ valet scrupula diei $16' 12'' 15'''$.

Vicissim $47' 39''$ scrupula diei quot valent horas?

In quinta $47' 38'' +$ valent horas $19^{\circ} 3' 22''$ — residuum secundum unicum seu $60'''$, quæsitæ ut 60' ostendunt $24''$ (duobus apicibus, quos illis $60'''$ detraxeramus, numero ostenso 24° superpositis.) Ergò proposita scrupula $47' 39''$ valent horas $19^{\circ} 3' 46''$ — Non prodit enim hic exactum, quia linea est scrupulosa, quarta scilicet fui quinariï.

Sed

Sed si memor eris eorum, quæ capite primo sunt dicta de his linearum quinariis, quod scilicet in primis intelligantur accedere quadrivicenario numero $24'''$; in secundo deficere $12'''$, in tertio accedere $12'''$, in quarto deficere $24'''$. Et sexagenario numero in prima linea deficere $24'''$, in secunda accedere $12'''$, in tertia deficere $12'''$, in quarta accedere $24'''$, poteris etiam ex quatuor prioribus quinariis lineis exactè excerpere conversiones istas. Nam exactè $47' 38'' 24'''$ valent $19^\circ 3' 21'' 36'''$. Ergò residua $36'''$, unius secundi quæsita, $36'''$ ostendunt horas $14^\circ 24'$, valentque $14'' 24'''$. Itaque totus $47' 39''$ valet exactè horas $19^\circ 3' 36'' 0'''$.

In tabulis Rudolphi exacta erit ista conversio in lineis omnibus, quia servabitur naturalis, & propria divisio numerorum 24° & 60° in lineis omnibus.

Alia quoque conversio est in usu crebro, horarum 24 in gradus æquatoris 360, & vicissim. Et potest hac quoque conversio perfici ope harum 2 tabularum in hunc modum.

Quæritur horæ $19 25' 37''$, quot faciunt gradus, & scrupula æquatoris. In columna igitur tertia $19^h 26' 24''$, dat $48' 36''$ exactè. Sed temporaria $47''$, quæsita ut temporaria $47'$ dant $1' 58''$, quæ valent $1'' 58'''$. Ergo propositæ $19^h 25' 37''$ dant $48' 34'' 2'''$, quæ multiplica in 6, & diminue apices, roveniunt $291^\circ 24' 12''$, gradus æquatoris.

Vicissim gradus æquatoris sic convertes in horas: divide eos per 6, & auge apices: quotiens in sexagenariam immisus, ostendit è regione horas quæsitas.

Quærantur $259^\circ 34' 17''$ quot sint horæ?

Sexta pars est — 43 15 43. Quære igitur $43' 15'' 43'''$, & monstrabunt in tertia $17^h 18' 14''$ —.

C A P U T V.

De Quadrivicenaria cum Logarithmis composita.

I. **T**ABULA diurnorum & horariorum solet Ephemeridibus præfigi, prolixæ admodum: ubi docemur, dato diurno excerpere horarium, ad datum horarum numerum. Excerptio ipsa licet non fatiget mentem multiplicatione, fatigat tamen & mentem additione, & oculos manusque excerptione. At hîc conjunctio columnæ quartæ cum tertia, & tollit necessitatem tabulæ diurnorum, & opus peragere docet longè facilius, quippe additione simplici (non logistica) logarithmorum.

Quære enim tam diurnum, quàm horas in quadrivicenaria, & logarithmos ab iis monstratos adde in unam summam; quæ inter logarithmos quæsita, monstrabit è regione in columna tertia portionem motus competentem horis.

Utile est hoc præceptum, ubi magnus est diurnus, non major tamen quàm gr. 4, ut in lunâ & cometarum nonnullis.

E X E M-

EXEMPLUM.

Sit diurnus lunæ $14^{\circ} 23'$ Quæritur quantum
 competat horis 19 42
 Logarithmus ad 14 23 0" est 51200 circiter.
 Logarithmus ad 19 42 0 est 19730 circiter.

Summa 70930

Hæc quæsitæ inter logarithmos, ostendit in quadrivicenariâ $11^{\circ} 48' 0''$.
 Tot gradus competunt horis 19 42'.

ALIUD EXEMPLUM.

Sit diurnus $2^{\circ} 15'$. Horæ 938'.
 Logarithmus 2 15 est 237000 circiter.
 Logarithmus 9 38 est 91200 circiter.

Summa 328200 dat $0^{\circ} 54' 0''$.

Potest verò hoc exemplum, ubi diurnus est adeò parvus, aliter etiam & faciliori viâ computari, quam referam in caput IX.

II. Huc pertinet etiam ratio computandi ad aspectus lunæ cum stellis. Ut si detur separatio diurna duorum planetarum, & distantia eorum, minor quàm illa; quæraturre quot horis ea conficiatur?

Tunc enim à logarithmo distantiae, auferatur logarithmus separationis diurnæ, residuum inter logarithmos quæsitum, ostendit è regione sub quadrivicenariâ horas competentes.

EXEMPLUM.

Sit mercurii diurnus retrogradus $2^{\circ} 15'$
 Lunæ directus - - 15 2
 Separatio igitur diurna 17 17
 Logarithmus est - - 32840 circiter.
 Sit distantia lunæ & mercurii 12 23
 Logarithmus - - 66170 circiter.

Residuum 33330 dat $17^{\circ} 12'$.

Tot igitur horis fit aspectus lunæ, ante vel post momentum, quo invenitur hæc distantia, prout luna post vel ante mercurium fuerit.

Si uterque directus, vel uterque retrogradus esset; minor diurnus à majori subtraheretur, ad eruendam separationem diurnam, uti docemur in astronomia. De aspectibus tardiorum inter se infra, cap. IX.

III. Diurnum indagare (minorem quidem quàm 24^h) ex motu aliquot horarum.

Tunc logarithmum horarum aufer à logarithmo motus competentis, restabit logarithmus motus diurni.

EXEMPLUM.

Promotum deprehendatur sidus aliquod in horis 4 27.

Gradus 1 53' quantus erit diurnus?

Gradus 1 53 logar. 254500 circiter

Hor. 4 27 logar. 168400

Residuum 86100 dat gr. 10 9' diurnum.

Infra cap. ix, erit alius modus pro diurnis parvis.

IV. Datis tribus stellis in unâ rectâ, datisque singularum latitudinibus, duarum verò solarum longitudinibus; indagare & tertiæ longitudinem.

Idem, si pro latitudinibus notæ fuerint declinationes, pro longitudine verò sumatur assensio recta.

Cùm hæ rectæ plerumque sint breves & infra 24° nihil nobis nocebit, abusus *ἁπλοῦς* curvilinearum ut rectilinearum triangulorum.

Pone in regula detri, ad sinistram, differentiam latitudinum duarum, in medio differentiam longitudinum earundem; ad dextram differentiam iterum latitudinis stellæ tertiæ & priorum unius; quæsitisque his arcubus in quadrivcenariâ, exscribantur logarithmi; additisque secundo & tertio, de summa dematur primus; restabit logarithmus differentiæ longitudinis stellæ tertiæ.

Ufus hujus præcepti peropportunos est in tractatione observationum per lineas rectas trium stellarum.

C A P U T VI.

De Sexagenariâ cum Logarithmis compositâ.

TABULÆ hexacontádôn usus est in logistica multiplex, præcipuus quidem ad multiplicationes & divisiones logísticas, scrupulositatis infinitæ, ubi fit progressio ad tertia, quarta, quinta, sexta, & sic deinceps. Tantam subtilitatem non proficitur usus Chiliadis nostræ. Maneat igitur hæc utilitas propria tabulæ hexacontádôn.

Sed quia hæc tabula præfigi solet etiam tabulis astronomicis & ephemeridibus,

dibus, in quibus non fit progressio ultra secunda, vel ad summum tertia: hunc ejus usum tabula nostra sexagenaria plenissime præstat, ejusque necessitatem penitus tollit: quod docebo præceptis sequentibus.

I. Multiplicationes logísticas perficere compendiosissime.

Primum capiant numeri, qui sunt in se mutuò multiplicandi, denominationes seu apices familiares tabulæ Chiliadis. Nam quia in logistica apices antecedentes integra 1° versus sinistram, sunt indices sexagenarum, sequentes verò integra 1° sunt indices sexagesimorum scrupulorum, seu fractionum unius integri: sciendum igitur, numeros in columnâ sexagenariâ propriè intelligi de sexagesimis scrupulis primis & secundis, tanquam fractionibus unius integri. Ergò si offerantur gradus & minuta in datorum alterutro, pro gradibus scribantur minuta, pro minutis secunda. Tunc iis in sexagenaria quæsitis, excerpantur logarithmi, addanturque in unam summam; quæ inter logarithmos quæsitæ exhibebit sub sexagenaria factum in primis & secundis. Sed si prius apices multiplicandorum fuerunt mutati, nunc etiam facti apices erunt mutandi in contrarium, nam quot apices utrique junctim additi sunt, totidem jam uni soli quotienti sunt adimendi, & vicissim.

EXEMPLUM.

Sint multiplicandi $25^{\circ} 35''$ log. 85240
& $49^{\circ} 50''$ log. 18566

Summa 103806 dat $21^{\circ} 14''$ factum.

ALIUD.

$15^{\circ} 38' 40''$ sint multiplicanda in $53' 49''$.

Scribe $15^{\circ} 38' 40''$ log. 134400. Hic apices aucti sunt unitate.
& $53' 49''$ log. 10900

Summa 145300 dat $14^{\circ} 2''$.

Sed scribe $14^{\circ} 2'$ detracta ab apicibus unitate vicissim.

ALIUD.

Sexagenæ.

Sexagesimæ.

$39^{\circ} 20'$ sint multiplicanda $46' 15''$

Scribe $39^{\circ} 20''$ log. 42227. Hic apices aucti unitate
 $46' 15''$ log. 26028. Hic apices aucti binario.

Summa 68255 dat $30^{\circ} 19' 0''$.

Sed scribe $30^{\circ} 19' 0''$ detracto ternario ab apicibus sexagesimariis.

Scilicet hic factus non est $30'$ scrupula sexagesima prima, & $19''$ secunda, sed 30 sexagenæ secundæ & 19 primæ, &c.

ALIUD EXEMPLUM.

Sint multiplicandæ sexagenæ $37^{\circ} 41'$ in sexagesima $32'' 29'''$.

Scribe $37^{\circ} 41''$ log. 46513. Hic apices sunt aucti binario
 $32' 29''$ log. 61362. Hic apices deminuti unitate

Summa 107875 dat $20^{\circ} 24''$
 Sed scribe $20^{\circ} 24'$.

Nam propter primum detrahendus esset ab apicibus binarius, propter secundum addenda unitas vicissim. Compensatione igitur contrariorum facta, adhuc detrahenda fuit à facti apicibus unitas.

II. Quadrare numerum logisticum facillimè.

Rursum esto memor, quod numeri in sexagenariâ sint fractiones, eoque numerus quadratus prodiens, quod æstimationem apicum attinet, minor seu minoris valoris sit, quàm quadrandus. Propriissimè quidem quadratus, de quo quæritur: sic est ad quadrandum, ut hic ad unitatem 1° seu $60'$ scr. ut sit quadrare logistice nihil aliud, quàm tertiam proportionalem à maximo columnæ sexagenariæ invenire.

Numeri igitur quadrandi in sexagenariâ quæsi logarithmum duplica, duplum in logarithmis quæsitum exhibet ex sexagenariâ quadratum, quod quærebatur.

EXEMPLUM.

Sit quadrandus $49^{\circ} 53''$. Ejus logarithmus 18450 dupletur ut sit 36900 hoc in logarithmis quæsitum exhibet $41^{\circ} 29''$. Dico igitur $41^{\circ} 29'' 0''' 0'''$ esse quadratum numeri $49^{\circ} 53''$.

Si quid mutandum in apicibus, ut quadrandus inveniri possit in Chiliade, tunc valent præcepta eadem, quæ prius in multiplicatione, tantummodò ut memineris, quadrandum esse vice duorum in se multiplicandorum.

III. Divisiones logísticas perficere compendiosissimè.

Primò observentur eadem de aptatione apicum, quæ prius circa multiplicandos. Deinde observa, num dividendus (sic aptatus si opus fuit) major fuerit, quàm divisor, an minor. Nam si major, non propriè pertinet operatio ad Chiliada; sed ejus loco dividendus est, vel excessus ejus supra divisorem, vel ejus pars aliquota, minor divisore.

Si igitur hoc pacto dividendus logisticus fuerit minor ipso divisore, tunc logarithmus divisoris, auferatur à logarithmo dividendi; restabitque logarithmus quotientis. Atque is quotiens vel erit excessus itidem, addendus dividendo, ut constituatur verus quotiens, vel erit itidem pars æquè multiplex quotientis, vel denique, si nulla permutatio facta in datis numeris, erit ipse quotiens.

At mutatio in apicibus facta adhuc est compensanda. Nam quantum fuit additum vel subtractum apicibus divisoris, tantum etiam addendum vel sub-

trahendum apicibus quotientis. Quantum verò apicibus dividendi vel additum, vel subtractum : tantum contrariâ ratione subtrahendum vel addendum apicibus quotientis, seorsim utrumque.

EXEMPLUM.

Dividantur $29^{\circ} 30''$ log: 70995
in $59^{\circ} 0''$ log. 1681

69314 dat $30^{\circ} 0''$

Hic nulla est facta mutatio apicum, nec in divisore, nec in dividendo, quippe dividendus erat minor divisore, casusque proprius Chiliadi. Ergò $30^{\circ} 0''$ erit quotiens justus. Quod mirari non debes $29^{\circ} \frac{1}{2}$ distributa in partes 59° , facere portiones magnitudine 30° majore, quàm erat totus dividendus. Debes enim cogitare, illa 59° non esse integra, sed fractionem unius integri, proinde quotiens 30° est portio debita non uni 1° scrupulo primo, sed uni 1 integro.

ALIUD.

Dividantur Gr. $6^{\circ} 0'$ | Scribe $6^{\circ} 0''$ log. 230258
In dies $59^{\circ} 3'$ | $59^{\circ} 3''$ log. 1550

Residuum 228708

Hoc residuum ostendit $6^{\circ} 8''$. Si ergò dati habuissent illos apices, cum quibus excerpimus logarithmos, tunc quotiens hic fuisset. At quia apicibus dividendi sunt adjectæ duæ unitates, sic ut ex 0° fierent sexagesima 0° ex sexagenis verò 6° fierent sexagesima 6° : Vicissim igitur quotienti huic $6^{\circ} 6''$ adimendi duo apices sexagesimarii, ut fiat $6^{\circ} 6^{\circ}$. Quia verò etiam divisoris apicibus fuit adjecta unitas sexagesimaria, ut pro 59° scriberentur 59° , & pro $3'$ scriberentur $3''$, rursus igitur idem est faciendum quotienti primo mutato, ut pro $6^{\circ} 6^{\circ}$ scribatur itidem $6^{\circ} 6'$.

EXEMPLUM UBI DIVIDENDUS MAJOR.

Dividantur $57^{\circ} 23''$ major
in $41^{\circ} 15''$ ut minorem

Subtrahe manet $16^{\circ} 8''$ residuum jam minus divisore
Ergò divide $16^{\circ} 8''$ log. 131350
in $41^{\circ} 15''$ log. 37470

Residuum 93880 ostendit $23^{\circ} 29''$

Cui adde divisorem ipsum totum semel, quia semel tantum erat subtractus à dividendo, colligitur $1^{\circ} 4' 44''$. Quotiens debitus uni integro, cujus divisor $41^{\circ} 15''$ erat pars seu fractio.

EXEMPLUM DE PARTE ALIQUOTA.

Sint dividendi $3^{\circ} 45' 13''$ per $57' 8''$. Cùm igitur contineatur divisor in dividendo crassâ æstimatione, minus quàm quater: operabor per dividendi partem quartam, quæ facilè sumitur, estque

$$\begin{array}{r} 56' 18'' 15''' \text{ log. } 6360 \\ \text{Divisoris } 57' 8'' \text{ log. } 4900 \end{array}$$

Residuum 1460

Hoc ostendit quotientem $59' 8''$. Hujus igitur sumendum est iterum quadruplum scilicet $3^{\circ} 56' 32''$.

Nota hoc exemplum potuisset etiam tractari aliter pro $3^{\circ} 45' 13''$ scribendo

$$\begin{array}{r} 3' 45'' 13''' \text{ log. } 277000 \\ \text{Divisor } 57' 8'' \text{ log. } 4900 \end{array}$$

Residuum 272100

Hæc enim ostendit $3' 56'' 30'''$. Quia verò apicibus dividendi fuit apposita unitas: detrahatur vicissim apicibus quotientis unitas, fietque quotiens $3^{\circ} 56' 30''$ ut prius.

Quomodo compendiosè sit agendum, si dividendus fuerit aliquoties major divisore, tradetur infra capite IX, modus etiam alius.

IV. Operari per regulam proportionum, detri dictam, in logistis.

Si trium datorum unus sit pura unitas, sic ut ea possit aptari pro 1° seu $60'$: siquidem hæc unitas fuerit primo loco ad sinistram; tunc operatio absolvitur per meram additionem logarithmorum, loco primo traditam: sin autem fuerit hæc pura unitas 1° seu $60'$ loco secundo vel tertio; tunc operatio perficitur per simplicem subtractionem logarithmorum loco secundo propositam. At si nusquam fuerit pura unitas, tunc sinistimi logarithmus, aufertur à summa duorum logarithmorum residuorum, si potest: vel quod idem est, sinistimi logarithmus aufertur ab uno duorum residuorum, si potest; residuum addetur logarithmo tertii: utroque modo conficitur logarithmus quotientis.

EXEMPLUM.

$29' 45''$ - dat - $15' 43''$ - quid - $58' 47''$?	Vel 133970
log. 70150	Aufer 70150
133970	
Adde 133970	Refid. 63820
Summa 136020	Adde 2050
Aufer 70150	
Residuum 65870	65870
Quotiens ostenditur $31' 3''$.	ut prius.

Si

Si finistimi logarithmus à summâ reliquorum subtrahi non potest, si nimirum finistimus datus; minor fuerit utroque reliquorum dato seorsim: id indicio est, quotientem excrecere supra 1° seu $60'$. Quare operare per secundi vel tertii partem aliquotam, quotientisque prodeuntis sume æquè multiplicem: vel operare per excessum alterutrius datorum, supra datum; finistimum; quotientique emergenti adijunge ipsam finistimum: uti in divisione doctus es.

V. A numero logistico proposito, radicem extrahere quadratam facillimè.

Ne turberis, quod radix sit major suo quadrato: fit enim hoc propterea, qui numeri sexagenariæ sunt fractiones unius integri: ut suprâ dictum. Radix enim logistica nihil est aliud, quàm medium proportionale inter integrum 1° & numerum logisticum integro minorem, vel etiam majorem.

Ergò numeri propositi logistici logarithmum bipartire, semissis enim quæsitus inter logarithmos in sexagenariâ ostendet radicem quæsitam.

EXEMPLUM.

Quadratus esto $50' 27''$ logar. 17360
semiss. 8680.

Hic semissis ostendit radicem $55' 0''$ eritque ut $1^{\circ} 0' 0''$ - ad $55' 0''$ - sic hoc ad $50' 27''$ ferè.

Quod si quadratus habuerit alios apices, quàm in sexagenariâ, quærat eus partis quadratæ, quæ minor fuerit integro, puta quartæ, vel nonæ, vel sedecimæ, &c. radix, eaque inventa vicissim duplicetur, triplicetur, vel quadruplicetur, &c.

Sit quadratum $1^{\circ} 39' 20''$. Hic cùm excurrat supra integrum, sic ut inveniri non possit in sexagenariâ, tantò ejus quadrantem $24' 50''$. Hic cùm jam inveniatur in sexagenariâ, ejus ergò log. 88190, semissis 44095 ostendit $38' 36''$ radicem quadrantis; ejus ergò duplum $1^{\circ} 17' 12''$ est radix quæsitæ; seu magis propriè est medium proportionale inter $1^{\circ} 0' 0''$ & $1^{\circ} 39' 28''$.

Mirabitur hoc imperitus, quomodo $1^{\circ} 17'$ sit radix de $1^{\circ} 39'$, reputans illic esse 77' hic 99', & verò radicem de 99' esse paulò minorem quàm 10'. At memineris integrum in sexagenariâ non esse unum scrupulum, sed unum gradum: proinde hujus unitatis linearis quadratum, itidem est unitas superficialia valens 1° seu $60'$. Et sic unitatis linearis, cum fractione appendice, per $17'$ expressâ, quadratum rectè fit unitas superficialia cum fractione appendice per $39'$ expressâ. Si verò cogitationem ab hac unitate gradus transferas ad unitatem scrupulariam; tunc unitas linearis scrupularia, quadratum habet, unitatem superficialiam, quæ valet scrupulum: Et sic prioris unitatis graduariæ $60'$ scrupula in formam redacta quadratam, latus habebunt paulò brevius 8, unitatibus lineæ scrupulariæ. Unde elucet consensui rei utriusque.

VI. Inter duos numeros logísticos medium proportionale constitutare.

Si datorum unus est $1^{\circ} 0' 0''$ jam modò doctus es idem facere per extractionem radicis. Hæc est enim medium proportionale. Si verò non est integrum sexagenariæ, scilicet 1° interdatos, adde datorum logarithmos, summæ semiffem quære inter logarithmos, & excerpes ex sexagenariâ quæsitum medium proportionale.

EXEMPLUM.

Sint dati $25' 35''$ logarith. 85240
& $49' 50''$ logarith. 18566

Summa 103806

Semiffis 51903 dat $35' 43''$

Erit igitur ut $25' 35''$ ad $35' 43''$, sic hoc ad $49' 50''$.

VII. In specie hîc docemur partem proportionalem venari, in tabulis æquationum & alibi.

Fit autem secundum præcepta præmissa multò compendiosissimè, quoties totum, quod debetur uni gradui, seu horæ, seu 60 minutis, non superat $60'$ minuta. Adduntur enim logarithmi 1. Differentiæ uni gradui, vel horæ respondentis, & 2 scrupulorum, integris gradibus vel horis adhærentium, summa inter logarithmos quæsitæ exhibet, ad latus sub sexagenariâ partem proportionalem quæsitam.

Sit anomalia $136^{\circ} 47' 14''$, & excerpatur cum integris 136° æquatio — $4^{\circ} 15' 23''$. Sitque differentia æquationum duarum vicinarum decrefcentium $37' 29''$. Quæritur pars proportionalis scrupulis $47' 14''$.

Logarith. $47' 14''$ est 23920 circiter

Logarith. $37' 29''$ est 47050 circiter

Summa 70970 dat $29' 30''$ partem proportionalem decrementi, ablata igitur hæc à $4^{\circ} 15' 23'$ relinquit $3^{\circ} 41' 55'$ æquationem correctam.

EXEMPLUM ALIUD.

Sit horarius $31' 24''$, sint minuta unius horæ $41' 48''$, quæritur quantum iis debeat de horario? Adde logarithmos 64770, & 36150. Summa 100920, ostendit partem proportionalem $21' 52''$.

VIII. Dato

VIII. Dato horario lunæ à sole, datisque scrupulis incidentiæ, moræ dimidiæ, vel durationis dimidiæ: eruere tempus incidentiæ, moram dimidiam, vel durationem dimidiam in eclipsibus.

VEL,

Dato horario, datoque arcu per currendo, indagare numerum horarum & minutorum, intra quos arcus percurritur.

Primum aufer horarium lunæ à sole quoties potes à datis scrupulis, totiesque scribe unam horam.

Deinde à logarithmo residui, quod minus erit horario, aufer logarithmum horarii, restabit logarithmus minutorum horis integris adjiciendorum.

Sint scrupula $53' 16''$
 Horar. lunæ à sole $31' 24''$ aufer semel & scribe hor. 1

Residuum $21' 52''$ logar. 100920

Horarii $31' 24''$ logar. 64770

Residuum 36150 dat $41' 48''$

Ergò tempus est $1^h 41' 48''$.

IX. Dato numero horarum & minutorum, cui respondeat datus minor numerus graduum & scrupulorum, inquirere horarium: oportet autem horarum numerum infra 60' esse.

Quære datos (diminutis signis) in sexagenariâ, & logarithmum horarum, aufer à logarithmo graduum; residuum est logarithmus horarii ex eadem sexagenaria excerpenti cum ipsis signis.

Ut si horis $50' 26''$ luna promoveatur $28^\circ 3'$

Diminue signa, ut stent illic $50' 26''$ hic $28' 3''$

Jam igitur à logarithmo $28' 3''$ sc. 76036

Aufer logarithmum $50' 26''$ sc. 17469

Residuum 58567 ostendit horarium
 $33' 24''$, retentis signis.

APPENDIX.

Si plures essent gradus quàm horæ, aufer numerum horarum à numero graduum quoties potes, totiesque scribe unum gradum in quotiente. Deinde cum eo, quod de gradibus fuerit residuum, operare, ut jam dictum, prodibuntque

Q

buntque scrupula & secunda, gradibus integris in quotiente scriptis, addenda.

Sed hæc cap. 1^x tradantur per alium modum simpliciore, & magis proprium, ut mutatione apicum non sit opus.

COMPENDIUM.

Conducit etiam ad brevitatem, si major numerus fuerit infra 30, uti tunc operemur per utriusque duplum.

Ut si horis 25 13' luna promoveretur gr. 14 1 $\frac{1}{2}$, hoc perinde esset ac si horis 50 26' responderent gr. 28 3'

X. Si triangulum rectangulum, seu planum, seu sphæricum, omnia latera minora habuerit, quàm 1° seu 60': tunc datis duobus lateribus circa rectum, per logarithmos invenire latus tertium recto subtensum, seu basin.

Facilius quidem fit, in sphæricis quidem per antilogarithmos canonis, seu ut cap. VII dicatur, per logarithmos complementi: si tamen per opportunitatem etiam sexagenariâ placeat uti: sic poterit operatio institui.

Logarithmos laterum excerptos singulos seorsim duplica, cum his duplis, ut logarithmis, excerpe laterum quadrata ex sexagenariâ, eaque in unam summam adde, summa quæsitâ in sexagenaria, exhibebit è regione logarithmum, cujus semissis, ut logarithmus quæsitus, ex eadem sexagenariâ exhibebit basin quæsitam.

Ut si sint latera 28' 17" & 50' 31", eorum logarithmi 75200 & 17200, dupli horum 150400 & 34400, ostendunt quadrata laterum 13' 20" & 42' 32". Summa utriusque est 55' 52"; hujus logarithmus 7140, et hujus semissis 3570, ostendit basin 57' 54".

XI. Vicissim datâ basi, in sic comparatis, & latere alterutro, invenire latus reliquum.

Rursum duplicia logarithmos basis & lateris dati; per hæc dupla excerpe quadrata ex sexagenariâ, minusque à majori subtrahe; residui logarithmus est duplus logarithmi lateris quæsitî: ut si basis 57' 54", latus 28' 17", logarithmi 3570, 7140, duplicati sunt 7140, 150400, quadrata 55' 52" & 13' 20", residuum 42' 32", logarithmus 34400, dimidium 17200, quod ostendit 50' 31" latus quæsitum.

CAPUT VII.

De Copulatione Columnæ Arcuum cum Columnâ Logarithmorum.

Per solos Logarithmos Arcuum & Angulorum, omnia Triangula Sphærica solvere, omnia scilicet quæsitæ ex tribus datis eruere.

PRIMA autem sunt rectangula, eorumque casus sedecim, in quibus inter tria data intelligitur semper ipse rectus angulus.

Hic autem vox complementi crebrò usurpanda, cùm sic solitariè sumitur, semper dat subintelligi ad quadrantem: Basis verò vox pro latere maximo sumitur, quod scilicet angulo recto opponitur. Et loquimur de talibus rectangulis, quorum latera sunt minora quadrante singula.

I. Ex basi & angulo latus oppositum.

Addantur invicem logarithmi datorum, conficitur logarithmus quæsitæ lateris.

II. Ex lateribus basin.

Addantur invicem logarithmi complementorum datorum laterum, conficietur logarithmus complementi quæsitæ basis. Vide cap. vi numero x modum alium.

III. Ex angulo & latere adjacente angulum reliquum.

Addantur invicem logarithmi anguli, & complementi lateris, conficietur logarithmus complementi quæsitæ anguli.

IV. Ex basi & latere angulum oppositum.

Auferatur logarithmus basis à logarithmo lateris, restabit logarithmus anguli oppositi.

V. Ex angulo & latere opposito basin.

Auferatur logarithmus anguli à logarithmo lateris oppositi, restabit logarithmus basis.

VI. Ex angulo & latere subtenso angulum reliquum.

Auferatur logarithmus complementi lateris dati à logarithmo complementi anguli dato oppositi, restat logarithmus ipsius anguli residui quæsitæ.

VII. Ex latere & basi latus reliquum.

Auferatur logarithmus complementi lateris à logarithmo complementi basis, restabit logarithmus complementi lateris reliqui. Vide suprà cap. vi num. xi modum alium.

VIII. Ex angulis latus.

Auferatur logarithmus anguli, qui lateri quæsito adjacet, à logarithmo complementi anguli oppositi, restat logarithmus complementi lateris quæsiti.

IX. Ex lateribus angulum.

Pro his compositis casibus, aptior est canon ipse logarithmorum quadrantis, quam Chilias nostra: quia in illo non opus est excerptione arcus, quippe cum ejus complementum statim cum logarithmo suo è regione in conspectum veniat. Ut jam non dicam quòd me-sologarithmi canonis, quibus caret Chilias nostra, præstent hos compositos etiam simplices.

Addantur in unam summam logarithmi complementorum datorum, laterum summa excerptat arcum: Ejus arcus complementi logarithmus, auferatur à logarithmo lateris quæsito angulo oppositi, restat logarithmus anguli quæsiti.

X. Ex basi & angulo residuus angulus.

Addantur in unam summam logarithmi datorum, summa excerptat arcum: ejus complementi logarithmus auferatur à logarithmo complementi anguli dati, restabit logarithmus anguli quæsiti.

XI. Ex angulo & latere adjacente, latus oppositum reliquum.

Addantur in unam summam logarithmi, anguli & complementi lateris, summa excerptat arcum: ejus complementi logarithmus auferatur à logarithmo complementi anguli dati, restabit logarithmus complementi lateris quæsiti.

XII. Ex basi & angulo latus adjacens.

Addantur logarithmi datorum, summa excerptat arcum; ejus complementi logarithmus auferatur à logarithmo complementi basis, restabit logarithmus complementi lateris quæsiti, angulo adjacentis.

XIII. Ex basi & latere angulum adjacentem.

Auferatur logarithmus complementi lateris à logarithmo complementi basis, residuum excerptat arcum, ab hujus complementi arcu, auferatur logarithmus basis, restabit logarithmus anguli adjacentis.

XIV. Ex

XIV. Ex angulo & latere opposito latus reliquum.

Auferatur logarithmus anguli à logarithmo lateris, residuum excerpit arcum, à cujus complementi logarithmo, auferatur logarithmus complementi lateris: restabit logarithmus complementi lateris reliqui.

XV. Ex angulo & latere adjacente basin.

Addantur invicem logarithmi anguli & complementi lateris; summa excerpit arcum; hujus complementi logarithmus auferatur à logarithmo lateris dati, restabit logarithmus basis.

XVI. Ex angulis basin.

Auferatur logarithmus anguli unius à logarithmo complementi anguli alterius; residuum excerpit arcum, ab hujus complementi logarithmo, aufer logarithmum ipsius anguli alterius, seu posterius hinc adhibiti, restat logarithmus basis.

EXEMPLUM CASUUM OMNIUM.

Sit Basis $72^{\circ} 0'$. Angulus $50^{\circ} 0'$.

				Logarithmī.	Complementa.	Logarithmī.
Basis	72°	$0'$	F	5030 A	$18^{\circ} 0'$ L	117440 O
Angulus	50	0	G	26650 B	40 0 M	44190 P
Latus oppositum	46	45	H	31680 C	43 15 N	37805 Q
Latus majus	63	12	I	11370 D	26 48 R	79635 T
Angulus major	69	48	K	6340 E	20 12 S	106350 V

Hoc exemplum sic fuit constructum.

F, G, sunt ex arbitrio, quæ dant L, M. Ergò habentur A, B, O, P, jam per I ex A, B, habetur C hinc H hinc N hinc Q. Tunc per VII. ex O, Q habetur T, hinc R hinc I hinc D, etiam per IV. ex A, D habetur E hinc K hinc S hinc V. Reliqua præcepta omnia erunt loco probationis, verbi causâ ex prim. Ex basi F & angulo K latus I illi oppositum, invenitur enim D qui dat I.

Ex primo.	Ex secundo.	Ex Tertio.
5030 A	37805 Q	26650 B
6340 E	79635 T	6340 E
11370 D	117440 O	79635 T
		106285 V
		pro 106350
		44145 P
		pro 44190

Ex

Ex quarto.		Ex quinto.		Ex Sexto.	
5030 A	26650 B	6340 E	44190 P	106350 V	
31680 C	31680 C	11370 D	37805 Q	79635 T	
26650 B	5030 A	5030 A	6385 E pro 6340	26715 B pro 26650	
Ex septimo.		Ex octavo.		Ex nono.	
117440 O	26650 B	6340 E	73805 Q	79635 T	
79635 T	106350 V	44190 P			
37805 Q	79700 T pro 79635	37850 Q pro 37805	117440 O 18° O'	117440 O	
			31680 C	11370 D	
			5030 A 72 O	5030 A	
			26650 B	3640 E	
Ex decimo.		Ex undecimo.		Ex duodecimo.	
5030 A	5030 A	26650 B	6340 E	5030 A	5030 A
26650 B	6340 E	79635 T	37805 Q	26650 B	6340 E
31680 C	11370 D	106285 V	44145 P	31680 C	11370 D
73805 Q	79635 T	6340 E	26650 B	37805 Q	79635 T
44190 P	106350 V	44190 P	106350 V	117440 O	117440 O
6385 E pro 6340	26715 B pro 26650	37850 Q	79700 T	79635 T	37805 Q
Ex decimo tertio.		Ex decimo quarto.			
37805 Q	79635 T	26650 B	6340 E		
117440 O	117440 O	31680 C	11370 D		
79635 T	37805 Q	5030 A	5030 A		
11370 D	31680 C	117440 O	117440 O		
5030 A	5030 A	37805 Q	79635 T		
6340 E	26650 B	79635 T	37805 Q		
Ex decimo quinto.		Ex decimo sexto.			
26650 B	6340 E	26650 B	6340 E		
79635 T	37805 Q	106350 V	44190 P		
106285 V	44145 P	79700 T	37850 Q		
6340 E	26650 B	11370 D	31680 C		
11370 D	31680 C	6340 E	26650 B		
5030 A	5030 A	5030 A	5030 A		

Primi

Primi igitur octo casus perficiuntur per operationem simplicem; reliqui octo per duplicem, in quibus scilicet semper aliquid aliud quæritur ante id, quod proponitur, per unum ex octo prioribus.

Resolvuntur itaque 9 in 2 & 4. Sic 10 in 1 & 6. Sic 11 in 3 & 8. Sic 12 in 1 & 7. Sic 13 in 2 & 14. Sic 14 in 5 & 7. Sic 15 in 3 & 5. Sic 16 in 8 & 5.

Excerpti autem studio logarithmos rudes, ut si arcus propositus caderet medius inter duos expressos in Chiliade, medium etiam aliquid inter duos illorum logarithmos eligerem, rotundo fine; ut monerem, etsi neque arcus, neque eorum logarithmi in Chiliade rotundi sint, nihil tamen opus esse, ut calculator sese ubique maceret, minutias confectando, quod equidem grave esset in hac chiliade: facilius aliquantò in canone quadrantis; in quo arcus rotundi sunt, & singula minuta suos habent logarithmos.

Aliud etiam compendium non contemnendum (ut ad marginem monui) habet canon præ meâ chiliade, quod is arcuum complementa cum suis logarithmis exhibet è regione. Itaque quod ego in chiliade cogor circumloqui longius, logarithmum scilicet complementi arcus, lateris, basis, vel anguli: id in canone brevius exprimimus, antilogarithmum arcus, lateris, basis & anguli dicentes: ut innuamus excerptum esse antilogarithmum è regione logarithmi.

Tertium compendium canonis est in eo, quod is habet etiam mesologarithmos, pro quos, casus octo posteriores sunt simplices, qui sunt hic duplices: verum hoc canonis compendium apud inexercitatos conjunctum est cum dispendio, quod distrahitur animus additionibus & subtractionibus mesologarithmorum coëcis, hoc est, multitudine cautionum, quibus arithmetica coëfica constat.

Sed quod attinet logarithmos, paulò exactiores ex ipsâ etiam chiliade excerptendos, ut etiam in hoc satisfiat curiositati quorundam: tradam cap. sequenti VIII modum elaborandi logarithmum cuique finui respondentem: capite verò IX modum alium subsidiarium, capiendi pro logarithmis partem proportionalem vulgarem, idque sine labore, mediantibus aliis logarithmis.

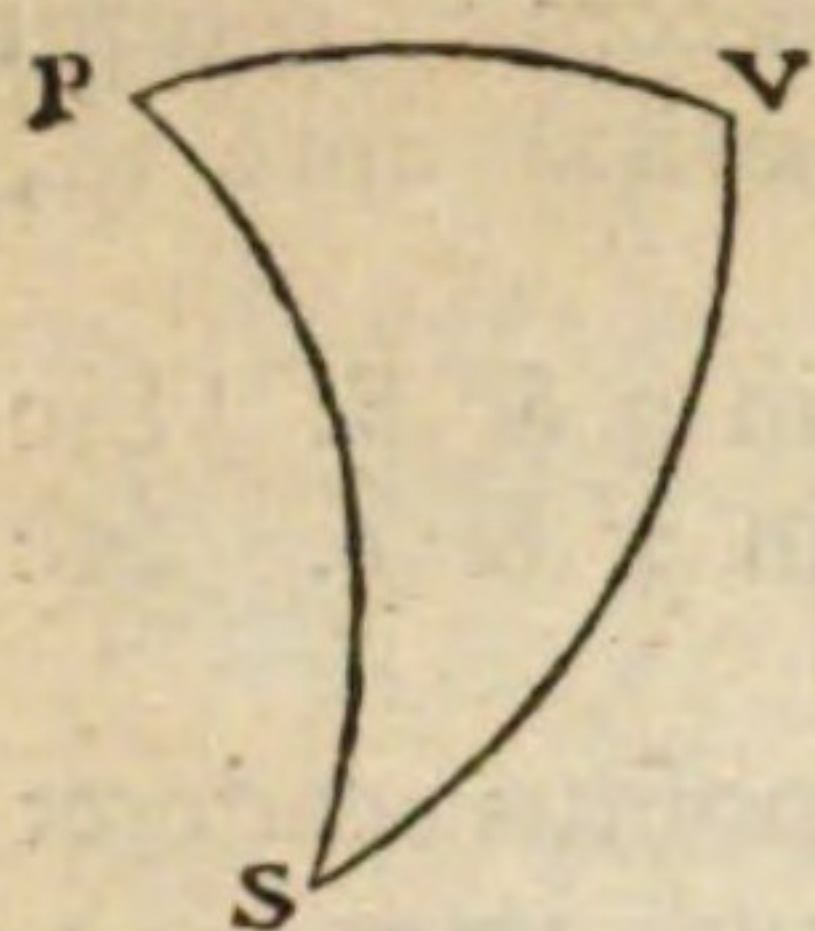
Hactenus igitur de triangulis sphaericis rectangulis egimus. Nunc ad obliquangula transeamus.

DE TRIANGULIS OBLIQUANGULIS SPHÆRICIS; IN QUIBUS CASUS
SUNT 12.

I. Si dentur duo latera, & angulus uni oppositus: quæraturne angulus alteri datorum laterum oppositus: ad logarithmum anguli, adde logarithmum lateris dati, quæsito oppositi, à summa aufer logarithmum lateris angulo dato oppositi, restabit logarithmus anguli quæsit, recto seu majoris, seu minoris.

EXEMPLUM CASUS PRIMI, LOGARITHMIS RUDIBUS SEU ROTUNDIS.

In triangulo PVS data sunt latera duo PV $38^{\circ} 30'$, VS $40^{\circ} 0'$, angulus VPS $31^{\circ} 34'$, oppositus ipsi VS. Quæritur angulus VSP lateri VP oppositus.



VPS $31^{\circ} 34'$ logarith. 64720

VP $38^{\circ} 30'$ logarith. 47480

Summa 112200

vs $40^{\circ} 0'$ logarith. 44190 subtr.

Residuum 68010 ut.

Logarithmus dat vsp $30^{\circ} 26'$ circiter.

Hic sitne ipse quæsitus, an ejus residuum ad semicirculum, facile discerni poterit ex habitudine laterum, præsertim oppositi PV. Non est enim mei instituti minimas cautiones hâc vice confectari.

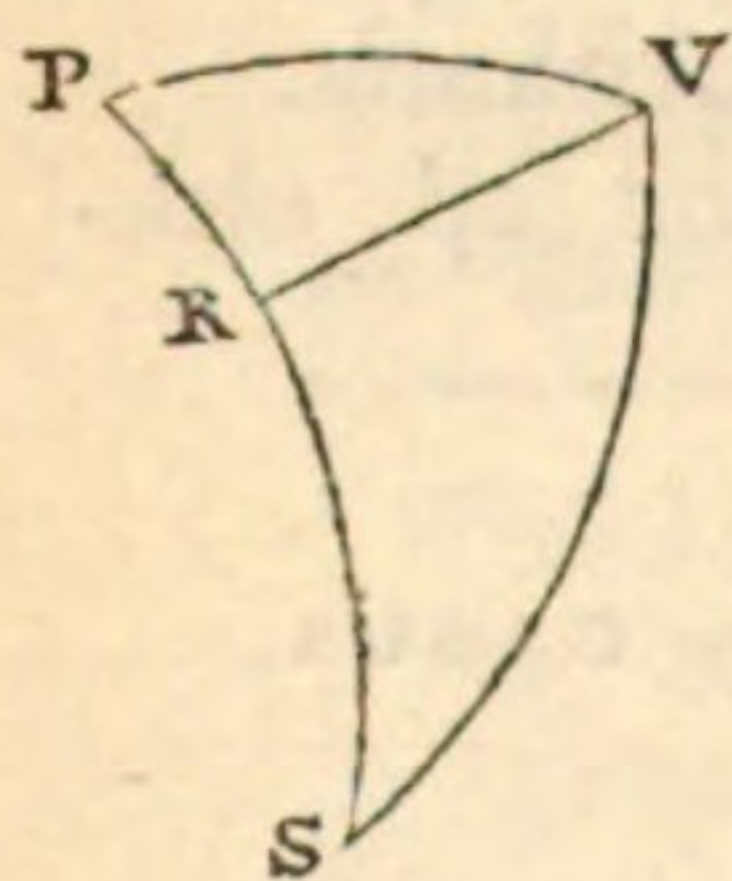
II. Si iisdem datis, quæraturs latus tertium: Tunc adde logarithmos anguli dati, & lateris attigui dati, prodit logarithmus perpendiculi dictus. Ejus perpendiculi logarithmus complementi ablatus à datorum laterum logarithmis complementi, relinquit logarithmos duorum arcuum, quorum vel differentia, vel complementorum summa, est latus quæsitum tertium.

III. Si iisdem datis, quæraturs angulus inter latera data comprehensus.

Tunc quæraturs latus tertium, ut præcepto secundo horum. Ejus logarithmo adde logarithmum anguli dati, à summâ aufer logarithmum lateris, cui datus angulus opponitur: restat logarithmus anguli comprehensi, ejusque complementi ad duos rectos.

EXEMPLUM II. ET III. CASUUM.

Iisdem datis, quæraturs PS & PVS.



	Logarith.		
VS $40^{\circ} 0'$	44190	Compl. $50^{\circ} 0'$	Log. 26660
VPS $31^{\circ} 34'$	64720		
VP $38^{\circ} 30'$	47480	Compl. $51^{\circ} 30'$	Log. 24510
VR $19^{\circ} 0'$	112200	Compl. $71^{\circ} 0'$	Log. 5600
SR $35^{\circ} 54'$		Compl. $54^{\circ} 6'$	Log. 21060
PR $34^{\circ} 8'$		Compl. $55^{\circ} 52'$	Log. 18910
SP $70^{\circ} 2'$	Log. 6190		
			70910
SVP $49^{\circ} 57'$	log. 26720		

Residuum hoc est, seu complementum anguli ad duos rectos, quòd facile patet ex magnâ proportionem PS ad PV, vs. Quando verò dubitari potest, tunc quærenda sunt ejus elementa PVR, RVs.

IV. Si

IV. Si datis duobus lateribus & angulo comprehenso, quærat^r latus tertium.

Tunc ad logarithmum lateris minoris, adde logarithmum comprehensi, prodit logarithmus perpendiculi in latus majus, quod perpendiculum constituit duo illius lateris elementa: Hujus enim perpendiculi logarithmus complementi ablatus à logarithmo complementi lateris minoris, dat logarithmum arcus, cujus complementum est alterum ex illis elementis, eoque vel addi, vel auferri debet à toto latere majori, summæ vel residui, ut elementi alterius, logarithmum complementi, adde logarithmo complementi perpendiculi, fietque logarithmus complementi lateris tertii quæsitⁱ. Prior casus est, quando angulus comprehensus & datus obtusus est; posterior, quando acutus.

V. Si datis duobus lateribus & angulo comprehenso, quærat^r angulorum residuorum unus.

Tunc quære latus tertium, ut in quarto præcepto horum. Ejus logarithmum aufer à summa logarithmorum anguli initio dati, & lateris angulo quæsit^o oppositi, restabit logarithmus anguli quæsitⁱ.

EXEMPLUM IV. ET V. CASUUM.

Dentur VP, PS & VPS, quærat^r VS & PVS.

			Logar.	Compl.	Logar.
	PV	30° 30'	47480	51° 30'	24510
	VPS	31 34	64720		
Perpendiculum	VR	19 0	112200	71 0	5600
Elementum unum	PR	34 8		55 52	18910
	PS	70 2	6190		
Elementum alterum	RS	35 54		54 6	21070
Quæsitum latus	VS	40 0	44190	50 0	26670
Summa log.	VPS & PS		70910		
Quæsit. angulus	PVS	49 57	26720.		

VI. Si datis tribus lateribus quærit^r angulus.

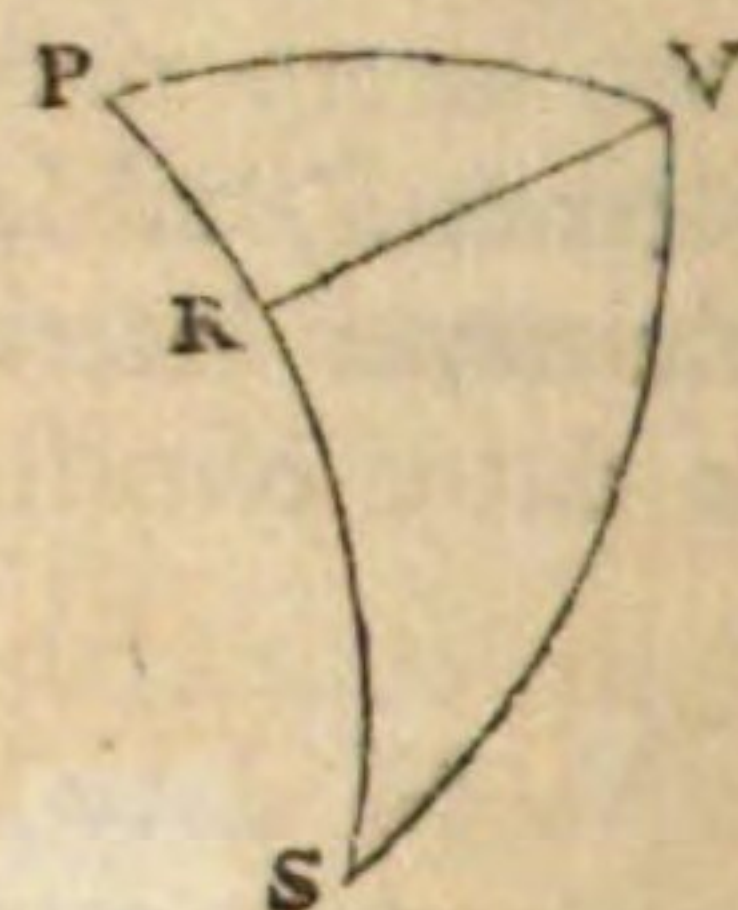
Tunc laterum circa quæsitum angulum exquire differentiam; illam à latere tertio aufer, eidemque adde, & tam residui quàm summæ constitue semisses. Hâc præparatione præmissâ, jam excerpe quatuor logarithmos, duos semissium, & duos laterum circa quæsitum angulum, summamque horum aufer à summa illorum:

illorum: Differentiæ, quæ futura est, semissis, ut logarithmus, ostendet semissimam anguli quæsitum.

EXEMPLUM SEXTI CASUS.

Dentur VP, PS, SV quæraturs VPS.

	VP	38°	30'	Log.	47480	
	PS	70	2	Log.	6190	
Diff.		31	32		53670	summa
	VS	40	0			
				Semiffes		
Residuum		8	28	4	14	Log. 260400
Summa		71	32	35	46	Log. 53700
						314100 summa
						260430 summar. diff.
Quæsitus	VPS	31	34	15	47	Log. 130215 semissis, quæraturs.



Quæraturs PVS.

	VP	38°	30'	Logarithmi.	47480	
	VS	40	0		44190	
		1	30		91670	
	PS	70	2			
				Semiffes		
		68	32	34	16	57440
		71	32	35	46	53700
						111140
						19470 summar. diff.
Quæsitus	PVS	130	14	65	7	9735 dimid.

Quæraturs

Quærat^r PSV.

			Logarithmi.	
SV	40°	0'	44190	
PS	70	2	6190	
	30	2	50380	
	38	30		
	8	28	4	14
	68	32	34	16
			Semiffes	
			260400	
			57440	
			317840	
			267460	summar. diff.
			133730	dimidium.

Quæsitus VSP 30 26 15 13

VII. Si datis duobus angulis, & latere alteri datorum subtenso, quærat^r latus reliquo subtensum: Ad logarithmum lateris dati, adde logarithmum anguli adjacentis dati, à summâ aufer logarithmum anguli, lateri dato oppositi; restabit logarithmus lateris quæsitⁱ, ejusque complementi ad semicirculum: ubi pro ratione quantitatis datorum, vel ipse logarithmi arcus, vel ejus complementum ad semicirculum sumendum est pro latere.

EXEMPLUM VII. CASUS.

Dentur PVS. VPS, VS, quærat^r PS.

			Logarith.	ALIUD.			
VS	40°	0'	44190	Latus	40°	0'	44194
PVS	130	3	26720	Ang.	30	28	67917
			70910				112111
VPS	31°	34'	64720	Ang.	80°	0'	1531
Quæsitum vs	70	2	6190	Lat. quæf.	19°	19'	110580

VIII. Si iisdem datis quærat^r angulus tertius.

Tunc adde logarithmos lateris dati, & anguli alterius adjacentis: prodit logarithmus perpendiculari ex angulo quæsit^o demissi; cujus logarithmus complementi ablatus à datorum angulorum logarithmis complementi, relinquit logarithmos duorum elementorum anguli quæsitⁱ, à perpendicularo constitutorum: quorum ideò vel summa constituit quæsitum, si perpendicularum cadit intra triangulum, vel differentia, si extra.

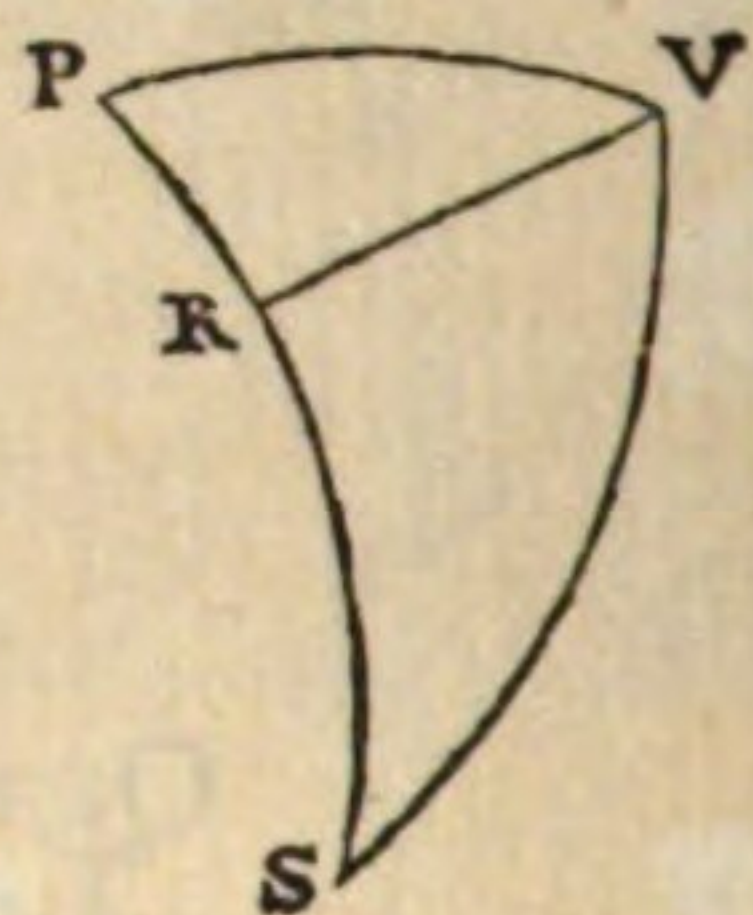
IX. Si iisdem datis, quærat^r latus interjectum datis angulis.

Tunc quærat^r angulus tertius, ut præcepto VIII. horum. Ejus logarithmo adde logarithmum lateris dati; à summa aufer logarithmum anguli, qui lateri illi dato opponitur, restabit logarithmus lateris interjecti.

EXEMPLUM VIII. ET IX. CASUUM.

Dentur PVS, VPS, VS, quærat^r VSP & VP.

			Logarith.			
VS	40°	0'	44190	Compl.		Logar.
PVS	130	3	26720	40°	3'	44080
Perpendic. ex angulo s	29	29	70910	60	31	13870
VPS	31	34		58	26	16000
Elementum unum			47	40		30210
Elementum alterum			78	14		2130
VSP quæsitus			30	34	per subtractionem.	



NOTA.

VSP	30° 34'	67630	Arcus	130° 3'	} Logarithmus est idem.
		111820	&	49 57	
VPS		64720	At	40 3	
Quæsitus VP	38 37	47100	est minoris qua- drante complementum; majoris qua- drante excessus.		

ALIUD EXEMPLUM.

			Logarith.	Compl.		Logarith.
Latus	40°	0'	44194	°		
Ang. adjac.	30	28	67917	59	32	14859
Perpendic. ex angulo	19	1	112111	70	59	5613
Ang. alter. lateri	80	0	1531	10	0	175072
oppof.						
Elementum unum				56	44½	99247
Elementum alterum				10	35	169470
Ang. quæsitus				76	19	per addit.
Angulus tertius	76°	19'	2879			
			47073			
Latus quæsitum angulo	39	22	45542			
tertio quæsito oppositum.						

X. Si

X. Si datis duobus angulis & latere interjecto, quærat^r angulus tertius, lateri dato oppositus. Tunc ad logarithmum anguli minoris, adde logarithmum lateris interjecti, prodit logarithmus perpendiculi ex majori angulo; cujus perpendiculi logarithmus complementi & affervari debet, et auferri ab anguli illius ejusdem logarithmo complementi, relinquetur logarithmus unius elementi illius anguli, unde perpendiculum est demissum. Hoc igitur ablatum ab angulo ipso toto, vel ille ab hoc, relinquit ejus elementum alterum.

Hujus iterum logarithmus addatur logarithmo affervato, ita fiet logarithmus complementi anguli tertii quæsi, vel ejus excessus supra quadrantem.

Si datorum angulorum alter est major quadrante, latus datum minus esse debet quadrante.

Si latus datum est quadrante majus, tunc elementum anguli, quod illi respondet, est etiam quadrante majus, & sic triangulum obtusangulum, ubi perpendicularis cadit extra.

XI. Si datis duobus angulis, & latere interjecto, quærat^r laterum residuorum unum: Tunc quære angulum tertium, ut in præcepto x horum, ejus logarithmum aufer à summa logarithmorum lateris initio dati, & anguli lateri quæsito oppositi, restabit logarithmus lateris quæsiti.

EXEMPLUM X. ET XI. CASUUM.

Dentur PVS, VPS, VP, quærat^r VSP & VS.

			Logar.	Compl.	
Ang. tot.	PVS	130° 3'	26720	40° 3'	Log.
	VPS	31 34	64720	58 26	16000
	VP	38 30	47480		
Perpendic. ex v		19 1	112200	70 59	5610 affervandus.
Unius elementi ang.		64 23		64 23	10390
Alterum elementum ang.		65 40	9300		
			5610 affervatus		
		59 29	14910		
Quæsitus	VSP	30 31	67780		
Quæsitum	VS	39 54	44420		

XII. Si datis tribus sphærici angulis, quærat^r laterum aliquod.

Tunc primum pro angulo maximo scribe & usurpa ejus complementum ad semicirculum, deinde angulorum latus quæsitum attingentium exquirat differentiam: illam ab angulo tertio aufer, eidemque adde, & tam residui quàm summæ constitue semisses.

Horum

Horum logarithmos adde in unam summam, à quâ aufer summam logarithmorum duorum angulorum latus attingentium: residui semiffis, ut logarithmus, dat semiffem lateris quæfiti ex columnâ arcuum; excipe casum unum, si nimirum in acutangulo triangulo, latus quæsitum subtenfum fuerit angulorum maximo: tunc operatione per ejus complementum ad semicirculum peractâ, ut prius, prodit semiffis, non ipsius lateris quæfiti, sed ejus itidem complementi ad semicirculum.

At in obtusangulo, etiam si latus quæsitum subtenfum fuerit obtuso, & sic maximo, valet nihilominus regula generalis, proditque semiffis lateris quæfiti ipsius.

EXEMPLUM XII. CASUS.

Dentur anguli PVS, VSP, SPV, quæraturs PS.

					Logarithmi.
		VSP	30° 28'		67910
		VPS	31 34		64720
	Differentia		1 6		132630 summa
Complementum ad semicirc.		PVS	49 57		
	Differentia		48 51	femiff.	88309
	Summa		51 3	24 25½	84187
				25 31½	
					172496 summa.
					39866 diff. sum.
Quæsitum	PS	70 0 35 0			19933 femiff.

Quæraturs PV.

					Logarithmi.
		PVS	130° 3'		26725
Compl. ad semicirc.			49 57		64724
		VPS	31 34		
			18 23		91449
		VSP	30 28	femiffes.	
			12 5	6 2½	225142
			48 51	24 25½	88309
					313451
					222002
Quæsitum	VP	38 29 19 14½			111001

Quæraturs

Quærat^r vs.



	PVS	130° 3'		Logarithmi.
Compl. ad femicirc.		49 57		26725
	VSP	30 28		67917
	VPS	19 29		94642
		31 34	femiffes	
		12 5	6 2½	225142
		51 3	25 31½	84187
				309329
Quæsitum	vs	39 57	19 58½	214687
				107343

ALIUD EXEMPLUM.

Sint anguli circa latus quæsitum.

	30° 28'		67917
	76 19		2879
Differentia	45 51		70796 summa
Ang. oppos.	80 0	maximus trium	
Compl. ad femicirc.	100 0		
	54 9	27° 4½	78712
	145 51	72 55½	4508
			83220 summa
			12424 diff. sum.
			6212 femiffis.
Quæsitus	140 2	70 1	
	39 58.		

Viciffim

Vicissim sint anguli circa quæsitum hi :

					Logarithmi.
Maximus trium	80°	0'			
Compl. ad semicirc.	100	0			1531
Alter	76	19			2879
Differentia	23	41			4410 summa
Tertius	30	28	semiffes		
Differentia	6	47	3	23 $\frac{1}{2}$	282750
Summa	54	9	27	4 $\frac{1}{2}$	78712
					361462 summa
					357052 diff. sum.
Quæsitum latus	19	19	9	39 $\frac{1}{2}$	178526 semiff.

Atque hæc de obliquangulis, & in universum de sphaericæ superficiei triangulis curvilineis, quæ formantur arcubus circulorum maximorum.

De rectilineis verò triangulis, seu quæ formantur in superficiei planâ, capite nono agetur.

C A P U T VIII.

De Copulatione Columnæ Sinuum, seu Numerorum Absolutorum, & Columnæ Logarithmorum.

P R Æ C E P T U M I.

Logarithmum excerpere accuratum Numeri scrupulosi propositi.

ET si rarissimè est opus tantâ scrupulositate, quoties tamen est opus, fit per prop. xxvii corollarium iii in hunc modum.

Propositi numeri scrupulosi excessum super rotundum chiliadis proximè minorem, continua, appositis versus dextram septem cyphris; sic continuatum divide per hunc ipsum proximè minorem, elaboratum tamen prius seu continuatum per unam insuper vel duas figuras scrupulosas, quæ contineant dimidium duarum primarum figurarum excessus, vel aliquid proximè minus dimidio :

dimidio : Quotientem aufer à logarithmo ad dictum rotundum proximè minore, restabit logarithmus propositi scrupulosi.

EXEMPLUM.

Propositus esto scrupulosus 23456,78
Chilias exhibet ad rotundum
proximè minorem 23400,00 Log. 145243,42
Igitur excessus scrupulosi est 56,78 242,36 quotiens auferatur

145001,66 hic est logarith. quæ-
situs accuratius

Ei apponantur 7 cyphræ 5678000000

Hunc divide per inventum rotun-
dum chiliadis proximi minorem 2340000

Sed elaboratum per duas figuras sig-
nificativas, quæ sint dimidium de
56 primis duabus figuris in excessu
scilicet per 28. Ut 2342800

Divisio compendiosa

56780	
23428	
46856	2
9924	
9371	4
553	
469	2
84	
70	3
14	
14	6

ALIUD EXEMPLUM.

Sing. gr. 81 scrupulosus 987 | 68,83 $\frac{2}{3}$ log.
Minor chiliadis 987 | 00,00,1308,52
Excessus appositis cyphris | 68834000000
Divisor | 98734 00 6
592404

959360
888606 9
707540
691138 7
164020
197468 2

Logarith. 1308,52
Quotiens 69,72
1238,80
quæsitus

Quotiens.

Hic prodeunt tantum quatuor figuræ.

ALIUD EXEMPLUM.

Sit finus 9°	15643,45	Log.	185789,93
Rotundi ergò	15600,00		
Excessus	43,45		278,15 quotiens subtr.
Divisor	15621		
	31242		185511,78 log. quæsitus.
		2	
	12208		
	109347		
		7	
	127330		
	124960		
		8	
	23620		
	15621		
		1	
	80000	5	

CAUTIO.

Rarò usu venit, ut opus nobis sit logarithmo tam scrupuloso & accurato: plerunque sufficit logarithmus rotundus, cui non sint plures à sinistris figuræ significantes, quàm in quot figuris logarithmi bini chiliadis, proximi invicem, nihil inter se differunt, unâ plus.

Ut quia logarithmus ad 23400,00 est 1452 | 43,40
 Sed logarithmus ad 23500,00 est 1448 | 16,97
 Sufficit igitur logarithmus rotundus 1450 | 00,00
 Quia 1450 est medium inter 1452 & 1448

CAUTIO ALIA.

Si vero omninò esset opus logarithmo accuratione, tunc scito, quod in logarithmis valdè magnis, ad numeros parvos chiliadis, divisor debeat esse vicinissimus medio geometrico, inter numerum datum scrupulosum, & inter proximè eo minorem chiliadis: paulò tamen major medio geometrico: ut docet allegata propositio.

EXEMPLI CAUSA.

Si quærat logarithmus numeri 150,00. Hic numerus scrupulosus est, quia cadit inter duos chiliadis proximos, scilicet inter 100,00 & 200,00. Excessus est 50,00. Si hunc divideres per 12500, quod est inter 15000 datum scrupulosum,

scrupulosum, & inter datum proximè minorem 10000 medium arithmeticum, prodiret quotiens 40000,00 auferendus à 690775,54; restaretque 650775,54. Sin autem dividas per 12000, geometricum medium inter 15000 & 10000 quotiens erit 41666,67, qui ablati à logarithmo proximè minoris, scilicet à 690775,54, relinquit 649108,87, quasi logarithmum ad propositum absolutum seu finem 125,00.

Quod verò logarithmus iste fit jam minor justo, sic patet per usum præcepti sequentis.

Pro 150,00 sume centuplum 15000,00, ejusque logarithmum 189712,00 adde logarithmo centuplicationis 460517,03. Sic colligitur logarithmus numeri 150,00, scilicet 650229,03.

Quare quotiens 4166667 quam subtrahebamus, erat major justo, ac proinde medium proportionale inter scrupulosum, & chiliadis proximè minorem, scilicet 120,00, erat divisor minor justo.

ALIA EXEMPLARIS OBSERVATIO,

Pro scrupulorum magnorum logarithmis accuratis indagandis.

Sit numerus scrupulosus		
Seu sinus 81°	98768,834	log. accuratus est 1238,80
Sagitta	1231,166	
Secantis complementi (9°) exc.	1246,513	
Summa	2477,679	
Horum med. arithmeticum	1238,840	
Differentia medii arith. & sagitt.	7,673	
Adde ad sagittam fit	1238,839	logarithmus ferè: qui differt
Vel differentia excess. & sag.	15,347	à logarithmo accurato solis 4.
Ejus dimidium	7,673	unitatibus loco secundo post
		punctum, scilicet quia logarithmus est adhuc parvus.

Sic per complementum scrupulosi parvum.

Sin $84^{\circ} 16'$	99499,762	
Chiliadis numerus major	99500,000	dat log. 501,25
Complementum scrupulosi par-		24 quotientem adde
vum	238000000	
Divisor	9950000	$\frac{2}{4}$ 501,49 log.
		quotiens
Hic sagitta	500,238	
& excess. secantis in canone	502,753	
Faciunt summam	1002,991	
Dimidium	501,496	paulò majus est logarithmo accurato.

Vide super hac re etiam prop. xxii corollaria & præcepta, fol. 21.

Ut compendiosè excerpas logarithmos numerorum quorundam scrupulorum, scilicet sine divisione.

C A S U S P R I M U S.

Si propositus numerus scrupulosus fuerit minor quàm 10000,00, minor scilicet quàm decima pars maximi in chiliade: sic ut logarithmus ejus sit valdè magnus, & differentiæ logarithmorum circa illa loca non magnæ tantum, sed insuper etiam notabilibus incrementis crescentes. Tunc logarithmo propositi decupli (qui facilius elaboratur) adde logarithmum decuplicationis, vel logarithmo centupli, logarithmum centuplicationis, &c; ita habetur logarithmus propositi numeri.

E X E M P L U M.

Propositus esto scrupulosus 2345,68. Hic cum sit minor quàm 10000,00, ergò quære logarithmum ejus decupli, sc. 23456,80. Is autem invenitur superiori methodo 145001,00. Huic igitur adde logarithmum decuplicationis

$$\begin{array}{r} 145001,00 \\ 230258,51 \\ \hline \text{Summa } 375259,51 \text{ est log. quæsitus numeri } 2345,68. \end{array}$$

A L I U D E X E M P L U M.

Propositus esto numerus scrupulosus 175,00 ut ejus logarithmus exactus habeatur: quære centupli 17500,00 logarithmum, qui est 174296,93
Et pro appositis 00 adde centuplat.

$$\begin{array}{r} 174296,93 \\ 460517,03 \\ \hline \text{Emergit numeri } 175,00 \text{ log. } 634813,96 \end{array}$$

ALIUD EXEMPLUM.

Sinus unius minuti solet exprimi hoc numero 29,09 quæritur hujus numeri logarithmus.

Scribe millecuplum	29090,00	
Hoc proximè minor invenitur	29000,00	log. 123787,43
Excessus	90,00.	Hic continuatus septem
cyphris, & divisus per 29045,00 medium arithmeticum inter datum & excerptum: seu ille continuatus 5 cyphris, & divisus per 29045, dat quotientem 309,87, qui ablatas ab exscripto logarithmo, relinquit logarithmum millecupli		
		123477,56
Huic igitur adde millecuplatorem		690775,54
Provenit logarithmus quæsitus		814253,10

Quod si numerus 29,09 exactè exprimeret finem unius minuti, tunc hic etiam exactus effet logarithmus unius minuti: at quia in 29,09 ultima unitas non est omninò plena, ideò etiam logarithmus unius minuti est paulò major: ut quidem Urfinus eum exprimit sic 814257.

CASUS SECUNDUS.

Idem ferè processus est, si datus scrupulosus fuerit major quidem quàm 10000,00 parvus tamen etiamnum & minor quàm maximus chiliadis. Tunc enim logarithmo dupli, vel tripli, vel quadrupli (quod quidem non excedat maximum chiliadis) additur logarithmus totuplans.

Ut si detur numerus scrupulosus 23456,78.

Hic etsi major est quàm 10000,00, est adhuc tam parvus, ut ejus quadruplus scilicet 93827,12 sit adhuc minor maximo chiliadis. Quærat igitur hujus logarithmus:

Adde	93800,00 est log. 6400,53	
Appendix scrupulosa	27,12 aufert	28,82 quotiens auferend.
		138629,44 log. 4 druplans add.
Hæc dividitur per	938135	
	18,76270 2	145001,15 logarithmus quæsitus.
	8357300	
	7505080 8	
	852220	
	835730 8	
	16490 2	

Hic

Hic casus generaliter proponi potest, ut circumspicias, in quâ proportionē (terminorum quidem rotundorum) sit numerus propositus ad aliquem proximè minorem maximo chiliadis. Tunc constituitur proportionis illius logarithmus, & per eum augetur logarithmus illius proximè minoris.

Sit sinus quicunque, puta 15643,45; hic continetur in 100000,00 sexies, sextuplum enim est 93860,70. Huc potest accedere triens 5214,48, & conficitur 99075,18. Est igitur 15643,45 ad 99075,18 ut 19 ad 3, logarithmus novem decupli est 299443,90. Hinc aufer triplicantem 109861,23

Logarithmus igitur proportionis est 189582,67.

Quare logarithmus illius multiplicis proximè minoris maximo chiliadis per divisionem elaboratus est iste :

Exc. 75180000000		99075,18	1005,03
Div. 99037000000		99000,00	75,91 quotiens
693259	7		
385410		Exc. 75,18	929,12
495185	5		184582,67
902250		Adde log. prop.	185511,79
891333	9	Log. quæsitus numeri absoluti 15643,45	
10917	1 quotiens.		

CASUS TERTIUS.

Si scrupulosus, minor quàm 50000,00 habuerit non plures quàm 4 figuras significativas, earumque ultima fuerit quinaris.

Tunc dupli vel quadrupli logarithmo in chiliade invento, adde logarithmum duplicantem vel quadruplicantem.

EXEMPLA.

Ut si datus sit numerus scrupulosus 49950,00 minor scilicet quàm 50000,00, ejus dupli 99900,00 logarithmo 100,05, adde logarithmum duplicationis 69314,72, fit 69414,77 logarithmus quæsitus.

ALIUD.

Suprà casu primo quærendus fuit logarithmus ad numerum 17500,00, propter scilicet ejus centesimæ 175,00 logarithmum indagandum. Hic si non positus effet in ipsâ chiliade, posset sic investigari.

Duplum

Duplum est 35000,00, quadruplum 70000,00. Hujus verò logarithmus est	35667,49.
Huic adde logarithmum quadruplatorem	138629,44
Prodit qui supra	174296,93

CASUS QUARTUS.

Si datus scrupulosus, duplicatus continuè, tandem fortiatur non plures figuras significativas, quàm tres primas à sinistris, sed tunc excefferit maximum chiliadis, tum logarithmo partes 10 vel 100 de hoc sic multiplicato, adde logarithmum duplicationis vel quadruplicationis, &c, à summâ aufer logarithmum decuplationis vel centuplationis, &c.

Ut si proponatur numerus scrupulosus	51250,00
Hujus duplus est	102500,00
Et hujus duplus est	205000,00 habens

significativas figuras non ultra tres.

Hic verò jam est major quàm 100000,00. Ergò ejus partis decimæ 20500,00

Logarithmo	158474,53
Adde logarithmum quadruplicantem hoc loco	138629,44
Et aufer logarithmum decuplantem hoc loco	230258,51

Restat numeri 51250,00 logarith.	66845,46
----------------------------------	----------

Potuit etiam sumi duplus ipsius 205, &c, scilicet 410, &c, & rursus hujus duplus 820, &c, hujus log.

Adde fedecuplatorem	19845,09
	277258,88

	297103,97
Et aufer decuplatorem	230258,51

Restat idem quod prius	66845,46
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CASUS QUINTUS.

Si datus scrupulosus habuerit partem aliquotam, quæ solas tres figuras à sinistris habeat significativas: Tunc logarithmo illius partis aufer logarithmum totuplantem.

Ut

Ut si proponatur numerus 19980,00 scrupulosus.
 Hujus semissis est 9990,00. Hujus numeri logarithmus per præ-
 ceptum primi casus invenitur sic :

Decupli 99900,00. Logarith. est	100,05
Huic adde logarithmum decuplationis	230258,51

Ita fit log. 9990,00 partis dimidiæ	230358,56
Hinc aufer logarithmum duplationis	69314,72

Restat log. numeri 19980,00 quæsitus	161043,84
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PRÆCEPTUM II.

Dato numero absoluto, qui excedit maximum chiliadis (ut sunt
 secantes arcuum) invenire ejus logarithmum.

Excedentis dati constitue partem aliquotam, quæ non excedat, puta decimam, centesimam, millesimam, &c; vel etiam dimidiam, tertiam, quartam, per Cas. II præced.

Ejus partis in chiliade quæsitæ logarithmus, auferatur à logarithmo totuplationis (ut decuplationis, centuplationis, millecuplationis, vel etiam duplationis, triplationis, quadruplationis) restabitque logarithmus excedentis privativus.

Ut si quærat logarithmus numeri 102400,00. Hic numerus habet loca octo, cum chiliadis maximus 100000,00, habens loca itidem octo, sit tamen minor. Quare sumatur numeri dati pars decima 10240,00 utpote quæ jam fit minor maximo chiliadis. Hic numerus cum jam sit in quarto loco scrupulosus, logarithmum exactum non invenit in chiliade, quare per præcept. præcedens elaborandus est ejus logarithmus sic :

Numerus 10240,00			
In chil. prox. mino.	10200,00	habens log.	228278,25
			Quotiens 391,39 subtr.
Exces. nostri prolong.	40,0000000000		
Divisor est	1022	000	
Medium scil.	3066		
Arithmeticum inter		3	
datum, & proximè	9340		
minorem chiliadis	9198	9	
	1420		
	1022	1	
	3980		
	3066	3	
	9140		
	9108	9	

227886,86. Hic est logarithmus partis decimæ: qui ablatus à logarithmo 230258,51 decuplationis, relinquit logarithmum 2371,65 privativum, signo —, numeri scilicet dati excedentis 102400,00.

In hoc exemplo, quia pars decima numeri propositi fuit scrupulosa, præstabat nos uti præcepti prioris casu v.

Ecc. numeri 102400,00 propositi.

Semissis 51200,00 rotundus, habet log. 66943,07 subtr.

Logarithmus duplicationis est 69314,72

Restat quod prius 2371,65

In genere, si fuerint tres numeri continuè proportionales, eorumque medius idem, qui & maximus chiliadis; tunc excedens maximum hunc chiliadis logarithmum habet eundem cum minimo trium, sed privativum, signo — præponendo, ne videatur nihil aliud significare, quàm defectum in figurâ ultimâ, ut cap. 1.

Ut si sit 80000,00 ad 100000,00, ut hic ad 125000,00.

Quia igitur numeri 80000,00, logarithmus est 22314,36 positivus, erit ergò & numeri 125000,00, logarithmus 22314,36 sed privativus.

Aliter & facilitate inopinabili.

A dato numero excedente rejice tres ultimas figuras ad dextram. Sic curtatum quære inter logarithmorum differentias interlineares: & logarithmum ipsum ei differentiæ proximum, planeque respondentem exscribe; erit enim hic ipse logarithmus numeri propositi excedentis, sed privativus.

Ut si quærat, quis sit logarithmus intervalli Martis & solis 152500, seu prolongati 152500,00, ut sit intervallum solis & terræ mediocri 100000,00,

T

deletis

deletis ergò tribus ultimis, restat 152,50. Huic verò numero proximus invenitur inter logarithmorum incrementa interlinearia iste 152,55, habens ante se logarithmum 42312,00, post se 42159,45, ut sic ipsi differentiæ 152,55 intermediæ, respondeat etiam intermedius 42230,00 circiter. Hic igitur est numeri 152500,00 logarithmus privativus.

ADMONITIO DE SECANTIBUS.

Atque hinc apparet, etiam secantes ipsos arcuum complementorum quodammodo haberi per differentias logarithmorum, ad illos arcus, prolongatas tribus locis, quod suprà cap. II. promisi me indicaturum.

ADMONITIO ALIA.

Hic potest vicem chiliadis implere, canonis Neperiani columna media, mesologarithmorum, juncto canone tangentum in hunc modum.

Datum numerum excedentem quære inter tangentes, et nota ejus arcum in gradibus & scrupulis: cum hoc excerpe ex canone mesologarithmorum, mesologarithmum privativum, habebis sic excedentis logarithmum privativum.

PRÆCEPTUM III.

Dato logarithmo, qui non exactè reperiatur in chiliade, assignare numerum absolutum justum cum scrupulositate suâ, sicubi eâ sit opus.

Dato logarithmo proximè majorem exscribe ex chiliade, cum numero absoluto rotundo respondente, factâque subtractione dati ab exscripto, residuum duc in absolutum exscriptum ut multiplicantem, factus, abscatis septem ultimis ad dextram locis, collocetur loco quatuor cyphrarum exscripti absoluti: Ita prodit quam proximè justus absolutus, dato logarithmo competens. Sed quia ei adhuc deest aliquid, corrigetur sic, si facti curtati dimidium colles locis ultimarum cyphrarum multiplicantis, & multiplicationem repetas.

EXEMPLUM.

Datus esto log. 145001,10 subtr.
Hoc prox. major
Chiliadis log 145243,42 ei resp. absol. 23400,00

Residuum 242,32
Duc in abs. exscript. 234,00,00

4846|4
726|96
96|9280000

Factus abscatis 7 ultimis 567c| Hic esset addendus ad exscriptum

tum absolutum, fieretque 23456,70. Sed quia pars addenda nondum est perfecta; ideò dimidium ejus 2835 appono ad multiplicantem, ut fiat 2342835, repetoque multiplicationem sic:

$$\begin{array}{r}
 242,32 \\
 234,2835 \\
 \hline
 4846 \overline{)4} \\
 726 \overline{)96} \\
 96 \overline{)928} \\
 4 \overline{)8464} \\
 1 \overline{)93856} \\
 72696 \\
 121160 \\
 \hline
 \end{array}$$

Factus correctior 5677

Compendiosè sic,
quia ultimi 35,
parum aut nihil
efficiunt.

$$\begin{array}{r}
 242,32 \\
 234,28 \\
 \hline
 4846 \overline{)4} \\
 727 \overline{)0} \\
 96 \overline{)9} \\
 4 \overline{)8} \\
 1 \overline{)9} \\
 1
 \end{array}$$

5677 |

Facilè autem videbit exercitatus calculator, non totam sibi multiplicationem esse repetendam, sed multiplicatione per anteriores figuras multiplicantis peractâ, & summâ constitutâ, jam id quod per figuras anteriores semissis de emergente facto, fit, tantummodò subjiciendum esse decenter facto priori, in hunc modum.

$$\begin{array}{r}
 2423,4 \\
 234 \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 4846 \overline{)4} \\
 727 \overline{)0} \\
 96 \overline{)9} \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 5670 \overline{)3} \\
 \hline
 28,4 \overline{)8} \\
 1 \overline{)9} \\
 \hline
 \end{array}$$

$$5677 |$$

Hoc igitur correctiori facto apposito

ad primo exscriptum 23400,00

Fit absolutus justus 23456,77

CAUTIO.

In logarithmis valdè magnis, non est accuratissima hæc ratio: id tamen, quod peccatur, utcunque magnum, in logarithmis magnis, non efficit tamen sensibile quippiam in iis, quorum causâ sunt logarithmi.

ALIA ADMONITIO.

Plerunque non est nobis opus scrupulositate numeri quæfiti: sic ut sufficiat exscribere aliquem absolutum medium inter duorum circumstantium logarithmorum absolutos. Et tunc etiam sufficit scribere tam numeros, quàm logarithmos usque ad punctum; quod quidem punctum huic ipsi compendio servit.

PRÆCEPTUM IV.

Dato numeri logarithmo privativo, invenire numerum.

Privativum datum aufer à multiplicatore aliquo majore, ut à decuplatore vel centuplatore, &c, vel etiam à duplicante, triplicante, &c; cùm residuo quære numerum in chiliade, qui erit quæfiti pars decima vel centesima, &c, vel etiam dimidia, tertia, &c.

Ut si detur logarithmus—2371,65 privativus: ut scias quis ei respondeat numerus: aufer eum à logarithmo decuplationis 230258,51, quippe qui jam est illo major, restabit 227886,86. Hic indicat in chiliade numerum 10240,00, qui erit pars decima quæfiti: quare quæsitus ipse erit 10240,00 ferè.

Vel aufer datum privativum à 69314,72 duplicationis logarithmo, quia etiam hic jam est major dato privativo, restabit 66943,07, qui ostendit in chiliade 51200,00, qui est pars dimidia quæfiti: quare ipse quæsitus est 102400,00.

Aliter & facilitate inopinabili.

Privativum datum quære inter logarithmos chiliadis, & incrementum duorum proximorum, illi respondens, exscriptum, auge tribus figuris ad dextram, sic formatus erit numerus logarithmi dati quæsitus.

Ut si datus sit idem, qui prius 2371,65. Hic inter logarithmos quæsitus, cadit medius inter 2429,27 & 2326,86 & differentia horum, itidem media, est 102,41. Hanc auge tribus locis, fiet 102410,00, numerus respondens, ferè.

ADMONITIO.

Atque hîc iterum vicem chiliadis supplet ex canone logarithmorum quadrantis, columna media mesologarithmorum. Quæsitus enim in eâ logarithmus excerpit arcum, gradus scilicet in calce (quippe cum sit privativus) minuta in margine dextro. Hic arcus translatus in canonem finium & tangentum, ostendit inter tangentes numerum quæsitum.

PRÆCEPTUM V.

De usu Differentiarum interlinearium.

Et si incrementa logarithmorum chiliadis respiciunt (intra eam) solos numeros absolutos, ad latus positos, eorumque logarithmos, inter quos sunt inserta: sciendum tamen est, illa eadem, eodem ordine à primo chiliadis ipsius (dissimulato jam ejus vestibulo) fervire cuicunque alii tabulæ construendæ, quæ alios numeros (progreffione tamen arithmeticâ æquabili surgentes), alios etiam eorum logarithmos habuerit, dummodò pauciores mille fuerint.

Verbi causâ, si pro numero maximo 100000,00 placeat statuere maximum 60' 0'', & hujus partes facere 720, sic ut una pars sit 0' 5'', pro eo quod in chiliade est 100,00 logarithmus in chiliade ad numerum 720 est 32850,41. Hunc aufer à logarithmo ad 100,00, scilicet 690775,57, residuus erit logarithmus ad 0' 5'' novæ tabulæ, scilicet 657925,13. Quod si jam subtraxeris ordine omnia incrementa (seu potius decrementa) logarithmica chiliadis ipsius (excluso jam ejus vestibulo) orsus à primo 69314,72 usque ad ultimum ex 720, scilicet 138,98, constituti erunt logarithmi ad omnes partium novarum collectiones, puta ad 0' 10'' & 0' 15'', &c. Idem fiet, si unum & eundem logarithmum, qui est ad numerum 720 seu 72000,00 subtraxeris ab omnibus & singulis 720 logarithmis, ordine invicem succedentibus in chiliade.

ALIUD EXEMPLUM.

Si quis in subsidium rei monetariæ vellet construere tabulam, cujus maximus numerus sit una marca, hoc est 16 uncia, hoc est 256 oboli; logarithmus ad absolutum 25600,00 chiliadis scilicet

136257,79

Ablatus à logarithmo primo chiliadis scilicet

690775,54

Relinquit logarithmum unius oboli scilicet

554517,75

Ab hoc logarithmo ablata prima 256 incrementa logarithmica chiliadis, quorum primum sit 69314,72, ultimum verò 391,39, constituent omnium numerorum monetariorum logarithmos.

Ufus verò talis tabulæ facillè intelligatur ex comparatione harum præceptionum.

Ego quidem illum hâc vice prætereo, cùm nulla pars chiliadis à re monetaria nomen fortiatur.

Alius usus indicatus est præcepto II.

PRÆCEPTUM VI.

Duos numeros absolutos in se mutuò multiplicare per logarithmos,
& factum invenire.

Multa quidem sequentium præceptorum, seipsis inutilia videbuntur, cum facilius & exactius operationes aliquæ peragantur viâ communi: at propter cognationem cum aliis utilioribus, negligenda non fuerunt, lucis causâ.

Quando igitur numeri duo absoluti sunt inter se multiplicandi, ut sciatur factus: tunc hoc est perinde, ac si in regula trium, poneretur primo loco ad sinistram unitas, loco secundo & tertio multiplicans, & multiplicandus; ut factus occupet locum quartum, veluti peractâ regulâ trium, quotientem.

Igitur adde logarithmos duorum numerorum, excerptos ex chiliade: summa quæsitâ rursus inter logarithmos, monstrat absolutum in illorum columnâ numerum, qui semper est quæsitâ pars multiplex, proportionis decuplæ continuæ. Nam si uterque numerorum absolutorum fuit minor maximo chiliadis; tunc numero per summam logarithmorum monstrato sunt apponendæ septem cyphræ ad dextram, ut ita factus hic rectè comparari possit cum facientibus.

EXEMPLUM.

Sint invicem multiplicandi	51200,00	log. 66943,07
	& 76800,00	log. 26396,55
	Summa	93339,62

Hæc summa dat absolutum 39330,00 circiter.

Igitur factus ex multiplicandis est 393300000000,00 circiter.

Sin autem facientes excedunt maximum chiliadis, tunc per eos excerpti non potest, sed per eorum partes decimas vel centesimas, &c. & tunc etiam factus, sic conformato, ut vult præceptum superius, apponendæ sunt insuper omnes cyphræ, quæ facientium utrique junctim fuerunt adimendæ, ut excerptio fieri posset.

Ut si numerorum superiorum alter fuisset uno loco longior, scilicet 768000,00, etiam factus prodiret uno loco longior, scilicet 393300000000,00. Sin autem etiam alter fuisset duobus locis longior, scilicet 5120000,00, factus prodiret in universum tribus locis longior, scilicet 39330000000000,00.

CAUTIO.

Si logarithmus ex additione excresceret supra maximum chiliadis logarithmum, ipsius etiam vestibuli: aufer ab eo decuplatorem vel centuplatorem, &c. sic ut remaneat minor maximo chiliadis: per hoc residuum excerptus numerus absolutus, & per præceptum conformatus, debet jam vicissim curtari unâ, duabus, &c, figuris.

P R Æ C E P T U M VII.

Numerum absolutum quadrare, seu ejus quadratum præter propter invenire.

Duplica logarithmum numeri quadrandi, duplum hoc inter logarithmos quæsitum, exhibet ex columnâ suâ numerum absolutum, cui sunt apponendæ ad dextram septem cyphræ; sic habebitur quadratum numeri propositi, saltem in primis ejus figuris ad sinistram. Nam loca reliqua ad dextram cyphris impleta, nullius sunt momenti, etiam si vel scrupulosissimè exprimerentur.

E X E M P L U M.

Quadrandus sit 3100,00 log. 347376,81
 Ejus duplum 694753,62
 Hoc duplum indicat absolutum 96,10
 Ergò quadratum quæsitum est 9610000,0000

V E L,

Sit quadrandus 31000,00 log. 117118,30
 Duplum 234236,60
 Hoc duplum indicat absolutum 9610,00
 Ergò quadratum quæsitum est 961000000,0000.

P R Æ C E P T U M VIII.

Numeri absoluti cubum invenire.

Triplica logarithmum numeri; triplum hoc inter logarithmos quæsitum, exhibebit primas cubi quæsitæ figuras, quibus apponendæ sunt aliæ 14.

E X E M P L U M.

Numeri absoluti 90000,00, logarithmus 10536,05 triplicatus facit 31608,15, qui ut logarithmus dat ex columnâ numerorum absolutorum 72900,00.
 Ergò cubus est 729000000000000,000000.

P R Æ C E P T U M IX.

Si duorum numerorum absolutorum major dividendus est per minorem; quotientem eruere per logarithmos.

Quando dividendus est major per minorem: tunc hoc est perinde ac si in regulâ trium, primo loco ad sinistram collocaretur minor, secundo major, tertio unitas, quartoque loco ad dextram quotiens. Nam ut minor est ad majorem, sic unitas ad quotientem.

Aufer igitur logarithmum minoris, à log. unitatis absolutæ in vestibulo chiliadis, scilicet ab 1611809,59, residuo adde logarithmum majoris, summa inter logarithmos quæsitæ ostendit inter absolutos quotientem.

E X E M P L U M.

EXEMPLUM.

Sit dividendus 99200,00 pro 3200,00. Hujus ergò logarithmum 344201,94 aufer, & vicissim illius logarithmum 803,22 adde, ad 1611809,59, conficietur logarithmus 1268412,87, qui ostendit absolutum 0,31. Ergò 0,31 est quotiens.

VEL,

Aufer logarithmum minoris à proximè majori logarithmo unitatis cujusque in chiliade, residuo adde logarithmum majoris, provenit logarithmus numeri, à quo sunt abscindenda totidem loca, quot cyphas habuit unitas illa, quæ logarithmum dedit.

Ut si in exemplo superiori logarithmum minoris 344201,94 abstulisses à 460517,03 logarithmo numeri 1000,00, residuus fuisset

116315,09

Cui additus majoris log.

803,22

Confecisset logarithmum

117118,31

Hic verò indicat 31000,00. Quia verò numerus seu unitas illa 1000,00 habet 5 cyphas, decurta igitur hunc vicissim 5 ultimis locis, manetque 31.

OBSERVATIONES.

Si per divisoris decuplum diviseris; prodibit pars decima quotientis quæfiti: & sic consequenter.

Si unus vel ambo excefferint maximum chiliadis: decurtatis ambobus æquali numero locorum, sic ut fiant minores maximo chiliadis, operatio peragatur, prodibitque quotiens justus.

EXEMPLA DIMISSIS FIGURIS POST PUNCTUM.

Dividendus fit 100000 per 4362.

Hic possum uti divisoris decuplo 43620, cujus log. cùm fit 82965, auferam eum à logarithmo proximè majori unitatis alicujus ex chiliade, scilicet à numeris 10000 logarithmo 230259, restat 147294, cui jam nihil additur, quia logarithmus numeri 100000 est 0. Hic ergò logarithmus ostendit absolutum 22950. Sed quia unitas usurpata habet quatuor cyphas, debent ergò vicissim rejici ab hoc absoluto, quatuor quidem loca, ut formetur pars decima quotientis, tria verò, ut ipse quotiens. Erit igitur ille 23 vel $22\frac{95}{1000}$.

Quod si dividendus fuisset 1000000, divisor 4362: tunc pro 1000000, chiliadis maximum (usque ad punctum) superante, scripsissem 100000, ut qui non superat, & pro $4362,436\frac{1}{3}$ utrumque scilicet curtassem æqualiter.

Tunc usus priore processu, pro $436\frac{1}{3}$ usus essem ejus centuplo 43620. Et prodisset $2\frac{259}{10000}$ centesima quotientis, ipse igitur quotiens $225\frac{9}{100}$.

ALIUS DIVIDENDI MODUS.

Dividendum majorem decurta, uti curtatus fiat minor divisore. Tunc à logarithmo curtati aufer logarithmum divisoris, residuum erit logarithmus numeri, qui decurtandus est tot locis, ut decurtatio utraque deleat loca 7 (vel, si omittimus figuras post punctum, loca 5.)

Ut si fit dividendus 99200 per 3200.

Rejectis à dividendo duabus figuris fit 992 jam minor divisore. Jam igitur per observationes superiores sume utriusque decuplum, ut fiant proximi maximo chiliadis.

$$\begin{array}{r} \text{Ergò } 32000 \text{ logar. } 113943 \\ \quad 9920 \text{ logar. } 231062 \\ \hline \text{Residuum} \quad 117119 \end{array}$$

Est logarithmus numeri 31000. Quia igitur à dividendo rejectæ sunt figuræ duæ, rejiciantur jam à quotiente tres, ut rejectarum sint 5 formabitur quotientis 31.

PRÆCEPTUM X.

Ex numero absoluto, ut quadrato, radicem extrahere quadrati.

A numero absoluto proposito resecta loca bina & bina, tantisper, quoad loca residua ad sinistram representent numerum non majorem maximo chiliadis: numeri sic decurtati logarithmum adde logarithmo unitatis puræ in vestibulo chiliadis: summæ semissis quæsitus inter logarithmos, ostendet è regione inter absolutos, primas figuras radices quæsitæ, cui numero pro binis locis prius resectis, restituendæ sunt singulæ cyphræ. Ita formatur radix quadrata quæsitæ.

EXEMPLUM.

Quadratum esto 961000000, quæritur ejus radix quadrata. Numerus igitur iste habet loca 9, cum maximus chiliadis habeat tantum 8. Abscinde duo ultima (numero pari) ut restent septem.

$$\begin{array}{r} \text{Ergò } 96100,00 \text{ habet log. } 3978,00 \\ \text{Sed puræ unitatis in vestib. log. } 1611809,59 \end{array}$$

$$\begin{array}{r} \text{Summa } 1615787,59 \\ \text{Semissis } 807893,80 \end{array}$$

Hic semissis indicat absolutum 31,00, huic verò pro duobus locis prius rejectis jam restitue unam cyphram, fietque vera radix 310,00.

U

ALIUS

ALIUS MODUS SINE LOGARITHMO UNITATIS.

A numero proposito rejice loca ad dextram numero impari, ut fiat minor maximo chiliadis :

Hujus numeri decurtati logarithmum ipsum bipartire, cum semisse inter logarithmos quaesito, excerpe ex columnâ absolutorum numerum competentem, qui offert primas ad sinistram figuras radice quaesitæ : sed nisi septem omnino loca dempta fuerint initio, nondum erit hic numerus radix ipsa. Pro binis enim minus quam septem illic ademptis, singulæ cyphræ hîc sunt adimendæ, pro binis plusquam septem ademptis, singulæ restituendæ.

Et nota, quod etiam si numerus ipse statim initio fuerit minor maximo chiliadis ; tamen unus illi locus sit demendus, ut radici demantur tria loca, vel unus adjiciendus, ut radici demantur quatuor, vel si tam est parvus, tres adjiciendi, ut radici demantur quinque.

EXEMPLUM.

Ut si quadratus proponatur 961000000. Hic cum habent loca 9, si auferres unum, adhuc haberet nimium, octo scilicet, & in primo novenarium, cum maximus chiliadis habeat octo quidem & ipse, sed in eorum primo unitatem. Aufer ergo loca tria (numero scilicet impari) restant sex, numerus scilicet 961000. Hujus logarithmus est 234236 usque ad punctum, semissis hujus 117118 ostendit 31000,00.

Quia igitur quadrato, loca sunt dempta 4 minus quam 7 demenda jam sunt huic numero duo ; formaturque radix 310,00.

Si quadratus fuisset 9610000000000, constans scilicet locis 13, ut igitur minor restet numerus, quam maximus chiliadis : oportet rescindere loca 7, numero scilicet impari : & tunc peractâ operatione de invento absoluto rescinderetur nihil, quia loca à quadrato rescissa, fuerunt numero septenario, quot cyphras habet maximus chiliadis.

Si verò quadratus habuisset loca 15, rescindenda fuissent loca 9, numero scilicet impari, ut semper in hoc modo : ut scilicet restaret numerus minor maximo chiliadis. Cum autem 9 excedat 7 binario, jam igitur ad numerum per operationem inventum, vicissim fuisset apponenda una cyphra.

Denique si quadratus fuisset 9610000, adhuc rejiciendus fuisset ab illo locus unus, semper enim in hoc modo aliquid impari numero mutandum est. Unum verò à 7 ablatum relinquit 6, tres igitur cyphræ tunc fuissent rejiciendæ, ut radix tunc fiat 31,00.

Et si quadratus fuisset 96100, rejici poterit unus locus, poterit & addi unus, ut prolongatus nihilominus sit minor maximo chiliadis. Sed tunc qui apponit unum, is demit uno minus quam nihil. Differentia verò inter 7, & inter unum minus quam nihil, est 8, quatuor ergo demeret loca de invento 31000,00, ut sic formeretur ipsius 96100 radix 310.

Ita si quadratus fuisset 9610, potuissent addi loca tria, fuissetque 96100,00

minor maximo chiliadis, habens logarithmum 3978. Semissis verò 1989 ostendisset absolutum 98029,00.

Tria verò loca prius apposita pro VII delendis, faciunt decem: quinque igitur hîc dele, ut restet radix 98 plus, numeri quadrati 9610 ferè.

P R Æ C E P T U M XI.

Medium proportionale inter duos absolutos datos invenire.

Quia hoc fit in arithmetica vulgari, multiplicatis in se mutuò duobus datis, factique radice quæsitâ; fit igitur per logarithmos similiter, conjunctione in unum, sexti & decimi præceptorum; in hunc tamen modum commodissimè.

Absolutus uterque acquirat loca 7, nisi fortè unus eorum fuerit ipse maximus chiliadis, huic relinqui possunt loca octo. Tunc sic aptatorum logarithmi adduntur, cum summæ semisse, ut logarithmo, excerpitur numerus absolutus; cui sunt restituenda loca prius abscissa, vel demenda quæ erant prius adjecta facientibus utrisque.

E X E M P L U M.

Sint duo numeri 987654321 & 59643, quæritur eorum medius proportionalis. A primo rejice loca duo ad dextram, ad secundum appone duas cyphas:

$$\begin{array}{rcl} \text{Ergò } 9876543 & \log. & 1242,25 \\ 5964300 & \log. & 51679,34 \end{array}$$

$$\begin{array}{rcl} \text{Summa} & & 52921,59 \\ \text{Semissis} & & 26460,80. \end{array}$$

Hic dat absolutum 76750,65. Cùm igitur uni datorum dempta sint loca duo, alteri totidem apposita: vicissim huic invento duo sunt apponenda, duo vicissim demenda, hoc est, compensatione factâ nihil mutandum. Ita hic ipse est medium proportionale quæsitum.

Hic operæ precium est videre, quantus error fuisset commissus, si logarithmi fuissent excerpti per proximè minores vel majores absolutos sine elaboratione scrupulosâ.

Numero 98765,43
Viciniior chiliadis & major 98800,00 dat log. 1207,26 minorem justo.
Et numero 59643,00
Viciniior chiliadis & minor 59600,00 dat log. 51751,46 majorem justo.

$$\begin{array}{rcl} \text{Summa} & & 52958,72 \text{ propè vera} \\ \text{Semissis} & & 26479,36 \end{array}$$

Jam logarithmus chiliadis hoc proximè major, ut qui propior, dat absolutum 76700,00. Ergò semisse ipsi respondebit plus aliquid. Itaque si
U 2 medium

medium eligam inter 76700 & 76800, puta 76750,00, id erit proximum vero. Sanè suprà vides inventum 76750,65.

P R Æ C E P T U M XII.

Propositi numeri radicem invenire cubicam, ejusque radicis quadratum.

Quia unitas est ad radicem cubicam, ut hæc ad quadratum ejus, & hoc ad cubum ejus: ergò à numero absoluto, qui ut cubus proponitur, rejice ad dextram loca terna & terna tantisper, quoad usque loca residua repræsentent non majorem maximo chiliadis: Numeri sic decurtati logarithmum, aufer à logarithmo unitatis puræ in chiliadis vestibulo, residui tertiam partem, si adjeceris logarithmo numeri, constitues logarithmum utilem indagando cubicæ radicis quadrato, si duas tertias adjeceris, fiet logarithmus pro ipsa radice cubicâ. Excerpe igitur absolutos, cùm utraque summâ constitutâ, inter logarithmos quæsitâ, & pro ternis locis prius demptis, appone radicis quidem numero cyphras singulas, ejus verò quadrati numero binas: ita formabitur, & radix cubica & ejus quadratum.

E X E M P L U M.

Propositus esto cubicus numerus 729 cum 18 cyphris, loca scilicet habens 21. Rejice loca 15 quinquies scilicet terna: quia si 12 rejiceres, restarent 9, plura scilicet quàm habet maximus chiliadis.

Jam igitur 72900,00 dat log. 261866,63

Sed unitatis primæ in vestib. log. est 1611809,59

Differentia 1349942,96

Seu quod idem est 72900,00

Dat log. 31608,15

Sed unitatis secundæ in vestib. log. 1381551,08

Differentia 1349942,93

Hujus pars tertia 449980,98

Summa pro quadrato 711847,61

Summa pro rad. cub. 1161828,59

Ergò summa prior dat 81,00, cui appone propter loca 15 prius rejecta, cyphras 10 fiet 81000000000000. Quadratum radicis cubicæ.

Sic posterior summa dat 90, cui appone pro 15 locis prius rejectis jam cyphras 5, ita fit radix cubica 90000,00.

P R Æ C E P T U M XIII.

Invenire logarithmum, indicem proportionis, inter duos numeros datos.

Magnum habet usum hoc præceptum ad triangula plana solvenda, de quibus agemus capite sequenti.

Sunt autem casus varii. Aut enim uterque datorum terminorum invenitur expressus in chiliade; aut alter solum, aut neuter. Et si uterque in chiliade; tunc ii aut proximi sunt invicem, ut 237, 238 seu 23700,00, 23800,00, & tunc differentia logarithmorum, numeris ad latus respondentium, est quæsitus proportionis index; ut in hoc exemplo 421,05 indicat proportionem inter 237, 238. Aut non sunt invicem proximi, sed distant interjectis aliis: tunc logarithmus majoris auferatur à logarithmo minoris, restabit logarithmus proportionis inter datos terminos.

E X E M P L A.

Sint termini proportionis 1 & 60.

Numeri 1000,00 logar. 460517,03

Numeri 60000,00 logar. 51082,56 auf.

Restat log. proportionis inter datos 409434,47 qui ideò & sexagecuplator dici potest.

Sic numeri 1000,00 logar. 460517,03

Et numeri 24000,00 logar. 372970,14

Restat log. viginti quadruplæ proport. 87546,89

Sin vel neuter ex terminis, vel alteruter solum invenitur expressus in chiliade: tunc vel cadit talis inter duos chiliadis, proximos invicem, estque scrupulosus, vel excedit maximum chiliadis.

In priori casu elaboretur prius logarithmus numeri scrupulosi; & tunc per eum operatio est eadem, quæ prius.

In posteriori casu fumantur numeri utriusque æquè multiplices partes, minores maximo chiliadis, & cum eorum logarithmis agatur ut suprà.

Ut si quæraturs proportio inter 102400,00
& 100000,00

Decurta eos æqualiter, ut sint eorum partes decimæ scilicet

10240,00 log. 227886,86

10000,00 log. 230258,51

Erit proportionis log. 2371,65

N O T A.

Si numerorum alter fuerit ipse maximus chiliadis, tunc ipse logarithmus alterius numeri, est index proportionis inter utrumque seu positivus fuerit logarithmus, seu privativus. Logarithmus enim definitus est post prop. xx. nihil aliud quàm hæc ipsa proportio.

P R Æ C E P T U M XIV.

Proportionem cum terminis suis datam fecare in partes, quæ sint ad invicem in alia proportionem propositâ.

Data fit proportio inter terminos 37 & 53.

Hæc proportio fit secunda in partes duas; quarum una fit ad alteram, ut 5 ad 2.

Igitur constituatur quantitas proportionis secundæ per præcedens præceptum.

	Logarithmi		Logar.
Termini	5300—293746	vel	53000—63488
	3700—329684		37000—99425
Proportio	35938		35937

Jam secundæ proportionis, secundum quam secunda est prima, termini sunt 5 & 2, summa 7. Divide igitur proportionis primæ quantitatem in 7, erit pars septima 5134, & duæ septimæ 10268. Ergo quinque septimæ 25670.

Quantitate partium inventâ, jam duplici viâ potest secari in has partes prima proportio: Aut enim ut pars major stet à plagâ termini majoris.

Igitur termini majoris logarithmo	63488
Adde partem majorem	25670
Summa	89158

Hæc ut logarithmus ostendit absolutum 41000 ferè. Ergo ut se habet 5 ad 2, sic se habere fecimus proportionem inter 53, 41 ad proportionem inter 41, 37.

Aut ut pars minor sectæ, stet à plagâ termini majoris.

Eidem ergo termini majoris logarithmo	63488
Adde partem minorem	10268
Summa	73756

Hæc ut logarithmus ostendit 47830 circiter.

Ergo ut se habet 2 ad 5, sic fecimus se habere proportionem inter 53, $47\frac{83}{100}$ ad proportionem inter $47\frac{83}{100}$, 37.

Idem efficiemus etiam per logarithmum termini minoris, subtrahentes ab eo.

ALIUD EXEMPLUM ET NOBILE QUIDEM.

Datur proportio inter dies 687 periodi Martis & $365\frac{1}{4}$ periodi solis (terræ Copernico), sit hæc proportio secunda in partes duas, sic ut major ad minorem se habeat, ut 2 ad 1, & ut major pars proportionis divisæ stet à plaga termini minoris, ut ita proportio tota sit ejus partis majoris sesquialtera.

Ergò termini majoris 68700 log. 37542
Termini minoris 36525 log. 100740

Quantitas ergò proportionis 63198

Proportionis verò divisoris termini 2, 1 faciunt 3, quâ summâ, divisâ quantitas proportionis, facit ejus trientem 21066, & duos trientes 42132
Hos aufer à logarithmo partis minoris 100740

Restat 58608

Hoc ut logarithmus ostendit numerum absolutum 55650. Ergò ut 1 ad 2, sic fecimus proportionem inter 68700, 58608 ad proportionem inter 58608, 36525.

Quod si ex termino minori 36525 fiat 100000, distantia mediocris solis & lunæ, tunc ex hoc invento termino majori 58608, fiat 1524,00 ferè : distantia solis & Martis mediocris.

Et fit manifestum, quod proportio periodicorum temporum sit sesquialtera proportionis intervallorum mediocrium : ut indicavi in Epitomen Astronomiæ Cop. libro iv.

PRÆCEPTUM XV.

Regulam proportionem seu detri absolvere per logarithmos, seu sine multiplicatione & divisione.

Positis in regula tribus datis numeris, in chiliade comprehensis, ut docet vulgaris arithmetica, aufer logarithmum sinistimi à logarithmo medii, vel hunc ab illo : differentiam logarithmo dextimi adde in primo casu, subtrahe in secundo, si potest : summa illic, vel hic residuum, est logarithmus quarti quæsitæ seu quotientis positivus.

Rursum autem si logarithmus tertii fuerit minor residuo, subtrahe vicissim illum ab hoc, remanebitque privativus logarithmus, numeri scilicet absoluti, seu quotientis, superantis maximum chiliadis.

EXEMPLUM PRIMI CASUS.

77100	dat 53200	quid 92500
	Log. 63111	
	Log. 26007 subtr.	
	Residuum 37105	
	Logar. 7796 adde	

Logarithmus 44901 quotientis 63800 plus.

EXEMPLUM SECUNDI CASUS.

876	dat 941	quid 765
	Log. 6081 subtrahatur ut minor	
	Log. 13239	
	Residuum 7158 subtrahe quia secundus subtractus	
	Logarithmus 26788	

Logarithmus 19630 quotientis 822 minus.

EXEMPLUM CASUS TERTII.

889	dat 996	quid 960
	Log 401 subtrahatur ut minor	
	Log. 11766	
	Residuum 13365 subtrahendum erat, quia secundus subtractus fuit	
	Logarith. 4082 subtrahe hunc	
	quia minor est residuo.	

Logarith. 7283 privativus, quia tertius subtractus fuit. Hic igitur privativus per iv præceptum, dat quotientem 1075 majorem maximo chiliadis 1000, ut hîc quidem operamur per curtatos.

NOTA.

Licet autem commoditatis causa, omnes tres in regulâ positos, vel prolongare, vel curtare, ut fiant proximè minores maximo chiliadis: & tunc quotiens, qui pro sic accommodatos elicitur, idem debet pati, quod passus est finistimus: eos verò locos, qui secundo & tertio fuerunt adempti vel appositi solus, quotiens omnes contrariâ ratione vel recipere debet, vel amittere.

CAPUT IX.

De Copulatione Columnæ Logarithmorum cum duabus aliis junctis.

PRÆCEPTUM GENERALE.

QUOTIESCUNQUE in regulam proportionum offeruntur diversarum columnarum numeri, sicut duo quidem ex iis sint ex unius columnæ genere, unus verò residuus cum quotiente quæsito competat in columnam alteram: tunc operatio per logarithmos perfici potest; tam in vulgaribus seu absolutis numeris, quàm in logistis.

Hujus mixturæ dabo aliquot exempla, tanquam præcepta particularia.

I. PRÆCEPTUM.

Dividere horas vel gradus per sexagenaria scrupula.

Quia enim est, ut divisor sexagenarius, ad dividendum, sic integrum seu 60' ad quotientem: Divisor igitur & integrum, sunt ex genere eodem sexagenario, dividendus verò & quotiens possunt esse ex quadrivicenariâ. Possunt igitur illa exempla capitis VI, præcepti III; ubi dividendus aliquot integra continet, possunt, inquam, plerunque per quadrivicenariam perfici.

Ut si dividendi, ut suprâ, $3^{\circ} 45' 13''$ in $57' 8''$, sic age:

$3^{\circ} 45' 13''$	log.	185470	circ.	ex quadrivic.
Scr. 57 8	log.	4900		ex sexagenar.

Residuum 180570 dat ex quadrivicenaria quotientem
 $3^{\circ} 56' 30''$ ut & cap. VI.

II. PRÆCEPTUM.

Partem proportionalem scrupulis sexagenariis, capere de differentiâ, excedente integrum.

Addantur logarithmi scrupulorum in sexagenaria, & differentiæ excedentis in quadrivicenaria, quæditorum summa ex quadrivicenariâ ostendit partem proportionalem.

Ut si uni gradui anomaliae mediae respondeant $1^{\circ} 24' 17''$ coaequatae, quid competit minutis 49,23.

$$\begin{array}{r} 1^{\circ} 24' 17'' \text{ log. } 284000 \text{ circiter} \\ \text{Sc. } 49 \ 23 \text{ log. } 19480 \end{array}$$

Summa 303480 dat $1^{\circ} 9' 15''$ tantum competit scrupulis 49,23.

III. P R A E C E P T U M.

Dato diurno solis, vel planetarum v unius, colligere aliquot horarum motum.

Si diurnus minor est quam $60'$, ut in Saturno, Jove, Marte, & in sole; quære eum in sexagenaria, numerum horarum in quadrivicenaria, & adde eorum logarithmos; summa in logarithmis quaesita, eruet ex sexagenaria partem horis competentem.

In sole, Venere, Mercurio, si diurnus est inter 1 & 2° , quære proportionalem partem de excessu supra 1° , eamque adde ad id, quod horis datis in sexagenaria respondet per caput iv. In Mercurio verò, si ejus diurnus superat 2° partem proportionalem de excessu, adde duplo illius, quod horis datis respondet in sexagenaria.

Aliter, quia diurnus plerunque constat integris scrupulis, sine appendice secundorum: quærere illa poteris non in sexagenaria, sed inter absolutos, prolongata prius, ut cap. viii docui: & tunc quæ ex logarithmis colligitur summa, quaesita in logarithmis, rursus inter absolutos ostendet partem proportionalem decurtandam iterum totidem locis.

E X E M P L A.

		V E L	
Sit diurnus $17'$, horæ $19 \ 48' \ 0''$		Ex absolutis 17000	19237
Horar. $19 \ 48'$ log.	19237	Log.	177200
Ser. $17'$ ex sexag. log.	126200		
Summa 145437		Summa	196437
Pars proportionalis ex sexagenaria $14' \ 1''$		Ex absol. colum.	14020

Jam si Veneris diurnus fuisset $1^{\circ} 17'$, tunc è regione $19^h \ 48'$ invenio scrupula $49' \ 30''$, quæ addenda sunt ad $14' \ 1''$ partem proportionalem de excessu, essetque arcus quaesitus $1^{\circ} 3' \ 1''$. Et si Mercurii diurnus fuisset $2^{\circ} 17'$, duplicanda fuissent illa $49' \ 30''$ addendaque $14' \ 1''$, ita collegissemus $1^{\circ} 53' \ 1''$.

IV. PRÆCEPTUM.

Horarium elicere ex motu aliquot horarum datarum, minore quàm 60'.

Aufertur logarithmus horarum datarum, in quadrivicenariâ quæsitæ, à logarithmo ex sexagenariâ scrupulorum, horis competentium; residuum, ut logarithmus, ex sexagenaria exhibebit horarium. Exempla sunt facilia & obvia.

V. PRÆCEPTUM.

Datâ separatione diurnâ minore quàm 60' & distantia planetæ, ab alterius aspectu, minore ea quàm est diurnus, prodere horarum intervallum respondens.

In sexagenariâ, vel etiam inter absolutos, quæsitæ diurna separatione & distantia, logarithmus illius aufertur à logarithmo hujus, residuum, ut logarithmus, exhibet horas ex quadrivicenariâ, respondentes distantia seu intervallo.

VI. PRÆCEPTUM.

Hoc modo etiam ingressus solis in signa computatur: Per diurnum scilicet solis, & per distantiam ejus à principio signi, in meridie vicino, in sexagenariâ quæsitæ, & logarithmo illius ablato à logarithmo hujus. Residuum enim, ut logarithmus, ostendit in quadrivicenariâ, horas ante vel post meridiem.

Plurima alia per hanc combinationem columnarum perfici possunt: sed sex ista exemplorum loco sufficiant. Jam enim ad rariora & scrupulosiora nonnulla transeundum, suprâ dilata, & in hunc locum rejecta.

VII. PRÆCEPTUM.

Cuilibet arcui finum suum scrupulosum assignare per logarithmos.

Subsidium, ut vides, paratur hîc capiti II, ut in ejus tractatione promiseram. Quanta igitur sit operationis certitudo, vide ibi: nunc modum doceo.

Ab arcu dato aufer proximè minorem chiliadis, & exscribe ejus finum rotundum, ut jam corrigatur, suamque acquirat scrupulositatem. Tunc in sexagenariâ, quæsitæ & excessu arcus dati, & differentiâ binorem arcuum chiliadis, inter quos arcus propositus intercudit, auferatur logarithmus hujus à logar. illius: Residuum immissum in logarithmos, ostendit inter absolutos numerum,

qui præcis tribus ultimis locis ad dextram, fit augmentum scrupulosum exscripti sinus rotundi.

E X E M P L U M.

Propositus esto arcus	37° 49' 53"	
Proximè minor chil.	37 48 26	dat rotund. 61300,00
Residui	0 1' 27"	log. 372900 circ.
Differentiæ inter duos chil.	4 21	log. 262400 circ. aufer.
Residuum 110500		

Hoc quæsitum, ut logarithmus, dat inter absolutos numerum 33190,00 circiter. Ergò præcis 3 locis, acquirimus 33,19, & fit sinus scrupulosus arcui proposito conveniens 61333,19.

C A U T I O.

In fine quadrantis, ubi differentia in chiliade superat 60', uti liceret columnâ quadrivicenariâ, nisi tum simul etiam proportionalitas, quàm hoc præceptum præsupponit, insigniter turbaretur.

VIII. P R Æ C E P T U M.

Vicissim cuilibet sinui scrupuloso, suum assignare arcum.

Primùm cum sinu rotundo, proximè minori, exscribe arcum & differentiam subjectam. Tunc ad 4 ultimas figuras sinus scrupulosi (ad excessum scilicet sinus dati suprâ rotundum chiliadis proximè minorem) appone tres cyphras, & quæsito numero sic formato inter absolutos, differentiâ verò in sexagenariâ, adde eorum logarithmos; summa ut logarithmus ostendet in sexagenaria, scrupula prima & secunda, addenda arcui exscripto.

E X E M P L U M.

Ut si detur sinus scrupulosus 61333,10, proximè minor rotundus 61300,00 dat arcum 37° 48' 26". Et differentiam 4' 21" cujus logarithmus 262400 circiter. Quatuor autem ultimæ sinus scrupulosi figuræ hic sunt 3310, prolongatus hic numerus, ut sit 33100,00 dat logarith. 110500, qui additus ad priorem, facit summam 372900 circ.

Hæc ostendit partem proportionalem 1' 27". Ergò arcus erit 37° 49' 53."

IX. P R Æ-

IX. PRÆCEPTUM.

Cuilibet arcui suum assignare logarithmum sine magno labore.

Rursum hic subsidium paratur doctrinæ triangulorum sphaericorum, cap. VII traditæ, pro curiosis, qui dedignantur operari cum logarithmis rudibus. Præceptio erit verbosa quidem; at non valdè necessaria, ne metuas calculator.

Cùm enim in hâc chiliade, tam arcus, quàm logarithmi exhibeantur scrupulosi; & differentiæ, inter arcus quidem interjectæ, crescant continuò inter logarithmos verò, decrescant; ut ita neutrobique maneat unus & idem numerus constans: laborem equidem haud levem sumpserit, qui, quotiescunque opus fuerit logarithmo exacto, toties multiplicatione scrupulorum & secundorum excessus arcus, in differentiam logarithmorum, ut in numerum apicibus carentem (diversi scilicet generis) & rursum, facti divisione in differentiam, ex scrupulis & secundis constantem, venari velit partem proportionalem.

Hic igitur juvent logarithmi seipsos, ut contribules. Cùm arcu chiliadis, qui proximè minor proposito arcu vel angulo fuerit, exscribe logarithmum exactum; eumque arcum subtrahe à proposito: & quæsitis in sexagenariâ, tam excessu propositi, quàm differentia inter duos arcus chiliadis, inter quos cadit propositus, logarithmum hujus ruditer saltem excerptum, aufer à logarithmo rudi illius: residuum junge logarithmo rudi differentiæ, vicinorum duorum logarithmorum prolongatæ secundum præcepta præcedentis capituli, & quæsitæ inter absolutos: Summa ut logarithmus ostendit, rursum inter absolutos, numerum qui decurtatus prodit partem proportionalem, auferendam à logarithmo exacto arcus proximè minoris, primum exscripto, ut justus constituatur logarithmus arcus propositi.

EXEMPLUM.

Cupio logarithmum arcus	71° 40' 10"	
In chiliade invenio proximè minorem	71 37 21	log. 5234,65
Excessus dati	2' 49"	log. 305800
Differentia duorum in chiliade	10 57	log. 170000 rudes.
	Residuum	135800

Differentia duorum logarithmorum 105,32 prolongata ut fiat 10532,00, inter absolutos quæsitæ dat logarithmum 225000 rudem.

Adde resid. 135800

Summa 360800

Hæc ut logarithmus ostendit inter absolutos 2725,00. Deletis verò iterum duobus ultimis, fit pars proportionalis 27,25, quibus ablatis ab exscripto 5234,65.

5234,65, restat 5207,40 logarithmus, quantum proportionalitas locum habet, exactus. Nam capite VIII veriores, & plerunque exactissimos docuimus assignare sinibus, qui per eos suis arcubus transcribi possunt.

X. P R Æ C E P T U M.

Vicissim logarithmo scrupuloso arcum suum assignare.

Primum cum logarithmo chiliadis, qui proximè major fuerit proposito, exscribe arcum scrupulosum, cum differentia subjecta. Tunc subtrahe logarithmum propositum ab exscripto proximè majore: Excessum illius, & simul differentiam duorum in chiliade, prolonga æquali locorum numero, ut sint tamen minores maximo chiliadis. Sic igitur formatis iis, & inter absolutos quæsitis, logarithmum hujus ab illius logarithmo aufer, residuo adde logarithmum differentiae arcuum in sexagenariâ quæsitâ; sic accumulatur logarithmus partis proportionalis ex sexagenariâ itidem excerpendæ, et addendæ arcui exscripto.

E X E M P L U M.

Sit logarithmus	5207,40	
Prox. major	5234,65	dat 71° 37' 21" & diff. 10' 57"
Excessus	27,15	
Diff. logarithmor.	105,32	
Hi num. prolongati	2715,00	dant rudes 360800
	10532,00	225000
	Differentia	135800
	Differentiæ arcuum log.	170000
	Summa	305800 ostendit partem proportionalem 2' 49"
	Ergò arcus	71° 40' 10"

Habent igitur curiosi quod agant, si rudibus logarithmis, minimo cum decimo, uti dedignantur. Veruntamen qui tales futuri sunt, iis consultum est, ut privatâ quilibet operâ, & in vestibulo chiliadis columnam arcuum hîc vacantem impleat, & in fine quadrantis, decem ultimas arcuum differentias, in denas subdividat, sinuum quidem differentias statuens æquales, eis verò sinibus ex canone sinuum exacto suos adscribens arcus, & logarithmos sinibus suos ex doctrina capitis prioris accommodans.

Nam pars proportionalis, usitato modo quæsitâ, locum non habet in fine quadrantis: ut capite secundo monui.

XI. PRÆCEPTUM.

Per solos logarithmos, absolutorum, & arcuum seu angulorum, omnia triangula plana solvere: omnia scilicet quæsitæ ex tribus datis eruere.

Ad hoc caput propriè pertinet hæc doctrina, quia jungi debet columna absolutorum (non ut ii sunt sinus arcuum) & columna ipsorum arcuum. Sunt autem casus sex.

I. Si datis angulis & uno latere, quæritur latus.

Tunc prolongato vel decurtato latere, ut capite VIII doctus es, adde logarithmos lateris dati & anguli quæsito lateri oppositi; à summâ aufer logarithmum anguli, dato lateri oppositi, restat logarithmus lateris quæsitæ, similiter prolongati.

Si subtractio fieri non potest, vicissim summam subtrahe, restabitque privativus logarithmus, lateris excedentis maximum chiliadis.

EXEMPLUM.

Dentur anguli $50^{\circ} 0'$ & $60^{\circ} 0'$, datur igitur etiam residuus, ut complementum illorum duorum ad semicirculum scilicet $70^{\circ} 0'$, detur & latus oppositum medio angulo 573, quæritur latus oppositum maximo.

Lateris 57300,00 (prolongati)	log.	55687
Anguli $70^{\circ} 0'$	log.	6220 circ.
	Summa	61907
Anguli $60^{\circ} 0'$	Log.	14380 circ.
	Residuum	47527
	dat 62175,00	
	Latus ergo est	622 ferè.

Quod si rectus fuerit angulorum unus, & detur ei subtensum latus, sufficiet sola additio.

Sin autem quærat latus recto subtensum, sufficiet sola subtractio log. anguli à log. lateris oppositi. Restabit enim log. lateris recto subtensi.

II. Si datis duobus lateribus, & angulo uni eorum opposito, quærantur anguli reliqui.

Tunc lateribus æqualiter prolongatis vel decurtatis, ut sint proximè minora
maximo

maximo chiliadis, adde logarithmos anguli & lateris unius adjacentis; à summâ aufer logarithmum lateris dato angulo oppositi, restat logarithmus anguli lateri alteri oppositi, qui angulus interdum tam potest esse minor quadrante, quàm major: & sic complementum ad semicirculum ejus, quem logarithmus excerpit: quantisper non plura dantur. Tertius verò angulus est duorum junctorum complementum ad semicirculum.

EXEMPLUM.

Ut si dentur latera 622,507. Et huic oppositus $50^{\circ} 0'$ quærantur anguli reliqui.

Anguli $50^{\circ} 0'$	log. 26650
Adjac. lateris 62200,00	log. 47527
Summa	74177
Oppositi lateris 50700,00	log. 67957

Residuum 6220 dat ang. $70^{\circ} 0'$ alteri lateri oppositum.

Si ergò duo sunt $50^{\circ} 0'$ & $70^{\circ} 0'$: tertius erit eorum summæ complementum ad duos rectos scilicet $60^{\circ} 0'$

Si verò angulus prodiens fumeretur non $70^{\circ} 0'$ sed complementum ad semicirculum

110 0
Tertius fiet 20 0

Si datus angulus fuerit rectus; sufficiet sola subtractio logarithmi lateris recto oppositi, à log. lateris reliqui dati: restabit enim logarithmus anguli ei oppositi.

Si datus dato angulo oppositum logarithmum habuerit æqualem summæ priorum: rectangulum erit triangulum.

III. Si datis duobus lateribus, & angulo uni eorum opposito, quæraturs latus tertium.

Tunc primùm, ut in secundo casu, quæraturs angulus reliquo lateri dato oppositus; additis logarithmis, anguli dati, & lateris ei adjacentis; à summâ verò dempto logarithmo lateris dato angulo subtenfi: cum residuo, ut logarithmo, exscribatur angulus & jam summa duorum angulorum à duobus rectis ablatâ, constituiturs angulus tertius, cujus logarithmus, ut in primo casu addendus est logarithmo unius ex lateribus, à summâ auferendus logarithmus anguli oppositi; restabitque logarithmus lateris tertii.

Ut

Ut in exemplo priori, sumpto angulo secundo $70^{\circ} 0'$ & invento angulo tertio $60^{\circ} 0'$. Ejus logarithmo 14380

Addatur lat. 622 log. 47527 vel lateris 507 log. 67957

A summâ 61907 vel 82337
Aufer anguli 70° log. 6220 vel ang. 50° log. 26650

Restat 55687 55687

Logarithmus lateris tertii quæsit 57300,00, sed decurtati, ut priora, 573.

At si angulus in priori præcepto prodians sumatur non $70^{\circ} 0'$ sed $110^{\circ} 0'$, & tertius ideò sit $20^{\circ} 0'$, hujus igitur logarithmus 107290, additis logarithmis datorum laterum, & ablati logarithmis angulorum oppositorum, dat logarithmum 148600 lateris tertii, angulo scilicet $20^{\circ} 0'$ oppositi, scilicet 22630,00, decurtati igitur 226 $\frac{3}{8}$.

Si angulus datus fuerit rectus, aufertur logarithmus lateris recto oppositi, à logarithmo lateris reliqui, restat logarithmus anguli: hujus complementi logarithmus, vicissim additus logarithmo primo, lateris recto oppositi, dat logarithmum lateris reliqui.

IV. Si datis duobus lateribus, et angulo comprehenso, quærantur anguli reliqui.

Hic via directa solvendi hunc casum utitur mesologarithmis, qui non aliter elici possunt ex chiliade, quàm si logarithmos arcus ejusque complementi ab invicem subtrahamus: ut sic datâ eorum differentiâ, ipsi logarithmi partium quadrantis non directè possint accipi, sed positione sit utendum.

Dimisso igitur hoc modo; ne tamen manca hîc sit chilia nostra: utamur positionibus, potius in aliâ viâ facili & indirectâ: et si & hæc aptior sit logarithmus canonicis quadrantis.

Est autem talis:

Datis lateribus quærat eorum proportio: fit enim facillimè per capitis VIII præceptum XIII.

Hæc verò proportio, cùm sit differentia inter logarithmos angulorum residuorum, quorum angulorum summa est in anguli dati complemento ad semicirculum: Ponatur igitur angulus minor esse notus, auferaturque à dati complemento ad semicirculum, residui logarithmo adde proportionem laterum inventam; summa ut logarithmus, si dat arcum eundem, quem posueras, felix fuit positio; sin discrepat, muta positionem primam, sumens vel aliquid intermedium, si residuum ex complemento ad semicirculum fuerit quadrante minus: vel aliquid longius à primâ positione recedens, quàm quod prodierat; si residuum ex complemento ad semicirculum fuerit majus quadrante; præsertim si valdè magnum.

A tali nova positione incipiat nova operatio: Id fiat tantisper, donec prodeat id, quod ultimò fuit positum. Ita habebis utrumque angulum ex quæsitis.

Conducit autem primas positiones sic moderari, ut remaneat de complemento ad semicirculum aliquid quod exactè reperiatur in chiliade.

EXEMPLA.

Dentur latera 507 & $226\frac{3}{10}$, comprehensus esto 110° ut fit ejus complementum ad semicirculum 70° . Quærantur duo anguli residui; quos junctos scio facere 70° .

Primùm quæro proportionem laterum.

50700	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	67957
22630	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	148600

Proportio 80643

Compl. ang. ad semic.	$70^\circ 0'$
Positio arbitraria	$10^\circ 0' 10''$

Residuum	59 59 50	log. 14387
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Prodit	22 45 circ.	95030
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Positio secunda	$17^\circ 3' 37''$
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Residuum	52 56 23	22565
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Prodit	30 52 circ.	103208
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Positio tertia	$19^\circ 0' 47''$
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Residuum	50 59 13	25231
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Prodit	20 17 circ.	105874
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Hoc modo cum medio arithmetico inter positum & prodeuntem semper possemus propius ad consensum venire. At quia statim in secunda positione apparuit, veritatem esse ultra medium arithmeticum: sumemus etiam in quarta positione

positione aliquid, quod propius fit prodeunti $20^{\circ} 17'$ quam posito $19^{\circ} 1'$,
tentabimus scilicet

Positio quarta	$20^{\circ} 0'$	proportio	80643
Residuum	$50 0$	logarith.	26650
Prodit	$20 0$		107293

Hæc ergò tandem felix fuit positio, & anguli quæsitæ sunt $20^{\circ} 0'$ & $50^{\circ} 0$.

EXEMPLUM ALIUD.

Dentur latera 507, 573 & comprehensus 70° ut residuum ad semicirculum
fit $110^{\circ} 0'$

Quæro proportionem laterum.

50700			67957
57300			55687
Compl. ang. ad semic.	110°	proportio	12270
Positio arbitraria	$29^{\circ} 56' 14''$		
Residuum	$80 3 46$	log.	1511
Prodit	$60 36$	circiter	13781

Positio secunda	$45^{\circ} 58' 22''$		
Residuum	$64 1 38$		10647
Prodit	$52 40$	circ.	22917

Positio tertia	$49^{\circ} 4' 13''$		
Residuum	$60 55 34$		13467
Prodit	$50 38$		25737

Apparet ponendum	$50^{\circ} 0'$		
Residuum	$60 0$		14380
Prodit	$50 0$		26650

Ergò anguli quæsitæ sunt $50^{\circ} 0'$, & $60^{\circ} 0'$.

EXEMPLUM TERTIUM.

Dentur latera, ut prius 507, 573, sed comprehensus sit 20° complementum ad semicirculum 160° , manente igitur laterum proportionem.

Compl. ad semic.	$160^\circ \ 0'$	12270
Sit positio arbitrar.	70° logarith.	
Residuum	90 logarith.	0
Prodit	$62^\circ \ 8'$ circiter	12270
Positio secunda	$54^\circ \ 47' \ 45''$	
Residuum	$105 \ 12 \ 15$	
	$74 \ 47 \ 45$ log.	3563
Prodit	$58 \ 35$	15833
Positio tertia	$60^\circ \ 0'$	
Residuum	$100^\circ \ 0'$	
	$80 \ 0$ logarith.	1530 circ.
Prodit	$60^\circ \ 35'$ circ.	13800
Positio quarta	$61 \ 0$	
Residuum	$99 \ 0$	
	$81 \ 0$	1240
Prodit	$60 \ 52$	13510
Ergò positio felix erit	$60^\circ \ 47'$ circiter.	

Hic cum in primâ positione 70° . Residuum fieret 90° prodiit aliquid minus, ipsâ positione, scilicet $62^\circ \ 8'$. Residuum igitur apparuit futurum majus quadrante. In hoc ergò casu positionem secundam longius à primâ distare feci, quàm id quod per primam prodiit.

Sic in positione secundâ $54^\circ \ 47'$ prodiit $58^\circ \ 35'$.

Ergò

Ergò positio tertia $60^{\circ} 0'$ longius distare iussa est, à positione præcedenti $54^{\circ} 47'$ quàm id quod prodierat $58^{\circ} 35'$, minus tamen quàm quod primò prodiit, scilicet $62^{\circ} 8'$. Et in positione tertia $60^{\circ} 0'$ prodiit major $60, 35$, feci ergò quartam adhuc majorem, scilicet $61^{\circ} 0'$. Per hanc autem quartam $61^{\circ} 0'$ prodiit jam minus $60^{\circ} 52'$.

Ergò quinta positio adhuc minor est facta, major tamen quàm $60^{\circ} 35'$, quod prius prodierat, scilicet $60^{\circ} 47'$. Est igitur angulus minor $60^{\circ} 47'$, major $99^{\circ} 13'$ ferè.

Si datus fuerit rectus : reliqui duo ex canone Neperiano facilè habentur, quæsitâ proportionem laterum inter mesologarithmos, inventa enim ea ostendit utrumque angulum.

V. Si datis duobus lateribus & angulo comprehenso quæraturs latus tertium.

Nulla est alia machina, quàm ut prius quærantur anguli, per præcedens. Tunc iis inventis, cadit quæstio in casum primum.

Si tamen angulus datus sit rectus, logarithmos laterum duplica, duplicatorum logarithmorum numeros absolutos adde, summæ logarithmum dimidia; semissis iste, ut logarithmus, dabit inter absolutos quæsitum latus.

VI. Si datis tribus lateribus quæraturs angulus.

Tunc ante omnia ex majori angulo ressecandum est triangulum æquicrurum, cujus crura sint linea sectionis & latus minus; basis verò segmentum lateris majoris. Id verò obtinetur in hunc modum.

Adde logarithmos & differentiæ, & summæ laterum minorum, à logarithmorum summâ aufer logarithmum majoris lateris, residuum ut logarithmus ostendet inter absolutos, partem de latere majori subtrahendam, ut restet basis æquicruri supradicti.

Constitutâ hac basi, maximus angulorum habetur per sua elementa, si exscripseris logarithmos tam semissis de basi, quàm residui de latere majori, post ablatum basis semissem; & ab illo quidem minoris, ab isto verò medii lateris logarithmos abstuleris, relinquuntur enim logarithmi duarum partium anguli maximi. Duo verò minores anguli sunt horum majoris elementorum complementa.

E X E M P L U M.

Sint latera 507, 573, 622, quæruntur anguli.

Latus minus 507

Medium 573

Differentia 66 scribo 66000 log. 41552

Summa 1080 scribo 10800 log. 222562

Lat. maxim. 622 scribo 62200 log. 47482

Prodit 11460 Residuum 216632

Ad primum tres cyphræ accesserunt, ad secundum una, quæ sunt à prodeunte removendæ: vicissim ad tertium duæ appositæ, sunt etiam ad quotientem apponendæ: duabus igitur apposis, & quatuor remotis, formatur pars lateris majoris $114\frac{6}{10}$.

Pars lateris majoris 114,60

Latus majus 622,00

Basis æquicruri 507,40

Semissis 253,70 log. 137160

Residuum ablato basis semisse 368,30 log. 99886

Latus minus 507,00 log. 67924

Latus medium 573,00 log. 55687

Residui manent duo logarithmi 69236, 44199

Duorum elementorum anguli $30^{\circ} 1' 40^{\circ} 0'$

Ergo totus angulus iste est $70^{\circ} 1'$

Duo verò minores sunt complementa elementorum, scilicet $59^{\circ} 59' \& 50^{\circ} 0'.$

Hæc igitur de usu chiliadis nostræ hâc vice sufficiant.

F I N I S.

LOGARITHMO-TECHNIA:

SIVE

METHODUS CONSTRUENDI

LOGARITHMOS

NOVA, ACCURATA, ET FACILIS;

SCRIPTO

ANTEHAC COMMUNICATA, ANNO SC. 1667, NONIS AUGUSTI.

CUI NUNC ACCEDIT

VERA QUADRATURA HYPERBOLÆ, ET INVENTIO SUMMÆ
LOGARITHMORUM.

AUCTORE NICOLAO MERCATORE, HOLSATO,
E SOCIETATE REGIA.

HUIC ETIAM JUNGITUR

MICHAELIS ANGELI RICCHII EXERCITATIO GEOMETRICA DE MAXIMIS ET MINIMIS:

Hic ob Argumenti Præstantiam & Exemplarium Raritatem recusa.

Prima hujus Tractatus editio impressa fuit Londini, Anno M,DC,LXVIII.

LOGARITHMO-TECHNICA:

1713

METHODUS CONSTRUENDI

LOGARITHMORUM

NOVA, ACCURATA, ET FACILIS

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L O G A R I T H M O - T E C H N I A.

LOGARITHMUS composito vocabulo dicitur à ratione & numero, quasi rationum numerus, id quod planè cum re consentit. Est enim logarithmus nihil aliud, quàm numerus ratiuncularum contentarum in ratione, quam absolutus quisque ad unitatem obtinet. In qua definitione rationes accipimus, tanquam magnitudines partibus constantes homogeneis toti; strictiori aliquantò notione, quàm vulgò solet. Quamvis enim ratum sit, rationem omnem ex comparatione quantitatum homogenearum oriri: certè nec quævis comparatio producit rationem; nec quarumvis quantitatum, homogenearum licèt, habitudo est ratio quanta, seu partibus constans. Nam æqualitatis, quæ dicitur, ratio, est illa quidem quantitatum homogenearum, atque æqualium, habitudo mutua, unde nec rationis appellatione privandam autumem, cui definitio Euclidea competit non minus ac aliis rationibus, quas inæqualitatis vocant: sed nihil obstat, quò minùs generalem istam rationis notionem porro dividamus ita, ut quantitatem habere putentur solæ rationes, ex inæqualium habitudine ortæ; at æqualitatis ratio in indivisibili consistat, habeatque se in rationibus, quemadmodum punctum in magnitudinibus, aut nullitas in numeris, quæ singula quantitate ac partibus carent. Componas enim sexcentas rationes æqualitatis, non augetur nec minuitur ratio, sed eadem manet æqualitas: secus atque in rationibus inæqualitatis, quæ additæ vel detractæ invicem, faciunt rationem majorem vel minorem. Quantum autem est, quod additis vel demtis partibus homogeneis augetur minuiturque. Sed nec quævis comparatio quantitatum homogenearum rationem producit. Veluti cum numerum dividimus per numerum; comparantur utique quantitates homogeneæ, spectando quoties altera contineatur in altera; sed quod inde oritur latus, nec ratio est ipsorum numerorum, nec sanè quantitatem exprimit rationis, quæ utrisque intercedit. Alioquin diviso numero quovis per æqualem, quæ inde oritur unitas, exprimeret quantitatem rationis æqualium, quam tamen quantitate carere suprà adstruximus. Quinimò, datis pluribus rationibus, v. gr. 4 ad 2, & 9 ad 3, si diviso utriusque antecedente per suum consequentem, exhiberent orti 2 & 3 veram quantitatem istarum rationum; oporteret, ut ex his ortis compositus numerus, nimirum 5, exhiberet quoque veram quantitatem rationis compositæ. Atqui ratio composita est 36 ad 6, cujus quantitatem jam exprimeret ortus 6, diversus sanè ab isto 5. Obtinuit tamen usus, ut rationes denominentur à latere orto; sic ratio 4 ad 2 dicitur dupla, & 9 ad 3 tripla: verùm hæc nomina arbitrio hominum imposita, retineri quidem possunt, veritati

autem derogare nullo modo debent. Quanquam nec utilitate caret iste modus denominandi rationes; siquidem arguit, rationes esse majores, quarum denominator est major, & contrà: eodem modo sinus majores congruunt majoribus arcubus, quorum tamen veram quantitatem exprimere nemini videntur. Cæterùm, ut linea est dupla lineæ, quam bis continet; ita, propriè loquendo, dupla foret ratio alterius rationis, quam bis continet; sed pro eo duplicatam dicere maluerunt scriptores, quorum arbitrio synonyma alioquin vocabula dupli & duplicati, res planè diversas significare intelliguntur. Verùm id quod est multò maximum, nimirum omnium quantitatum mensuram esse quantitatem homogeneam, & in divisione genuina fieri applicationem mensuræ homogeneæ ad quantitatem mensurandam, ortum vero ex tali applicatione, nihil aliud esse, quàm numerum arithmeticum, exprimentem quoties mensura continetur in mensurato; hoc scilicet est, quod omnem dubitationem excludit. Ita falluntur profectò, qui, applicatâ lineâ rectâ illatabili ad aream datam, putant inveniri latitudinem rectanguli: quasi non potiùs secundum veras divisionis leges applicaretur rectangulum æquè longum divisoris, & æquè latum unitati assumptæ, ad aream extensam quoque ad longitudinem divisoris; & quasi non ortus ex ista applicatione, numerus esset arithmeticus, exprimens quoties rectangulum mensurans contineatur in mensurato, vel (cùm per 1, vi eadem sit ratio) quoties latitudo rectanguli applicati contineatur in latitudine rectanguli mensurati; quò scilicet divisio verè opponatur multiplicationi, quæ resolvat hujus productum in sua elementa. Quemadmodum enim omnis multiplicator est numerus arithmeticus (ut habet Stevinus in Arithmetica Practica), ita omnis ortus à divisione est similiter numerus arithmeticus. Quæ quidem omnia facillimè præsentis negotio aptantur. Nam multiplicare rationem nihil est aliud, quàm replicare aliquam rationem toties, quot sunt unitates in numero aliquo arithmetico, qui dicitur factor. Et dividere rationem, est applicare rationem aliquam ad aliam rationem, ut inveniatur numerus arithmeticus, exprimens quoties mensurans ratio contineatur in mensurata. Id si hîc fieret, nihil dubium, quin vera patefieret rationis quantitas. At enimverò, cùm applicatur terminus alicujus rationis ad alterum; num putamus rationem applicari ad rationem? quo pacto igitur ortus ex tali applicatione potest exprimere quantitatem datæ rationis? Verum est quidem, quòd ortus ex applicatione termini ad terminum, rationem habet ad unitatem eandem, quam dividuus ad divisorem; sed hoc modo eadem prodit ratio, quæ ante divisionem fuerat, aliis tantum terminis expressa; nec proinde quantitas rationis datæ invenitur in mensura aliqua priùs nota, quemadmodum in aliis magnitudinibus divisis affolet, & instituti nostri ratio postulat: Siquidem tum demum quantitatem rationum exactè determinâsse videbimur, cùm eas omnes in una aliqua ac eâdem communi mensurâ æstimare noverimus; id quod logarithmorum ope præstari, definitione modò traditâ innuere volui. Ex qua porro intelligitur, cùm singuli logarithmi numerent particulas rationum inde ab unitate ad singulos ordine absolutos procedendo coacervatarum; fieri non posse, quin æqualibus logarithmorum differentiis (id est, æqualibus particularum incrementis) congruant quoque æquales rationes absolutis intercedentes (cùm integra ex æquali numero particularum æqualium conflata, inter se sint æqualia); adeoque logarithmos esse in proportionem arithmetica, cùm eorum

absoluti sunt in geometrica; idcirco posse operationem regulæ proportionum in compendium redigi, substitutâ additione & subtractione loco multiplicationis & divisionis: denique rationis cujusque bipartitionem, tripartitionem, &c, quæ alioquin requireret extractionem radicis quadratæ, cubicæ, &c, consistere in bipartitione, tripartitione, &c, differentiarum logarithmorum datis terminis congruentium (hoc est, ratiuncularum in ratione data comprehensarum). Qui usus cum sit eximius, patet postremo, quo pacto tam utiles numeri artificiales, seu logarithmi concinnari possint; nimirum investigando, quot ratiunculæ, assumptæ magnitudinis, contineantur in ratione cujusque absoluti ad unitatem. Sic enim unitatis logarithmus evadit 0, cum unitatis ad unitatem ratio sit æqualitatis, quam quantitate carere supra asserui. Ita nimirum fiet, ut cum inter multiplicandum vel dividendum unitas nihil mutet; hujus logarithmus 0 (dum additio & subtractio substituuntur multiplicationi & divisioni) nihil quoque additione vel detractioe sui mutet. Numerus autem ratiuncularum in ratione decupla contentarum commodissimè assumitur 1,0000000 (hoc est, una decupla ratio in numerum partium decimalium rotundum distributa). Ita enim fiet, dum inter unitatem & 10 intercedit una ratio decupla, & porro inter 10 & 100 altera, inter 100 & 1000 tertia, & deinceps; ut in centupla quidem ratione contineantur ratiunculæ 2,0000000 (hoc est, duæ decuplæ rationes in numerum partium rotundum distributæ), in millecupla verò ratione contineantur 3,0000000 (hoc est, tres rationes decuplæ in numerum partium rotundum distributæ); & deinceps. Unde primum hoc commodi consequimur, ut absolutis, iisdem characteribus expressis, iisdem competant logarithmi; veluti si absoluto 2 competat logarithmus 0,3010299; etiam absolutis 20, 200, 2000 competent logarithmi 1,3010299; 2,3010299; 3,3010299. Etenim si rationes 2 ad 1, 20 ad 1, 200 ad 1, & deinceps, intelligantur partitæ, illa quidem in rationes 2 ad 1 & 1 ad 1; ista in 20 ad 10, & 10 ad 1; hæc in 200 ad 100, & 100 ad 1; apparet quod excessus 2 ad 1, quo illa superat rationem 1 ad 1 æqualis sit excessui 20 ad 10, quo ista superat rationem 10 ad 1, idemque æqualis excessui 200 ad 100, quo hæc superat rationem 100 ad 1. Ergo si ratio 2 ad 1 præter æqualitatis rationem (quam innuit characteristica 0) contineat ratiunculas 3010299, qualium ipsa decupla continet 1,0000000; certè ratio 20 ad 1, præter unam decuplam (quam innuit characteristica 1) continebit eundem numerum ratiuncularum excurrentium 3010299; & ratio 200 ad 1, præter duas decuplas (quas innuit characteristica 2) continebit similiter eundem numerum ratiuncularum excurrentium 3010299. Unde porro & hoc consequimur, ut ex inspecta characteristica, uniuscujusque absoluti valorem æstimare queamus. Nam cum in decupla contineantur 1,0000000 ratiunculæ numero rotundo, & in centupla 2,0000000 ratiunculæ numero itidem rotundo; oportet, ut quæcunque sunt inter decuplam & centuplam contineant plus quam unam decuplam, minus autem quam duas; quamobrem characteristica omnium absolutorum, qui sunt inter 10 & 100, ipsiusque adeo denarii (hoc est, omnium numerorum, qui scribuntur duobus characteribus) erit 1; & sic deinceps.

His ita ordinatis, proximum est, ut ostendamus, quomodo inveniat mensura rationis, quam quisque absolutus obtinet ad unitatem, in partibus, qualium decupla continet 1,0000000 (hoc est, quomodo cujusque absoluti logarithmus

investigandus fit.) Verbi gratiâ: Scire velim, ratio 100[5 ad 1 quot contineat ratiunculas, qualium decupla continet 1,0000000. Dispesco igitur rationem datam 100[5 ad 1 in suas partes, nimirum 100[5 ad 100, 100 ad 10, & 10 ad 1, quarum posteriores duæ constituunt duas decuplas (unde patet characteristicam fore 2); itaque restat, ut investigemus, quota pars sit reliqua ista ratio 100[5 ad 100 ipsius decuplæ. Quod si igitur termini 100[5 & 100 ducantur uterque in sese, producti exhibebunt rationem duplicatam rationis 100[5 ad 100, cujus (duplicatæ sc. rationis) termini rursus in se ducti procreabunt duplicatam duplicatæ, id est, quadruplicatam rationis 100[5 ad 100: atque ita continuatâ multiplicatione terminorum, donec is, qui gignitur ex ductu continuo termini 100[5 in seipsum, evadat decuplus ejus, quem ductus continuus termini 100 in seipsum producit; denominator potestatis postremò genitæ ostendet, quot integris vicibus ratio 100[5 ad 100 contineatur in decupla. Et cum alter terminorum sit 100, cujus potestates omnes constant unitate & certo numero cyphrarum; omnis labor reliquus occupabitur circa elevandum alterum terminum 100[5 ad eam potestatem, quæ prioris termini (nimirum 100ⁱⁱⁱ) æquè altam potestatem excedat decuplo; cujus operationis compendium exemplo, quàm verbis docere præstat.

100	5000	(1)
	5001	(1)
100	5000	
	5025	
101	0025	(2)
52	00101	(2)
101	0025	
	10100	
	20	
	5	
102	0150	(4)
051	0201	(4)
102	0150	
	20403	
	102	
	51	
104	0706	(8)
607	0401	(8)
108	3068	(16)
860	3801	(16)
117	3035	(32)
530	3711	(32)
137	6011	(64)
110	6731	(64)
189	3406	(128)

1893406	(128)
6043981	(128)
3584985	(256)
5894853	(256)
12852116	(512)

Hæc potestas plusquam decuplo excedit potestatem æquè altam 100ⁱⁱⁱ; ergo resumo 256^{am}, eamque duco, non in sese, ut modò, sed in proximè præcedentem, nimirum 128^{am}, hoc modo:

3584985	(256)
6043981	(128)
6787831	(384)
1106731	(64)
9340130	(448)
5303711	(32)
10956299	(480)

Hæc potestas denuò excedit æquè altam 100ⁱⁱⁱ plusquam decuplo; ergo eandem 448^{am} duco, non in 32^{am}, ut modò, sed

in proximè præcedentem, hoc modo:

9340130	(448)
8603801	(16)
10115994	(464)

Ubi rursus nimium colligitur; ergo eandem adhuc 448^{am} duco, non in 16^{am}, ut modò, sed in proximè præcedentem, nimirum 8^{am}, hoc modo:

9340130	(448)
6070401	(8)
9720329	(456)
0510201	(4)
9916193	(460)
5200101	(2)
10015603	(462)

Quæ potestas rursus excedit limitem; quare eandem 460^{am} duco, non in 2^{am}, sed in 1^{am}, hoc modo:

9916193	(460)
5001	(1)
9965774	(461)

Cum

Cùm igitur 462^{da} potestas termini $100\frac{1}{5}$ excedat æquè altam 100^{iii} plus quam decuplo; at 461^{ma} ejusdem termini $100\frac{1}{5}$ excedat æquè altam 100^{iii} minùs quàm decuplo: aio, rationem $100\frac{1}{5}$ ad 100 contineri in decupla plus quam 461 vicibus, minùs autem quàm $462^{b^{us}}$.

Cæterùm

$$\text{Cùm potestas } \left\{ \begin{array}{c} 460 \\ 461 \\ 462 \end{array} \right\} \text{ fit } \left\{ \begin{array}{c} 9916193 \\ 9965774 \\ 10015603 \end{array} \right\} \text{ \& differentiæ } \left\{ \begin{array}{c} 49581 \\ 49829 \end{array} \right\} \left\{ \begin{array}{c} \text{propemodum} \\ \text{æquales;} \end{array} \right.$$

Itaque partem proportionalem, quâ potestas justa, nimirum 10000000, excedit proximè minorem 9965774, per regulam auream facilè ac tutò reperire datur, fumendo nimirum,

$$\begin{array}{r} \text{justæ} \\ \text{\& proximè minoris} \end{array} \quad \begin{array}{r} 10000000 \\ 9965774 \\ \hline \end{array}$$

differentiam

34226, & dicendo :

Ut differentia inter proximè minorem & majorem
(nimirum 49829.)

Ad differentiam inter proximè minorem & justam.
(putà 34226;)

Ita 10000, ad 6868; quæ sunt partes decimales unius vicis, adeò ut ratio $100\frac{1}{5}$ ad 100 contineatur in decupla $461686\frac{1}{8}$ vicibus. Porro, si decupla (five ratio $100\frac{1}{5}$ ad 100 sumta 4616868 vicibus) continet ratiunculas 1,0000000; quot ejusmodi ratiunculas continebit ratio $100\frac{1}{5}$ ad 100 semel sumta? Prodeunt $21659\frac{1}{7}$ ratiunculæ, quæ sunt exacta mensura rationis $100\frac{1}{5}$ ad 100, quibus si addas rationes 100 ad 10, & 10 ad 1, hoc est bis decuplam, constantem ratiunculis 2,0000000; fit integra mensura rationis $100\frac{1}{5}$ ad 1 (five logarithmus absoluti $100\frac{1}{5}$), hic scilicet $2,0021659\frac{1}{7}$.

Pari modo invenietur logarithmus absoluti $99\frac{1}{5}$, vel ratio absoluti $99\frac{1}{5}$ ad 1, si ex ratione 100 ad 1 (quæ æquipollet bis decuplæ), auferas rationem 100 ad $99\frac{1}{5}$, hoc est, ex ratiunculis 2,0000000 auferas numerum similium ratiuncularum in ratione 100 ad $99\frac{1}{5}$ contentarum. Quærat igitur primùm, quoties ratio 100 ad $99\frac{1}{5}$ contineatur in decupla. Ubi rursus alter terminorum cùm sit 100, operationis haut indiget; alter verò $99\frac{1}{5}$ continuo ductu in seipsum elevandus est ad eam potestatem, quæ decuplo minor sit potestate 100^{iii} æquè altâ. En operationem :

995000 (1)	8518016 (32)	æquè altâ; ergo resume
599 (1)	6108158 (32)	
8955000	7255660 (64)	1058613 (448)
895500	665527 (64)	139229 (16)
49750		
	5264459 (128)	977026 (464)
9900250 (2)	9544625 (128)	
520099 (2)		Quæ etiam plusquam
	2771452 (256)	decuplo minor est potes-
8910225	2541772 (256)	tate 100 ⁱⁱⁱ æquè altâ;
891023		ergo resume
198	554290 (512)	
49		1058613 (448)
	Hæc potestas plusquam	396069 (8)
9801495 (4)	decuplo minor est potes-	
5941089 (4)	tate 100 ⁱⁱⁱ æquè altâ;	1017002 (456)
	ergo resume	5941089 (4)
8821345	2771452 (256)	
784120	9544625 (128)	996814 (460)
980		
392	1459018 (384)	Hæc quoque plusquam
88	665527 (64)	decuplo minor est po-
5		testate 100 ⁱⁱⁱ æquè al-
	1058613 (448)	tâ; ergo resume
9606930 (8)	6108158 (32)	
396069 (8)		1017002 (456)
	901728 (480)	520099 (2)
9229310 (16)		
139229 (16)	Quæ potestas rursus	1006857 (458)
	plusquam decuplo mi-	599 (1)
8518016 (32)	nor est potestate 100 ⁱⁱⁱ	
		1001823 (459)

Cùm igitur 460^{ma} potestas termini 99|5 deficiat ab æquè altâ 100ⁱⁱⁱ plusquam decuplo; at 459^{na} ejusdem termini 99|5 deficiat ab æquè altâ 100ⁱⁱⁱ minùs quàm decuplo; aio, rationem 100 ad 99|5 contineri in decupla plusquam 459 vicibus, minùs autem quàm 460.

Tum, ut differentia potestatum 459^{nae} & 460^{mae} (nimirum 5009) ad differentiam 459^{nae} & justæ (putà 1823): Ita 10000, ad 3639. Quare ratio 100 ad 99|5 continetur in decupla 459|3639 vicibus.

Porrò, si decupla (sive ratio 100 ad 99|5 sumta 459|3639 vicibus) continet ratiunculas 1,0000000; quot ejusmodi ratiunculas continebit ratio 100 ad 99|5 semel sumta. Prodeunt 21769|3 ratiunculæ, quæ sunt exacta mensura rationis 100 ad 99|5, quâ scilicet ratio 99|5 ad 1 deficit à ratione 100 ad 1, hoc est, à bis decuplâ, quæ cùm constet ratiunculis 2,0000000, demtis hinc

hinc 21769[3, restat mensura rationis 99[5 ad 1, hæc scilicet 1,9978230[7, qui proinde est logarithmus absoluti 99[5.

Atque hoc modo æstimated rationum quantitibus in communi quadam mensura, non solum natura & usus logarithmorum clarius elucescit; sed et constructio eorundem multò facilius evadit. Id quod magis perspicuum fiet, cùm ostendero alterum etiam longè promptiorem modum rationes æstimandi. Sed amolienda est prius difficultas, quæ, haut scio an cuiquam detecta, plures utique in errorem induxit. Cum enim ratio duobus terminis intercedens vulgò haut aliter consideretur, quàm accipiendo alterutrum terminum ut antecedentem, & alterum ut consequentem; unde cùm ratio est quanta (hoc est, cùm termini sunt inæquales) vel major terminus est antecedens, & dicitur ratio majoris inæqualitatis, vel minor est antecedens, & dicitur ratio minoris inæqualitatis: Aio ego, eandem rationem iisdem terminis conceptam posse ac debere (saltem in Musicis, atque in hac nostra logarithmo-technia) alio etiamnum modo considerari ita, ut neuter terminorum existimetur tanquam antecedens, vel consequens, sed uterque capiatur simul pariter tempore atque ordine. Sic v. gr. in Musicis intervallum diapente, sive ratio $\frac{3}{2}$ vel $\frac{2}{3}$, potest quidem accipi ita, ut numerus undationum ab acutiori phthongo in aëre excitatarum, nempe ternarius, sit antecedens, & binarius, exhibens numerum undationum pari temporis spatio à graviore phthongo effectarum, sit consequens, dum intelligatur acutior phthongus tempore (vel saltem cogitatione) præcedere graviorem; & vice versâ: sed nihil vetat, quò minus etiam ambo isti phthongi simul atque eodem tempore consonent, adeoque neuter altero sit vel tempore vel naturâ prior. Cæterum nihilo majus ob hoc vel minus evadit intervallum diapente (ratione sesquialterâ constans) sive acutior phthongus præcedat graviorem, sive contra, seu denique ambo simul consonent. Ita, licet utilis sit demonstrationibus geometricis consideratio vulgaris, quâ minor terminus antecedens ad majorem consequentem dicitur minorem rationem habere, quàm idem ille major tanquam antecedens ad eundem minorem tanquam consequentem obtineat: Negari tamen haut potest, eundem numerum ratiuncularum contineri in sesquialtera ratione, sive ternarius sit antecedens, sive binarius, sive neuter; adeoque considerationem antecedentis & consequentis in æstimanda mole vel mensura rationum nullum instar habere. Non secus ac quinaris negatus (-5) mole haud differt à quinario affirmato ($+5$), cùm uterque constet quinque unitatibus; dissimulatâ nimirum affectione, propter quam negatus vel affirmatus censetur, & solâ mole vel quantitate simpliciter æstimatâ: cùm tamen accipiendo quinarium negatum, prout signo negationis affectus est, verum sit, eum minorem esse, non modò quovis numero affirmato, sed & omni negato, qui à nihilo minùs differat, quàm ipse, quales sunt binarius vel ternarius negatus (-2 , vel -3 .) Ubi præter molem numeri consideratur quoque, utram in partem eadem abeat à nihilo, affirmatam an negatam. Ita quoque sive unisonum (vel æqualitatis rationem, quæ quantitate caret, atque ideò rectè componitur nihilo) ponas in phthongo graviore (vel in binario), indèque ascendas ad phthongum intervallo diapente acutiorem (vel ad ternarium); sive contrà ponas unisonum in acutiori (vel æqualitatis rationem in ternario), indèque descendas ad phthongum intervallo diapente graviorem (vel ad binarium): certè eadem est utrobique quantitas intervalli Musici (atque

(atque idem numerus ratiuncularum intercedentium), licet ab unifono (vel ab æqualitatis ratione, tanquam nihilo) in diversas planè partes abeat. Unde si moles sola, aut quantitas rationis æstimetur, dissimulando utram in partem (majorisne, an minoris inæqualitatis) vergat ab æqualitate; nihilo major est ratio ternarii ad binarium, quàm binarii ad ternarium. Sed si cum mole unà includas quoque considerationem processus à majori termino ad minorem, vel contrà; non eo inficias, minorem esse quamvis rationem minoris inæqualitatis non modò quâvis ratione majoris inæqualitatis; sed & quâvis aliâ minoris inæqualitatis, quæ ab æqualitate minùs absit. Ita ratio antecedentis 5 ad consequentem 8, non modò minor est ratione antecedentis 8 ad consequentem 6 (vel 5), sed eadem quoque minor est ratione antecedentis 6 (vel 7) ad consequentem 8: licet sepositâ vel neglectâ notione antecedentis & consequentis, eadem sit moles rationis $\frac{5}{8}$ atque $\frac{8}{5}$. Distinguemus igitur deinceps inter quantitates mole-majores, ut affectione-majores: ita ut in rationibus notio inæqualitatis majoris vel minoris nil nisi affectionem innuat. Eas porrò rationes appellamus mole-majores, quarum major terminus divisus per minorem, dat quatum majorem; & vice versâ. Præterea majoris inæqualitatis rationum quæcunque mole, eadem & affectione majores sunt; at minoris inæqualitatis rationes quò sunt mole majores, eò affectione minores sunt. Quibus præmissis, digeremus ea, quæ restant, in propositiones.

PROPOSITIO I.

Si duæ quantitates ejusdem affectionis auferantur ab invicem (affirmata sc. ab affirmata, vel negata à negata), sitque quantitas reliqua ejusdem affectionis cum duabus ab initio datis; quantitas ablata mole-minor est quantitate ex qua auferebatur. Sin quantitas reliqua diversæ sit affectionis à duabus initio datis; quantitas ablata mole-major est quantitate ex qua auferebatur. Sit exempli gr.

ratio subducenda	ratio ex qua	ratio reliqua	
$\frac{5}{8}$)	$\frac{3}{5}$	$(\frac{24}{25})$	tres scilicet rationes ejusdem affectionis, putà minoris inæqualitatis singulæ; ergo, inquam, ratio subducta $\frac{5}{8}$ mole-minor est ratione ex quâ $\frac{3}{5}$. Sit rursus

ratio subducenda	ratio ex qua	ratio reliqua	
$\frac{8}{5}$)	$\frac{3}{2}$	$(\frac{15}{16})$	quarum priores duæ sunt ejusdem affectionis, nimirum majoris inæqualitatis ambæ, at tertia $\frac{15}{16}$ diversæ est affectionis, putà inæqualitatis minoris; ergo, inquam, ratio subducta $\frac{8}{5}$ mole-major est ratione ex quâ $\frac{3}{2}$.

PROPOSITIO II.

Si sint quotcunque rationes continuæ & terminorum æquidifferentium, v. gr. $\frac{a}{a+b}, \frac{a+b}{a+2b}, \frac{a+2b}{a+3b}$, & deinceps, faciendo scilicet antecedentem cujusque ex posterioribus rationibus æqualem consequenti proximè præcedentis, & à minoribus progrediendo ad majores: Erit quælibet præcedentium rationum mole-major qualibet sequente; sed & differentiarum inter ipsas rationes tam primarum, quàm secundarum, tertiarum, cæterarumque adeò omnium in infinitum, semper præcedens mole-major est sequente. Sin à majoribus terminis progrediare ad minores; contrarium eveniet.

Patet ex collatione sequentis tabellæ cum propositione prima.

Rationes. Diff. primæ.

Differentiæ secundæ.

a		
$a + b$	$aa + 2ab + bb$	
	$aa + 2ab$	
$a + b$		$a^4 + 6a^3b + 12aabb + 8ab^3$
$a + 2b$		$a^4 + 6a^3b + 12aabb + 10ab^3 + 3b^4$
	$aa + 4ab + 4bb$	
	$aa + 4ab + 3bb$	
$a + 2b$		$a^4 + 10a^3b + 36aabb + 54ab^3 + 27b^4$
$a + 3b$		$a^4 + 10a^3b + 36aabb + 56ab^3 + 32b^4$
	$aa + 6ab + 9bb$	
	$aa + 6ab + 8bb$	
$a + 3b$		$a^4 + 14a^3b + 72aabb + 56ab^3 + 128b^4$
$a + 4b$		$a^4 + 14a^3b + 72aabb + 162ab^3 + 135b^4$
	$aa + 8ab + 16bb$	
	$aa + 8ab + 15bb$	
$a + 4b$		
$a + 5b$		

PROPOSITIO III.

Si quotcunque rationum continuarum, quarum termini sint æquidifferentes, prima eademque minima vocetur a , differentia autem primæ & secundæ rationis vocetur b , tum ex differentiis secundis prima vocetur c ; atque ex tertiis prima vocetur d ; & sic porrò: Aio secundam rationem fore $a + b$, tertiam $a + 2b + c$, quartam $a + 3b + 3c + d$, quintam $a + 4b + 6c + 4d + e$; atque ita deinceps, componendo singulas rationes ex prima & tot differentiis, quot quæque locis abest à prima, ipsis autem differentiis jungendo coëfficientes numeros figuratos, primis quidem radices, secundis trigonales, tertiis pyramidales, atque ita porrò, singulosque adeò, prout naturali serie ordinantur in subjècta tabella:

A a

Unitates

Unitates.	radices.	trigo- nales.	pyrami- dales.	trigono- trigona- les.	trigono- pyrami- dales.	pyrami- di-pyra- midales.	trigono- trigono- pyramid.	trigono- pyrami- di-pyra- mid.	pyrami- di-pyra- midi-py- ramida.
1	1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9	10
1	3	6	10	15	21	28	36	45	
1	4	10	20	35	56	84	120		
1	5	15	35	70	126	210			
1	6	21	56	126	252				
1	7	28	84	210					
1	8	36	120						
1	9	45							
1	10								

Nimirum eodem modo, quo iidem figurati numerant complementa potesta-
tum à radice binomia genitarum; observato ascensu obliquo à sinistra dex-
trorsum.

Sic undecima ratio constat ex $a + 10b + 45c + 120d + 210e + 252f + 210g + 120h + 45i + 10k + l$; sumtis ordine numeris imam tabellæ
basin occupantibus.

Exhibet autem tabella non modò quotam velis rationem, sed & summam
quotcunque continuè sequentium, pari fermè negotio methodoque. Sic sum-
ma quinque rationum est $5a + 10b + 10c + 5d + e$, observato ascensu
obliquo, ut antè.

Rationes.

Rationes.	Demonstratio.			
	Diff. Primæ.	Secundæ.	Tertiæ.	Quartæ.
a	b			
$a + b$		c		
	$b + c$		d	
$a + 2b + c$		$c + d$		e
	$b + 2c + d$		$d + e$	
$a + 3b + 3c + d$		$c + 2d + e$		
	$b + 3c + 3d + e$			
$a + 4b + 6c + 4d + e$				

Cùm enim per præcedentem, progrediendo à majoribus terminis ad minores (vel, quod idem est, à minoribus rationibus ad majores), non modò secunda ratio excedat primam, sed & primarum, secundarum, cæterarumque differentiarum secunda quæque excedat primam; licebit fanè istos excessus vocare b , c , d , & deinceps.

Cùm a differentiarum tertiarum prima fit
Et quartarum prima (quâ secunda tertiarum excedit primam) d } Ex hypothesi;
Erit fanè secunda tertiarum e }
 $d + e$

Rursus cùm secundarum differentiarum prima fit
Et tertiarum prima (quâ altera secundarum excedit primam) c } Ex hypothesi;
Erit fanè altera secundarum d }
 $c + d$

Sed 2da 3tiarum (quâ tertia secundarum excedit alteram) erat $d + e$
Ergo tertia secundarum erit $c + 2d + e$

Porrò cùm primarum differentiarum prima fit b } Ex hypothesi;
Et secundarum prima (quâ secunda primarum excedit primam) c }
Erit fanè secunda primarum $b + c$

Sed altera secundarum (quâ tertia primarum excedit 2dam) erat $c + d$
Ergo tertia primarum $b + 2c + d$

Sed & 3tia 2darum (quâ quarta primarum excedit 3tiam) erat $c + 2d + e$
Ergo quarta primarum erit $b + 3c + 3d + e$

Denique cùm prima ratio fit a }
Et differentia inter primam & secundam rationem b } Ex hypothesi;
Erit fanè secunda ratio $a + b$

Et cùm differentiarum primarum secunda foret $b + c$
Erit tertia ratio $a + 2b + c$

Sed & differentiarum primarum tertia erat $b + 2c + d$
Ergo quarta ratio $a + 3b + 3c + d$

Tandem differentiarum primarum quarta erat $b + 3c + 3d + e$
Ergo quinta ratio $a + 4b + 6c + 4d + e$

Postremò rationes

Prima	a			
Secunda	$a +$	b		
Tertia	$a +$	$2b +$	c	
Quarta	$a +$	$3b +$	$3c +$	d
Quinta	$a +$	$4b +$	$6c +$	$4d + e$

} addantur,

fit summa quinque rationum $5a + 10b + 10c + 5d + e.$

A a 2

Rationes

Rationes vel Magnitudines.

Prima a

$$\frac{2}{1} a + b$$

$$\frac{3}{1} a + 2b + c$$

$$\frac{4}{1} a + 3b + 3c + d$$

$$\frac{5}{1} a + 4b + 6c + 4d + e$$

$$\frac{6}{1} a + 5b + 10c + 10d + 5e + f$$

$$\frac{7}{1} a + 6b + 15c + 20d + 15e + 6f + g$$

$$\frac{8}{1} a + 7b + 21c + 35d + 35e + 21f + 7g + h$$

$$\frac{9}{1} a + 8b + 28c + 56d + 70e + 56f + 28g + 8h + i$$

$$\frac{10}{1} a + 9b + 36c + 84d + 126e + 126f + 84g + 36h + 9i + k$$

PROPOSITIO IV.

Si quotcunque rationum continuarum, quarum termini sint æquidifferentes, prima eadēque maxima vocetur a , differentia autem primæ & secundæ vocetur b , tum differentiarum secundarum prima vocetur c , tertiarum prima d , & sic deinceps; aio secundam rationem fore $a - b$, tertiam $a - 2b + c$, quartam $a - 3b + 3c - d$, quintam $a - 4b + 6c - 4d + e$; atque ita deinceps, alternatis semper signis affirmatis & negatis. Demonstratur ut præcedens.

PROPOSITIO V.

Data rationis multiplicem invenire prope verum.

Constructio. Differentiam terminorum datæ rationis duc in denominatorem multiplicis dati, & à facto aufer ipsam differentiam, reliqui semissem adde termino majori, & detrahe minori; ita prodibunt duo termini exprimentes rationem paulò minorem quæsitâ. Tum si termini prodeuntes sint fortè numeri mixti ex integris & fractis; reducantur ad purè fractos, quorum denominatoribus omissis, ratio quæsitâ censebitur in numeratoribus integris & à fractione liberis. v. gr. Quæratu rationis $\frac{25}{8}$ quadruplum. Differentia terminorum 3 ducta in 4 exhibet 12, unde ablatis tribus restant 9, cujus semis $4\frac{1}{2}$ additus termino majori 28, facit $32\frac{1}{2}$, detractus autem minori 25, relinquit $20\frac{1}{2}$; erit igitur ratio $20\frac{1}{2}$ ad $32\frac{1}{2}$ paulò major quadruplo rationis $\frac{25}{8}$. Reductis terminis $20\frac{1}{2}$ & $32\frac{1}{2}$ ad purè fractos, fiunt $\frac{41}{2}$ & $\frac{65}{2}$, omissisque denominatoribus, erit ratio $\frac{41}{65}$ paulò major quæsitâ.

Demonstratio hujus & sequentis propositionis constabit ex propositione VII.

PROPOSITIO VI.

Data rationis partem imperatam invenire prope verum.

Constructio. Differentiam terminorum datæ rationis divide in partes totidem, quot denominator partis quæsitæ constat unitatibus, atque ex iis partibus exemptâ unâ, reliquarum semissem adde termino minori, & detrahe quoque majori; ita pro-

prodibunt duo termini exprimentes rationem paulò minorem quæsitâ: Tum si termini prodeuntes sint fortè numeri mixti ex integris & fractis; reducantur ad purè fractos, quorum denominatoribus omissis, ratio quæsitâ censebitur in numeratoribus integris, & à fractione liberis. v. gr. Oporteat rationis $\frac{3}{5}$ invenire partem quintam. Differentia terminorum 2 divisa quinquariam exhibet $\frac{2}{5}$, quæ est una pars quinta, eximenda ex integra summa quinque partium, quæ erat 2, & restant $\frac{8}{5}$, quarum semis $\frac{4}{5}$ additus termino minori 2, facit $3\frac{4}{5}$; deductus verò ex majori 5, reliquum facit $4\frac{1}{5}$; erit igitur ratio $3\frac{4}{5}$ ad $4\frac{1}{5}$ paulò minor, quàm pars quinta rationis $\frac{3}{5}$. Reductis terminis $3\frac{4}{5}$ & $4\frac{1}{5}$ ad purè fractos, fiunt $\frac{19}{5}$ & $\frac{21}{5}$, omissisque denominatoribus, erit ratio $\frac{19}{21}$ paulò minor quæsitâ. Rursus inveniendus sit rationis $\frac{8}{11}$ semis. Differentia terminorum 3 bipartita exhibet $1\frac{1}{2}$, qui est unus semis, eximendus ex integra summa duarum partium 3, & restat $1\frac{1}{2}$, cujus semis $\frac{3}{4}$ additus termino minori 8, facit $8\frac{3}{4}$; deductus autem ex majori 11, relinquit $10\frac{1}{4}$; erit igitur ratio $8\frac{3}{4}$ ad $10\frac{1}{4}$ paulò minor semisse rationis $\frac{8}{11}$. Reductis terminis $8\frac{3}{4}$ & $10\frac{1}{4}$ ad purè fractos, fiunt $\frac{35}{4}$ & $\frac{41}{4}$, omissisque denominatoribus, erit ratio $\frac{35}{41}$ paulò minor quæsitâ.

PROPOSITIO VII.

Invenire, quantum pars rationis imperata, quæ per præcedentem invenitur, deficiat ab exactiori.

Constructio. Primò, si partis imperatæ denominator sit numerus impar; sume rationes, quæ sunt rationi per præcedentem inventæ utrinque vicinæ & æquidifferentes, ita habebis tres rationes, quarum minimam aufer à media, & mediam à maxima, prodibunt duæ differentiæ, quarum differentiam de-nuò investigabis, tantisper asservandam. Deinde partis imperatæ denominatori unitatem detrahe, reliqui semissem in tabula figuratorum insertâ prop. III, quære inter radices, & invento congruentem numerum trigonalem excerptum tripartire, sic invenies, quoties sumenda sit differentiarum differentia supra asservata, ut acquiras particulam, quâ pars imperata, quæ per præcedentem inveniebatur, deficit ab exactiori. v. g. Scire velim, ratio $\frac{19}{21}$ per præcedentem inventa quantum deficiat ab exactiori quinta parte rationis $\frac{3}{5}$. Rationi $\frac{19}{21}$ utrinque vicinæ & æquidifferentes sunt $\frac{17}{19}$ & $\frac{21}{23}$. Differentia minimæ à media $\frac{437}{441}$ & mediæ à maxima $\frac{357}{361}$, & harum differentiarum differentia $\frac{157437}{157757}$ asservanda. Tum partis imperatæ denominatori 5 detraho 1, restant 4, cujus semissi 2 inter radices invento congruit trigonalis numerus 3, cujus triens est 1, indicans differentiarum differentiam supra asservatam $\frac{157437}{157757}$ semel sumtam exhibere particulam, quâ ratio $\frac{19}{21}$ deficit ab exactiori quinta parte rationis $\frac{3}{5}$, adeo ut hujus exactior pars quinta sit ratio $\frac{19}{21} +$ ratione $\frac{157437}{157757}$.

Sin partis imperatæ denominator sit numerus par; sume semissem differentiæ terminorum rationis per præcedentem inventæ, quem ejusdem termino minori detrahes, & majori addes pariter ac detrahes; ita obtinebis quatuor rationes continuas terminorem æquidifferentium, ex quibus minorum duarum differentiam auferes ex majorum duarum differentia, & emergentem differentiarum differentiam asservabis. Deinde partis imperatæ denominatorem bipartire, & invento

invento semissi congruentes in tabella propositioni III subjuncta species excerpere, saltim usque ad c speciem, positoque $a = \frac{1}{2}$, $b = 2$, $c = 1\frac{1}{3}$; duc cujusque speciei valorem in suum coefficientem, collectisque omnibus in unam summam, habebis, quot vicibus sumenda sit differentiarum differentia supra asservata, ut acquiras particulam, quâ pars imperata, quæ per præcedentem inveniebatur, deficit ab exactiori. Ex. gr. Rationis $\frac{8}{11}$ octans per præcedentem inventus sit $\frac{146}{149}$; scire velim, quantum is deficiat ab exactiori. Differentia terminorum est 6, cujus semis 3 detractus minori termino, relinquit 146; additus autem majori, facit 158; & detractus majori, relinquit 152. Sunt ergo quatuor rationes continuæ terminorum æquidifferentium $\frac{146}{149}$, $\frac{149}{152}$, $\frac{152}{155}$, $\frac{155}{158}$. Differentia duarum minorum rationum $\frac{24016}{24025}$ ablata à differentia duarum majorum $\frac{22192}{22201}$ relinquit differentiarum differentiam $\frac{1753825}{1753879}$ asservandam. Partis imperatæ denominator est 8, cujus semissi 4 congruunt in tabella propositioni III subjuncta species istæ: $a + 3b + 3c$; sed $a = \frac{1}{2}$, & $3b = 6$, & $3c = 4$. quæ juncta faciunt $10\frac{1}{2}$. Ergo differentiarum differentiæ $\frac{1753825}{1753879}$ supra servatæ sumendum est decuplum cum semisse, ut acquiramus particulam, quâ octans per præcedentem inventus deficit ab exactiori. Atqui rationis $\frac{1753825}{1753879}$ decuplum per v hujus est $\frac{1753582}{1754122}$ vel $\frac{876791}{877061}$, & semis per vi est $\frac{3507677}{3507731}$; adeò ut rationis $\frac{8}{11}$ octans exactior præter rationem per præcedentem inventam $\frac{146}{149}$ contineat etiamnum ratiunculas $\frac{876791}{877061}$, & $\frac{3507677}{3507731}$.

DEMONSTRATIO.

Cùm per III hujus summa trium rationum continuarum, terminis æquidifferentibus contentarum, sit

Erit ejusdem summæ triens

Rursus differentia primæ & secundæ rationis est
secundæ autem & tertiæ

Ergo differentiarum differentia

Cujus triens

Quem si addas mediæ trium rationum

Erit summa

Æqualis nempe trienti trium rationum supra notato literâ α . Ergo discerptâ ratione quavis in tres continuas terminorum æquidifferentium, ut jubet propositio VI, erit media ex iis paulò minor triente totius discerptæ, & quidem tantò minor, quantus est triens differentiæ differentiarum intercedentium inter rationem primam & secundam, nec non inter secundam & tertiam, quod innuit propositio VII.

Sic

Sic quoque per III hujus, summa sedecim rationum continuarum, terminis æquidifferentibus contentarum, est

$$16a + 120b + 560c + 1820d$$

Et ejusdem summæ octans

$$2a + 15b + 70c + 227\frac{1}{2}d (\beta)$$

Tum per tabellam propositioni III subjunctam, quatuor mediæ ex istis sedecim, nimirum 7^{ma}, 8^{va}, 9^{na}, 10^{ma}, sunt

$$\left\{ \begin{array}{l} a + 6b + 15c + 20d \\ a + 7b + 21c + 35d \\ a + 8b + 28c + 56d \\ a + 9b + 36c + 84d \end{array} \right.$$

Differentia duarum priorum

$$b + 6c + 15d$$

posteriorum

$$b + 8c + 28d$$

Differentiarum differentia

$$2c + 13d$$

Hujus decuplum

$$20c + 130d$$

& semis

$$c + 6\frac{1}{2}d$$

Unà cum summa duarum ex quatuor istis mediis. $2a + 15b + 49c + 91\frac{1}{2}d$

Facit

$$2a + 15b + 70c + 228d$$

Æqualem octanti sedecim rationum suprâ notato literâ β . Ergo si ratio data discerpatur in partes sedecim, erunt duæ mediæ ex iis simul, paulò minores octante totius discerptæ; & quidem tantò minores, quantum est differentiarum (intercedentium inter rationes ex sedecim istis septimam & octavam, nec non inter 9^{nam} & 10^{nam}) decuplum cum semisse. q. e. d.

PROPOSITIO VIII.

Rationes terminorum æquidifferentium sunt propemodum, ut reciprocè ipsorum terminorum media arithmetica.

Explicatio. Sumatur per VI hujus rationis cujusvis, v. gr. $\frac{8}{9}$, pars quævis, v. gr. semis $\frac{3\frac{3}{4}}{2}$, tum pars quævis alia, v. gr. triens $\frac{2\frac{5}{6}}{3}$, & ut fiant terminorum æquidifferentium, pro $\frac{8}{9}$ sumatur $\frac{1\frac{6}{8}}{8}$, & pro $\frac{1\frac{5}{6}}{6}$ æquipollens $\frac{5\frac{0}{2}}{2}$. Medium arithmeticum terminorum rationis totius est 17, semissis 34, trientis 51. Liqueat igitur, ut tota ratio $\frac{1\frac{6}{8}}{8}$ est ad semissem suum $\frac{3\frac{3}{4}}{2}$; ita reciprocè semissis medium arithmeticum 34 esse ad medium totius 17: & ut tota ratio $\frac{1\frac{6}{8}}{8}$ est ad trientem suum $\frac{5\frac{0}{2}}{2}$; ita reciprocè trientis medium arithmeticum 51 esse ad medium totius 17: ideòque etiam, ut semis $\frac{3\frac{3}{4}}{2}$ est ad trientem $\frac{5\frac{0}{2}}{2}$; ita reciprocè trientis medium 51 esse ad medium semissis 34: Tantum porrò hanc analogiam abire à vero, quantum semisses, trientes, partesve aliæ rationum per VI hujus inventæ deficiunt ab exactis. Quamobrem id agendum, ne defectus ille instituto nostro officiat. Cæterum minor erit defectus, minùsque adeò officiat, quò rationes in analogiam adscitæ minores fuerint. Cùm enim secundum demonstrationem præcedentis, octans exactus sedecim rationum foret $2a + 15b + 70c + 227\frac{1}{2}d$, at summa duarum ex sedecim istis mediarum $2a + 15b + 49c + 91\frac{1}{2}d$, qui est octans per VI inventus; patet differentiam horum octantium consistere in contentiori parte secundarum & tertiarum differentiarum. Atqui rationum continuarum minores habent differentias prius minores, ac proinde differentiarum secundarum & tertiarum partem exiliorem multò etiamnum minorem. Sed exemplo res fiet illustrior. Nam rationis

$\frac{100}{101}$ femis, per VII hujus, est $\frac{199}{201} + \frac{7999399}{7999401}$; ubi ratio $\frac{199}{201}$ ab exactiori femisse deficit ratiunculâ superbipartiente $\frac{7999399}{7999401}$. Rursus rationem $\frac{199}{201}$ (quæ prius assumptæ $\frac{100}{101}$ propemodum femis est) si denuò bipartiamur per VII hujus, habebimus $\frac{399}{401} + \frac{127997599}{127997601}$; ubi ratio $\frac{399}{401}$ ab exactiori femisse deficit ratiunculâ superbipartiente $\frac{127997599}{127997601}$. Minor est igitur defectus, cum bipartimur rationem minorem $\frac{100}{101}$, quàm si bipartiamur majorem $\frac{99}{101}$, quantò scilicet ratiuncula superbipartiens $\frac{127997599}{127997601}$ minor est superbipartiente $\frac{7999399}{7999401}$, hoc est propemodum, quantò 8 milliones minores sunt 128 millionibus, nimirum sedecim vicibus. Sed & ejusdem rationis quò minor pars fumetur per VI hujus, eò minus deficiet à vero. Sic rationis $\frac{99}{101}$ femis, per VII hujus, erat $\frac{199}{201} + \frac{7999399}{7999401}$, & ejusdem triens per eandem est $\frac{299}{301} + \frac{8099459197}{8099460797}$; ubi quidem triens $\frac{299}{301}$ (qualis per VI invenitur) minùs deficit ab exactiori, quàm femis $\frac{100}{101}$ (per eandem inventus); quantò ratiuncula $\frac{8099459197}{8099460797}$ minor est alterâ $\frac{7999399}{7999401}$. Unde sequitur, cum bipartitâ ratione $\frac{100}{101}$ secundum VI hujus, non nisi binarium quasi perdamus in octo millionibus, vel unitatem in quatuor millionibus; futurum, ut istius femisse $\frac{199}{201}$ (five $\frac{99.5}{100.5}$) diminuto quovis modo per analogiam VI^{te} hujus superstructam, minùs etiam perdamus; adeoque à ratione $\frac{99.5}{100.5}$ nos analogicè argumentari posse ad quamvis minorem terminorum æquidifferentium, ita ut minùs quàm unitatem perdamus in quatuor millionibus; quòque ratio, ad quam argumentamur, minor fuerit, eò jacturam fore minorem.

PROPOSITIO IX.

Data mensurâ rationis $\frac{99.5}{100.5}$ in particulis, qualium decupla continet 1,000,000; invenire mensuram cujusvis minoris rationis terminorum æquidifferentium, in particulis similibus.

Rationis $\frac{99.5}{100}$ mensura suprâ inventa fuit 2176913, & rationis $\frac{100}{100.5}$ mensura ibidem 2165917, quarum summa exhibet rationem $\frac{99.5}{100.5} = 4342910$. Dehinc oporteat nos invenire mensuram rationis $\frac{100}{101}$. Ergo per præcedentem dic:

Ut medium Arithmeticum terminorum $\frac{100}{101}$ (nimirum 100.5) ad medium Arithmeticum terminorum $\frac{99.5}{100.5}$ (nimirum 100); ita mensura hujus rationis (putà

(putà 43429) ad mensuram istius 43213. Tot igitur particulis logarithmus absoluti 101 excedit logarithmum absoluti 100. Quare, cùm logarithmus absoluti 100 fit 2,0000000; oportet, ut logarithmus absoluti 101 fit 2,0043213.

Porrò invenienda fit mensura rationis $\frac{101}{102}$. Dic :

Ut medium arithmeticum terminorum $\frac{101}{102}$ (nimirum 101½) ad medium arithmeticum terminorum $\frac{99\frac{1}{2}}{100\frac{1}{2}}$ (nimirum 100); ita mensura hujus rationis (putà 43429) ad mensuram istius 42787. Tot igitur particulis logarithmus absoluti 102 excedit logarithmum absoluti 101. Quare, cùm logarithmus absoluti 101 foret 2,0043213; oportet, ut logarithmus absoluti 102 fit 2,008600.

Cùm autem in omnibus hisce analogiis terminus secundus fit 100, & tertius 43429; liquet, ad inveniendam mensuram cujusvis rationum sequentium nihil amplius restare, quàm ut dividamus numerum 43429 per medium arithmeticum terminorum rationis datæ. Cæterùm invenimus nos quidem rationis $\frac{99\frac{1}{2}}{100\frac{1}{2}}$ mensuram 43429, quæ fortè debebat esse unitate major, putà 43430; sed facilè intelligit quivis, si pro ratione $\frac{99\frac{1}{2}}{100\frac{1}{2}}$ assumissemus $\frac{999\frac{1}{2}}{1000\frac{1}{2}}$, & in cumulandis horum terminorum potestatibus calculum ad plures locos extendissemus, ad majorem utique præcisionem perveniri potuisse, adeoque huic methodo ad accuratam facilitatem nihil quicquam deesse. At non deest modus etiam hoc ipso facilius, qui post acquisitos paucos logarithmos solâ additione rem peragit, & præterea probam suam secum fert, quem propositionibus sequentibus breviter exponemus.

PROPOSITIO X.

Rationum duarum continuarum differentia est ad aliarum duarum continuarum differentiam; ut harum communis termini quadratum, ad istarum communis termini quadratum; dummodo singularum termini sint æquidifferentes.

Sint duæ rationes continuæ $\frac{a}{a+b}$, & $\frac{a+b}{a+2b}$, quarum terminus communis est $a+b$, & hujus quadratum $aa+2ab+bb$; sint verò & aliæ duæ continuæ $\frac{a+3b}{a+4b}$, & $\frac{a+4b}{a+5b}$, quarum communis terminus est $a+4b$, & hujus quadratum $aa+8ab+16bb$. Singularum termini differunt communi excessu b . Differentia duarum priorum est $\frac{aa+2ab}{aa+2ab+bb}$, & posteriorum duarum $\frac{aa+8ab+15bb}{aa+8ab+16bb}$. Atque hæ differentiæ sunt quoque terminorum æquidifferentium. Ergo per VIII hujus sunt propemodum, ut reciprocè ipsorum terminorum media arithmetica; hoc est, ut prior differentia $\frac{aa+2ab}{aa+2ab+bb}$, ad posteriorem $\frac{aa+8ab+15bb}{aa+8ab+16bb}$: ita horum terminorum medium arithmeticum

B b

aa +

$aa + 8ab + 15\frac{1}{2}bb$, ad medium arithmeticum istorum $aa + 2ab + \frac{1}{2}bb$; hoc est propemodum, ut communium terminorum quadrata, nimirum $aa + 8ab + 16bb$ ad $aa + 2ab + bb$; quæ ab istis mediis arithmeticis non nisi quantitate $\frac{1}{2}bb$ differunt, exiguâ fanè, & in rationibus minoribus (ubi sc. differentia terminorum b ad ipsos terminos exiguum instar habet) facile contemnendâ.

PROPOSITIO XI.

Rationum trium continuarum differentiarum differentia, est ad aliarum trium continuarum differentiarum differentiam; ut cubus medii arithmetici mediæ ex his, ad cubum medii arithmetici mediæ ex istis; dummodo singularum rationum termini sint æquidifferentes.

Sint tres rationes continuæ $\frac{a}{a+b}, \frac{a+b}{a+2b}, \frac{a+2b}{a+3b}$, quarum differentia differentiarum est $\frac{a^4 + 6a^3b + 12aabb + 8ab^3}{a^4 + 6a^3b + 12aabb + 10ab^3 + 3b^4}$, & mediæ illarum medium arithmeticum est $a + \frac{3b}{2}$, cujus cubus $a^3 + \frac{9aab}{2} + \frac{27abb}{4} + \frac{35b^3}{8}$; sint verò & aliæ tres continuæ $\frac{a+2b}{a+3b}, \frac{a+3b}{a+4b}, \frac{a+4b}{a+5b}$, quarum differentia differentiarum est $\frac{a^4 + 14a^3b + 72aabb + 160ab^3 + 128b^4}{a^4 + 14a^3b + 72aabb + 162ab^3 + 135b^4}$, & mediæ illarum medium arithmeticum $a + \frac{7b}{2}$, cujus cubus $a^3 + \frac{21aab}{2} + \frac{63abb}{4} + \frac{343b^3}{8}$. Cæterum singulæ rationes differunt communi excessu b ; at differentiæ differentiarum non sunt terminorum æquidifferentium, siquidem differentia terminorum prioris est $2ab^3 + 3b^4$, at posterioris $2ab^3 + 7b^4$: secus ac in præcedenti propositione. Quare cum illic res expediretur regulâ proportionum simplici inversâ, hic opus est duplici inversâ; nimirum:

Ut medium arithmeticum prioris differentiæ differentiarum (nimirum $a^4 + 6a^3b + 12aabb + 9ab^3 + \frac{3b^4}{2}$) ductum in differentiam terminorum posterioris ($2ab^3 + 7b^4$) ad medium arithmeticum posterioris differentiæ differentiarum (nimirum $a^4 + 14a^3b + 72aabb + 161ab^3 + \frac{263b^4}{2}$) ductum in differentiam terminorum prioris ($2ab^3 + 3b^4$): Ita differentia differentiarum prior ad posteriorem: Ita quoque cubus medii arithmetici mediæ trium priorum rationum (nimirum $a^3 + \frac{9aab}{2} + \frac{27aab}{4} + \frac{35b^3}{8}$) ad cubum medii arithmetici mediæ trium posteriorum rationum (nimirum $a^3 + \frac{21aab}{2} + \frac{63abb}{4} + \frac{343b^3}{8}$).

Quæ analogia vera esse deprehendetur, si productum extremorum æquale sit producto mediorum.

Atqui primus terminus $a^4 + 6a^3b + 12aabb + 9ab^3 + \frac{3b^4}{2}$ in $2ab^3 + 7b^4 = 2a^5b^3 + 19a^4b^4 + 66a^3b^5 + 102aab^6 + 66ab^7 + \frac{21b^8}{2}$, si ducatur in quantum

quartum $a^3 + \frac{21aab}{2} + \frac{63abb}{4} + \frac{343b^3}{8}$; productum est $2a^8b^3 + 40a^7b^4 + 297a^6b^5 + 1180a^5b^6 + 2991\frac{1}{8}a^4b^7$, &c.

Rurfus secundus terminus $a^4 + 14a^3b + 72aabb + 161ab^3 + \frac{263b^4}{2}$ in $2ab^3 + 3b^4 = 2a^5b^3 + 28a^4b^4 + 144a^3b^5 + 322aab^6 + 263ab^7 + \frac{789b^4}{2}$, si ducatur in tertium $a^3 + \frac{9aab}{2} + \frac{27abb}{4} + \frac{35b^3}{8}$; productum est $2a^8b^3 + 40a^7b^4 + 339a^6b^5 + 1593a^5b^6 + 4558\frac{1}{8}a^4b^7$, &c.

Hoc igitur productum cum consentiat cum isto, non modò in primis & secundis speciebus, sed & in maxima parte tertiarum & quartarum; aio analogiam in propositione memoratam, veram esse. Nam defectus, qui hîc apparet in productis terminorum, in ipsis terminis longè minor erat, quippe qui multiplicando crevit. Ut taceam in minoribus rationibus differentias secundas & tertias nullius ferè momenti esse*.

Simili modo ostendetur, differentias tertias rationum continuarum & terminorum æquidifferentium, esse ut quadrato-quadrata; quartas, ut quadrato-cubos terminorum, qui singulis in tabellâ propositioni 11 subjunctâ, è regione opponuntur. Atque ita deinceps.

PROPOSITIO XII.

Numerorum in progressionem arithmetica ordinatorum quadrata conveniunt in differentiis secundis, cubi in tertiis, quadrato-quadrata in quartis, & sic deinceps.

Patet ex inspectione tabellarum subjectarum:

Numeri Quadrata diff. 1. diff. 2.

1	1			
2	4	3	2	
3	9	5	2	
4	16	7		
Numeri	Cubi	diff. 1.	diff. 2.	diff. 3.
1	1			
2	8	7	12	
3	27	19	18	6
4	64	37	24	6
5	125	61		

* See the note at the end of this tract, p. 196.

Numeri	Quadrato-quadrata	diff. 1.	diff. 2.	diff. 3.	diff. 4.
1	1				
2	16	15			
3	81	65	50		
4	256	175	110	60	24
5	625	369	194	84	24
6	1296	671	302	108	

Hinc patet, datis v. gr. cubis quatuor, vel quadrato-quadratis quinque, quo pacto cæteri continuâ additione succenturiari possint. Sint enim dati

Cubi	diff. 1.	diff. 2.	diff. 3.
$a = 8$			
$b = 27$	$e = 19$	$b = 18$	
$c = 64$	$f = 37$	$i = 24$	$k = 6$
$d = 125$	$g = 61$	$l = ..$	$k = 6$
$n = ...$	$m = ..$		

Dic: $k + i = l, l + g = m, m + d = n.$

PROPOSITIO XIII.

Logarithmos quotvis locorum continuâ ac solâ additione producere ita, ut ultimo existente probo, cæteri omnes sint probi.

Constructio. Cùm sit $\frac{a}{b-c} = \frac{a}{b} + \frac{ca}{bb} + \frac{cca}{b^3} + \frac{c^3a}{b^4}$, & deinceps continuando progressionem in infinitum; si ponamus $a = 100 =$ medio arithmetico rationis $\frac{99\frac{1}{2}}{100\frac{1}{2}}$ & $b = 100000$, & $c = 0\frac{1}{2}$, adeò ut $b - c$ fit $99999\frac{1}{2} =$ medio arithmetico rationis datæ $\frac{99999}{100000}$, cujus mensuram invenire oportet, erit $\frac{a}{b-c}$ (nimirum $\frac{100}{99999\frac{1}{2}}$) $= \frac{a}{b}$ (five $\frac{100}{100000}$ vel $0\frac{1}{1000}$) $+ \frac{ca}{bb}$ (five $\frac{0\frac{1}{2} * 100}{1000000000}$ vel $\frac{50}{1000000000}$ vel $0\frac{1}{2000000000}$) $+ \frac{cca}{b^3}$ (five $\frac{0\frac{1}{2} * 100}{1000000000000000}$ vel $\frac{25}{1000000000000000}$ vel $0\frac{1}{4000000000000000}$) $+ \frac{c^3a}{b^4}$ (five $\frac{0\frac{1}{2} * 100}{1000000000000000000000}$ vel $\frac{1\frac{1}{4}}{1000000000000000000000}$ vel $0\frac{1}{800000000000000000000}$) $=$

0[001000005000025000125: fin manentibus a & b , ut antè, ponatur $c = 1[5$; erit $\frac{a}{b-c} = 0[001000015000225003375$: vel si rursus manentibus a & b , ponatur $c = 2[5$; erit $\frac{a}{b-c} = 0[001000025000625015625$. Liquet ex præcedenti, quo pacto datis numeri 5 quadrato & cubo, fequentium quoque numerorum 15, 25, cæterorumque quadrata & cubi additione continuâ inveniri possint. Istis igitur numeris cum suis quadratis & cubis ritè dispositis ita; ut post unitatem in 0[001 sextum locum occupet ultima figura numeri 5 (vel 15, vel 25); & sextum ab hac locum ultima figura quadrati 25 (vel 225, vel 625); & ab hac rursus sextum locum ultima figura cubi 125 (vel 3375, vel 15625); quemadmodum exempla superiora ostendunt: conflati erunt numeri, qui in tertium regulæ aureæ terminum 43429 ducendi sunt singuli, ut prodeat mensura rationum datarum $\frac{99999}{100000}$, $\frac{99998}{99999}$, $\frac{99997}{99998}$. Atque hæc multiplicatio, ut solâ quoque additione perficiatur, digerendus est tertius terminus 43429 in tabellam subsidiariam, hoc modo:

1	43429
2	86858
3	130287
4	173716
5	217145
6	260574
7	304003
8	347432
9	390861

Acquisitâ hoc modo rationum omnium mensurâ à $\frac{99999}{100000}$ usque ad $\frac{10000}{10001}$ in particulis, qualium decupla continet 1,0000000; mox addendo has ordine retrogrado concinnabimus logarithmos singulorum absolutorum à 10000 ad 100000; quorum ultimus si sit probus, præcedentes omnes erunt probi; nisi fortè error posterior quasi ex condicito corrigat priorem, quod in hac non magis quàm omnibus aliis probis, nec nisi rarissimè usuvenire potest.

ATQUE ita expositâ methodo construendi logarithmos novâ, accuratâ, & facili, haud scio, an opus sit monere lectorem, si ad praxin accedere luberet, non requiri compositorum numerorum logarithmos; ideoque omnes pares primum excludendos, deinde omnes à quinario productos; ita ut restent soli logarithmi numerorum in unitatem 3^{ium}, 7^{ium}, 9^{ium} exeuntium, atque horum quoque tertium quemque, cùm sit ex ternario compositus, omitti posse: sic acquisitis logarithmis absolutorum 10003 & 10013, omisso logarithmo absoluti 10023 ex ternario compositi quærendus est logarithmus absoluti 10033, utpote tricesimi ab absoluto 10003; tum logarithmus absoluti 10043, tricesimi ab absoluto 10013: tum rursus omisso logarithmo numeri 10053 quærantur logarithmi

garithmi numerorum 10063, 10073; atque ita deinceps. Quare dicendum per regulam propositione VII. traditam:

Ut 10048, nimirum medium arithmeticum inter 10033 & 10063,
ad 10018, nimirum medium arithmeticum inter 10003 & 10033:

Ita 13006, nimirum differentia logarithmorum congruentium absolutis 10003
& 10033,
ad 12967, differentiam logarithmorum competentium absolutis 10033 &
10063.

Dico:

Ut $\left\{ \begin{array}{l} 10048 \\ 10078 \\ 10108 \\ \&c. \end{array} \right\}$ ad 10018; ita 13006, ad $\left\{ \begin{array}{l} 12967 \\ \&c. \end{array} \right\}$

Item:

Ut $\left\{ \begin{array}{l} 10058 \\ 10088 \\ 10118 \\ \&c. \end{array} \right\}$ ad 10028; ita 12992, ad $\left\{ \begin{array}{l} 12953 \\ \&c. \end{array} \right\}$

At tertius qui sequitur ordo, nimirum:

Ut $\left\{ \begin{array}{l} 10068 \\ 10098 \\ \&c. \end{array} \right\}$ ad 10038; ita 12979, ad $\left\{ \begin{array}{l} 12940 \\ \&c. \end{array} \right\}$

hic ipse est, quem omittendum indigeto.

Pari modo tertius quisque in unitatem 7^{riam}, 9^{riam} exeuntium omittetur. Itaque fiet, omissis paribus lucrificamus semissem operæ, & detractis quinariis rursus partem decimam, denique excluso tertio quoque in 1, 3, 7, 9, deficientium, trientem laboris residui; unde non nisi $\frac{4}{15}$, nonaginta chiliadum, quæ sunt à 10000 ad 100000, industriam nostram expectant. Hoc est, de 90 chiliadibus restant solum 24 concinnandæ. Cæteros compositos à 7^{rio}, 11^{rio}, aliisve primis genitos, non est operæ pretium fecernere in methodo tam proclivi; præsertim cùm probando calculo inservire possint.

Cæterum ex iis, quæ hætenus differuimus, satis liquet, naturam logarithmorum geometriæ nullo modo obnoxiam esse; sed verius ac liquidius ex proprio suo fonte manare. Intercedit tamen utrisque cognatio suavis, & contemplatione dignissima, quam deinceps paucis exponere non gravabor.

PROPOSITIO XIV.

Sit hyperbole MBF, cujus latus rectum KL æquale sit transverso BC; erunt asymptoti AN & AE ad angulos rectos, & quadratum DF æquale rectangulo CDB per XXI. Conicor. I. Ex B & F cadant perpendiculares ad asymptoton BI & FH. Dico, AH esse ad AI, ut BI ad FH.

DEMONSTRATIO.

Sit $AB = I = AC$

$$BC = AB + AC = 2$$

$$BD = a$$

$$AD = AB + BD = 1 + a = DE$$

$$CD = BC + BD = 2 + a$$

$$CD \times BD = 2 + a \text{ in } a = 2a + aa = Q: DF$$

$$DF = \sqrt{2a + aa} = DG$$

$$AG = AD + DG = 1 + a + \sqrt{2a + aa}$$

$$\sqrt{2} . I :: AG . AH$$

$$A_H = \frac{1 + a + \sqrt{2a + aa}}{\sqrt{2}}$$

$$EF = DE - DF = 1 + a - \sqrt{2a + aa}$$

$$\sqrt{2} \cdot I :: EF \cdot FH$$

$$FH = \frac{1 + a - \sqrt{2a + aa}}{\sqrt{2}}$$

Ducatur AH in FH ;

ponendo $1 + a = c$, & $\sqrt{2a + aa} = d$

erit $1 + 2a + aa = cc$] subtraho

& $2a + aa = dd$ } tubthane

$$\mathbf{I} = cc - dd$$

$$\frac{cc - dd}{\sqrt{2} \times \sqrt{2}} = \frac{1}{2} = AH \times FH \quad \alpha$$

$$\sqrt{2} . I :: AB . AI$$

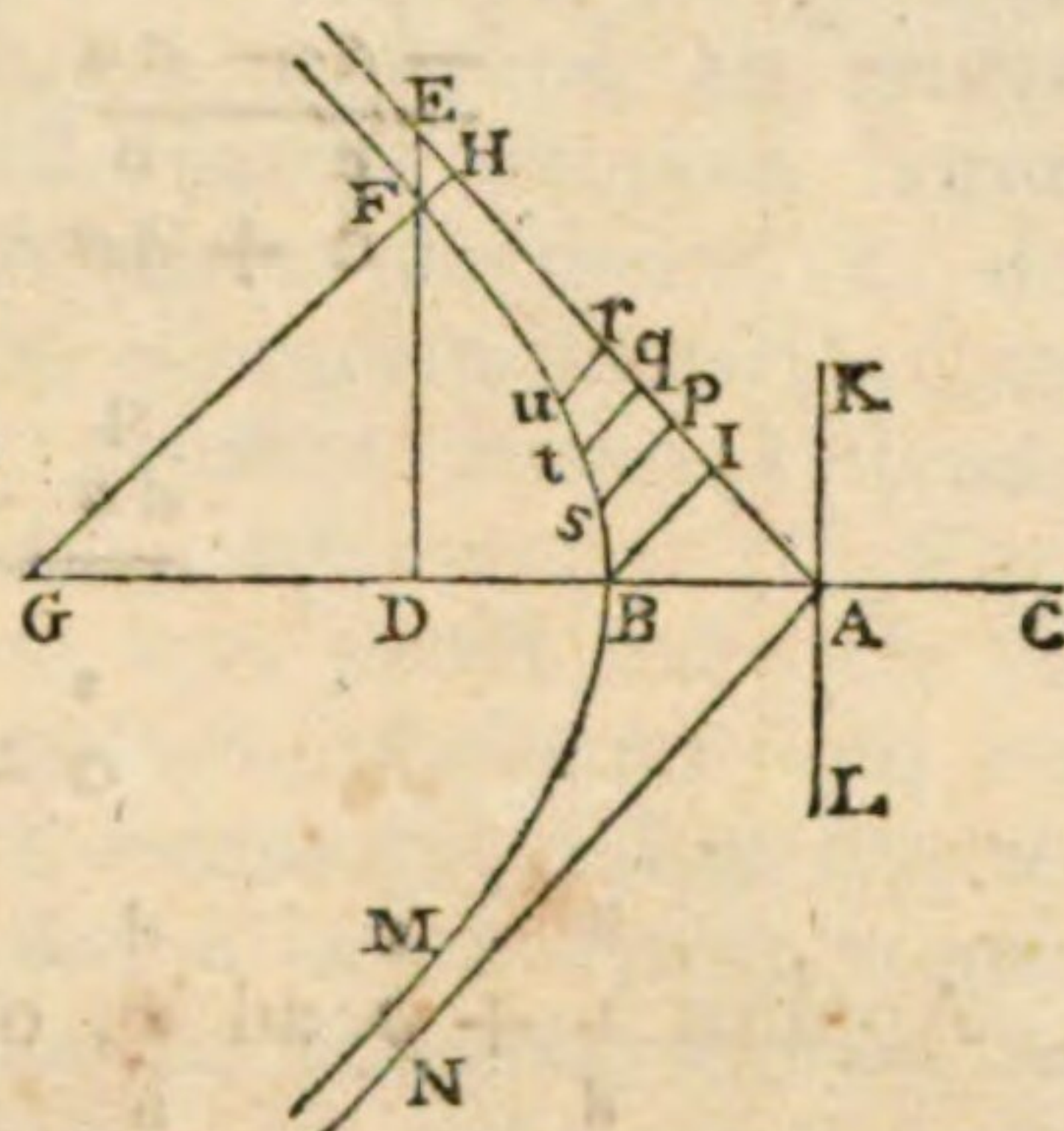
$$AI = \frac{I}{\sqrt{2}}$$

$$A I \times B I = \frac{I}{\sqrt{2}} \times \frac{I}{\sqrt{2}} = \frac{I}{2} \quad \beta$$

Ergo per α & β , $AH \times FH = AI \times BI$

$$\& AH. AI :: BI. FH$$

q. e. d.



PROPOSITIO XV.

In diagrammate præcedenti, positâ $AI = BI = 1$, & $HI = a$; oporteat invenire FH .

Dic per præcedentem : ut AH ad AI, ita BI ad FH; hoc est, $1 + a \cdot 1 :: 1$.
 $\frac{1}{1+a}$; nimirum FH æqualis est unitati divisæ per $1 + a$. Perficitur autem divisio ipso opere sic :

$\begin{array}{r} \overset{b}{1} + \overset{c}{a} \overline{) \overset{d}{1}} \\ \underline{\overset{f}{1} + \overset{g}{a}} \\ \overset{h}{0} - \overset{i}{a} \\ \quad \underline{\overset{l}{a} + \overset{m}{aa}} \\ \quad \overset{n}{0} + \overset{o}{aa} \end{array}$	$\begin{array}{c} \overset{e}{1} \\ \\ \\ \overset{k}{-a} \\ \\ \\ \overset{p}{+aa} \\ \\ \\ \overset{u}{-a} \end{array}$
$\begin{array}{r} \overset{q}{aa} + \overset{r}{a} \\ \underline{\hspace{1cm}} \\ \overset{s}{0} - \overset{t}{a} \end{array}$	

Applica $\overset{b}{1} + \overset{c}{a}$ ad $\overset{d}{1}$, oritur $\overset{e}{1}$; tum $\overset{e}{1}$ in $\overset{b}{1} + \overset{c}{a}$ producit $\overset{f}{1} + \overset{g}{a}$ subducendum ex $\overset{d}{1}$, & restat $\overset{h}{0} - \overset{i}{a}$. Rursus $\overset{b}{1} + \overset{c}{a}$ applicetur ad $\overset{h}{0} - \overset{i}{a}$, oritur $\overset{k}{-a}$; tum $\overset{k}{-a}$ in $\overset{h}{0} - \overset{i}{a}$ producit $\overset{l}{-a} - \overset{m}{aa}$, subducendum ex $\overset{h}{0} - \overset{i}{a}$, & restat $\overset{n}{0} - \overset{o}{aa}$.

Ad hoc applica $\overset{b}{1} + \overset{c}{a}$, oritur $\overset{p}{+aa}$; quod ductum in $\overset{b}{1} + \overset{c}{a}$ gignit $\overset{q}{aa} + \overset{r}{a}$, subducendum ex $\overset{n}{0} + \overset{o}{aa}$, & restat $\overset{s}{0} - \overset{t}{a}$. Atque ita continuatâ operatione, deprehenditur $\frac{1}{1+a} = 1 - a + aa - a^3 + a^4$ (&c.) = FH.

PROPOSITIO XVI.

Quovis numero in partes æquales innumeras discerpto; invenire summam quarumvis potestatum ab innumeris istis numeris genitarum.

Numeri dati potestas proximè superior potestatibus quæsitis, si dividatur per exponentem suum, extabit summa potestatum quæsitæ.

V. gr. Numerus datus sit 21; hic si discerpatur in partes innumeras, continebit non modò hos numeros 20, 19, 18, 17, &c. sed & innumeros interjectos, quorum quisque intelligitur ductus in unam partem infinitissimam numeri 21. Horum igitur omnium productorum summam si quæras; quoniam ipsa producta sunt potestates primæ (sive lineæ); erit potestas proximè superior quadratica; & ejus exponens 2. Ergo dati numeri 21 quadratum 441, si dividatur per exponentem

exponentem 2, extabit summa omnium primarum potestatum, genitarum ab innumeris istis numeris, qui in dato numero 21 continentur, nimirum 220[5]. Rursus quævis potestas prima intelligatur ducta in seipsam, & oporteat nos invenire summam omnium istorum quadratorum. Potestas proximè superior est cubica, & ejus exponens 3. Ergo dati numeri 21 cubus 9261, si dividatur per exponentem 3, extabit summa omnium quadratorum 3087. Horum quadratorum quodvis ducatur in suum latus, & oporteat nos invenire summam omnium istorum cuborum. Potestas proximè superior est quadrato-quadratica, & ejus exponens 4. Ergo dati numeri 21 quadrato-quadratum 194481, si dividatur per exponentem 4, extabit summa omnium cuborum 48620[5].

Demonstratio. Summa omnium ab unitate imparium æqualis est quadrato numeri terminorum; sic numerus terminorum omnium imparium ab unitate usque ad 21 est 11, cujus quadratum 121 æquale est summæ omnium horum imparium; 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21. At idem quadratum 121 duplicatum, nimirum 242, excedit summam omnium eorundem imparium unâ cum paribus inclusis ipso numero terminorum 11; deficit autem à summa omnium parium æque ac imparium eodem numero terminorum 11. Ergo quadratum duplicatum numeri terminorum imparium non potest excedere vel deficere à summa omnium, tam parium quàm imparium, plusquam ipso numero terminorum imparium, hoc est (si termini sint innumeri) eodem numero terminorum sive dimidio termini maximi, ducto in partem infinitissimam numeri dati. Quod productum si quis putet, adhuc rationem aliquam obtinere ad summam omnium terminorum; nondum utique divisus est numerus datus in partes innumeras, quod est contra hypothesin. Ergo quadratum dimidii numeri terminorum (tam parium quàm imparium) duplicatum; vel, quod idem est, dimidium quadrati numeri omnium terminorum (tam parium quàm imparium) æquale est summæ omnium terminorum.

Rursus; numerus pyramidalis ultimi ab unitate imparium, æqualis est summæ omnium quadratorum ab iisdem imparibus factorum. Sic numeri 21, tanquam ultimi imparium, pyramidalis 1771, æqualis est summæ omnium quadratorum factorum ab his imparibus; 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21. Unde haud fecûs, ac modò, conficietur eundem pyramidalem duplicatum (si termini sint innumeri); vel, quod idem est, trientem cubi facti à numero dato, æqualem esse summæ omnium quadratorum ab imparibus æquè ac paribus factorum.

Item; ultimi cujusvis trigonus in se ductus, æqualis est summæ omnium cuborum ab imparibus æquè ac paribus factorum. Sed trigonus iste, sive summa terminorum, suprâ æqualis erat $\frac{\text{Quadrato}}{2}$, ergo ejusdem. Trigoni quadratum æquale est $\frac{\text{Quadrato quadrato}}{4} = \text{summæ omnium cuborum}$. Atque ita deinceps.

PROPOSITIO XVII.

Quadrare Hyperbolam.

In diagrammate præcedenti, posita $AI = 1$; intelligatur asymptoto inde ab I versus E divisa in partes æquales innumeras, quæ sint, v. gr. $ip = pq = qr = a$. Erit, per XIV & XV hujus, $ps = 1 - a + aa - a^3 + a^4$ &c, & $qt = 1 - 2a + 4aa - 8a^3 + 16a^4$ &c, & $ru = 1 - 3a + 9aa - 27a^3 + 81a^4$ &c. Sed $ps + qt + ru = \text{areæ } BIRU =$

$$= \left\{ \begin{array}{l} 1 - a + aa - a^3 + a^4 \\ 1 - 2a + 4aa - 8a^3 + 16a^4 \\ 1 - 3a + 9aa - 27a^3 + 81a^4 \end{array} \right\} \&c =$$

$$= 3 - 6a + 14aa - 36a^3 + 98a^4,$$

hoc est, = numero terminorum contentorum in linea ir , minus summâ eorundem terminorum, plus summâ quadratorum ab iisdem, minus summâ cuborum, plus summâ quadrato-quadratorum, &c.

Hinc posito, ut antè, $IA = 1$; sed $ip = ol\ 1 =$ numero terminorum: invenio, per XV & XVI hujus, aream $BIPS =$ numero terminorum = $ol\ 1$, minus summa eorundem terminorum = $ol\ 005$, plus summa quadratorum ab iisdem = $ol\ 000333333$, minus summa cuborum = $ol\ 000025$, plus summa quadrato-quadratorum = $ol\ 000002$, minus summa quadrato-cuborum = $ol\ 000000166$, plus summa cubo-cuborum = $ol\ 000000014$, &c.

$$+ \left\{ \begin{array}{l} ol\ 1 \\ ol\ 000333333 \\ ol\ 000002 \\ ol\ 000000014 \end{array} \right\} - \left\{ \begin{array}{l} ol\ 005 \\ ol\ 000025 \\ ol\ 000000166 \end{array} \right.$$

$$+ \frac{ol\ 100335347}{ol\ 005025166} - \frac{ol\ 005025166}{ol\ 005025166}$$

$$+ ol\ 095310181 = \text{areæ } BIPS.$$

Sic posito $iq = ol\ 21 =$ numero terminorum: invenio, per XV & XVI hujus, aream $BIRT =$ numero terminorum = $ol\ 21$, minus summa eorundem terminorum = $ol\ 02205$, plus summa quadratorum ab iisdem = $ol\ 003087$, minus summa cuborum = $ol\ 000486202$, plus summa quadrato-quadratorum = $ol\ 000081682$, minus summa quadrato-cuborum = $ol\ 000014294$, plus summa cubo-cuborum = $ol\ 000002572$, minus summa quadrato-quadrato cuborum = $ol\ 000000472$, plus summa quadrato-cubo-cuborum = $ol\ 000000088$.

$$\begin{array}{rcl}
 + \left\{ \begin{array}{l} 0[21 \\ 0[003087 \\ 0[000081682 \\ 0[000002572 \\ 0[000000088 \\ 0[000000003 \end{array} \right. & | & - \left\{ \begin{array}{l} 0[02205 \\ 0[000486202 \\ 0[000014294 \\ 0[000000472 \\ 0[000000016 \end{array} \right. \\
 + 0[213171345 & | & - 0[022550984 \\
 - 0[022550984 & | & \\
 + 0[190620361 & | & = \text{areæ BIQT.}
 \end{array}$$

Unde apparet, ut ratio AI ad AD (1 ad 1[1] est dimidiata rationis AI ad AQ (1 ad 1[21]); ita aream BIPS esse dimidiam areæ BIQT. Cæterum proclive est hunc calculum extendere ad quotvis loca, quod mihi tentanti, prodiit area BIPS = 0[09531017980432486004395212328076509222060534; & area BIQT = 0[19062035960864972008790424656153018444121072, quam cum exactè duplam deprehenderem istius superioris scivi indeme calculum rectè posuisse.

PROPOSITIO XVIII.

Comparare Areolas hyperbolicas cum Ratiunculis absolutorum Æquidifferentium.

In diagrammate præcedenti, positâ AI = 1, & asymptoto inde ab I versus B divisâ in partes æquales innumeras, quæ sint, v. gr. ip, pq, qr: erit areola BIPS mensura ratiunculæ, quam AI obtinet ad ap; & areola spqt mensura ratiunculæ, quam ap obtinet ad aq; & areola tqru mensura ratiunculæ, quam aq obtinet ar, &c. Atque areolæ istæ supputantur prorsus eodem modo, quo suprâ propositione VIII & IX rationes terminorum æquidifferentium. Id quod paucis indicare opportunum duxi.

PROPOSITIO XIX.

Invenire summam Logarithmorum.

In eodem diagrammate, positâ AI = 1, & asymptoto inde ab I versus E divisâ in partes æquales innumeras, quæ sint, v. gr. ip, pq, qr, oportet invenire summam areolarum; BIPS + BIQT + BIRU (&c) — summa logarithmorum = solido, constanti ex areola BIPS perpendiculariter insistente lineæ ps, & areola BIQT perpendiculariter insistente lineæ qt, & areola BIRU perpendiculariter insistente lineæ ru &c, ductis nimirum singulis in unam infinitissimam lineæ datæ.

Constructio hujus problematis congruit cum constructione propositionis XVII, substituendo nimirum, pro numero terminorum, summam eorundem; & pro summa terminorum, summam quadratorum; & pro summa quadratorum, summam cuborum; &c. Sic positâ AI = 1, & ip = 0[1, oporteat nos invenire summam omnium logarithmorum inde ab 1 ad 0[1.

$$\begin{array}{rcl}
 + \left\{ \begin{array}{l} 0[005 \\ 0[000008333 \\ 0[000000033 \end{array} \right. & | & - \left\{ \begin{array}{l} 0[000166666 \\ 0[0000005 \\ 0[000000002 \end{array} \right. \\
 + 0[005008366 & | & - 0[000167168 \\
 - 0[000167168 & | & \\
 0[004841198 & | & = \text{summæ omnium logarithmorum.}
 \end{array}$$

Hinc patet, quomodo productum continuum omnium à 0 ad numerum datum arithmetice progredientium inveniri queat. Nam summa logarithmorum, est logarithmus producti continui.

Patet quoque ex præcedentibus, quo pacto problema Mersennianum, si non geometricè saltem in numeris, ad quotvis usque locos solvi possit. Atque hîc jam filum abrumpere cogor, tantisper dum otium pertexendi reliqua largiatur Deus.

Note to prop. XI, referred to in p. 187.

* In demonstratione hujus propositionis undecimæ auctor noster in plures calculi errores lapsus est. Etenim dicit cubum quantitatis binomiæ $a + \frac{3b}{2}$ esse $a^3 + \frac{9aab}{2} + \frac{27abb}{4} + \frac{35b^3}{8}$, cum reverà iste cubus sit $a^3 + \frac{9aab}{2} + \frac{27abb}{4} + \frac{27b^3}{8}$; & dicit cubum quantitatis binomiæ $a + \frac{7b}{2}$ esse $a^3 + \frac{21aab}{2} + \frac{63abb}{4} + \frac{343b^3}{8}$, cum reverà iste cubus sit $a^3 + \frac{21aab}{2} + \frac{147abb}{4} + \frac{343b^3}{8}$. Hi autem errores non sunt necessarii ad stabilienda auctoris ratiocinia, quæ aliquanto propiùs ad veritatem accedent, si hi errores corrigantur. His autem correctis, demonstratio tota erit ut sequitur.

Sint tres rationes continuæ $\frac{a}{a+b}$, $\frac{a+b}{a+2b}$, $\frac{a+2b}{a+3b}$, quarum differentia differentiarum est $\frac{a^4 + 6a^3b + 12aabb + 8ab^3}{a^4 + 6a^3b + 12aabb + 10ab^3 + 3b^4}$, & mediæ illarum medium arithmeticum est $a + \frac{3b}{2}$, cujus cubus $a^3 + \frac{9aab}{2} + \frac{27abb}{4} + \frac{27b^3}{8}$; sint verò & aliæ tres continuæ $\frac{a+2b}{a+3b}$, $\frac{a+3b}{a+4b}$, $\frac{a+4b}{a+5b}$, quarum differentia differentiarum est $\frac{a^4 + 14a^3b + 72aabb + 160ab^3 + 128b^4}{a^4 + 14a^3b + 72aabb + 162ab^3 + 135b^4}$, & mediæ illarum medium arithmeticum $a + \frac{7b}{2}$, cujus cubus $a^3 + \frac{21aab}{2} + \frac{147abb}{4} + \frac{343b^3}{8}$. Cæterùm singulæ rationes differunt communi excessu b ; at differentiæ differentiarum non sunt terminorum æquidifferentium, siquidem differentia terminorum prioris est $2ab^3 + 3b^4$, at posterioris $2ab^3 + 7b^4$: secus ac in præcedenti propositione. Quare cum illic res expediretur regulâ proportionum simplici inversâ, hic opus est duplici inversâ; nimirum:

Ut medium arithmeticum prioris differentiæ differentiarum (nimirum $a^4 + 6a^3b + 12aabb + 9ab^3 + \frac{3b^4}{2}$) ductum in differentiam terminorum posterioris ($2ab^3 + 7b^4$) ad medium arithmeticum posterioris differentiæ differentiarum (nimirum $a^4 + 14a^3b + 72aabb + 161ab^3 + \frac{263b^4}{2}$) ductum in differentiam terminorum prioris ($2ab^3 + 3b^4$): Ita differentia differentiarum prior ad posteriorem: Ita quoque cubus medii arithmetici mediæ trium priorum rationum (nimirum $a^3 + \frac{9aab}{2} + \frac{27aab}{4} + \frac{27b^3}{8}$) ad cubum medii arithmetici mediæ trium posteriorum rationum (nimirum $a^3 + \frac{21aab}{2} + \frac{147abb}{4} + \frac{343b^3}{8}$).

Quæ analogia vera esse deprehendetur, si productum extremorum æquale sit producto mediorum.

Atqui primus terminus $a^4 + 6a^3b + 12aabb + 9ab^3 + \frac{3b^4}{2}$ in $2ab^3 + 7b^4 = 2a^5b^3 + 19a^4b^4 + 66a^3b^5 + 102a^2b^6 + 66ab^7 + \frac{21b^8}{2}$ si ducatur in quartum $a^3 + \frac{21aab}{2} + \frac{147abb}{4} + \frac{343b^3}{8}$; productum est $2a^8b^3 + 40a^7b^4 + 339a^6b^5 + 1579a^5b^6 + 4377\frac{1}{2}a^4b^7$, &c.

Rursus secundus terminus $a^4 + 14a^3b + 72aabb + 161ab^3 + \frac{263b^4}{2}$ in $2ab^3 + 3b^4 = 2a^5b^3 + 31a^4b^4 + 186a^3b^5 + 538a^2b^6 + 746ab^7 + \frac{789b^8}{2}$, si ducatur in tertium $a^3 + \frac{9aab}{2} + \frac{27abb}{4} + \frac{27b^3}{8}$; productum est $2a^8b^3 + 40a^7b^4 + 339a^6b^5 + 1591a^5b^6 + 4527a^4b^7$, &c.

Hoc igitur productum cum consentiat cum isto, non modò in primis & secundis & tertiis speciebus, sed & in maximâ parte quarumarum & etiâ quarumarum; aio analogiam in propositione memoratam esse veram. Nam defectus, qui hîc apparet in productis terminorum, in ipsis terminis longè minor erat; quippe qui multiplicando crevit. Ut taceam in minoribus rationibus differentias secundas & tertias nullius ferè momenti esse.

MICHAELIS ANGELI RICCII

EXERCITATIO GEOMETRICA

DE MAXIMIS ET MINIMIS.

MICHAELIS ANGELI RICCI

EXERCITATIO GEOMETRICA

DE MAXIMIS ET MINIMIS

A B B A T I
S T E P H A N O G R A D I O

MICHAEL ANGELUS RICCIUS, S. P. D.

SCRPTIONEM hanc meam argumenti, ut vides, inter Mathematica difficillimi, sed æquè ad difficiliora quæque Problematum efficienda, & obscuriora Theorematum cognoscenda utilissimi, cùm in Geometrico, tùm in Analytico pulvere, ad te mittendam duxi, vir ornatissime, Stephane Gradi, quem ego unum omnium hujus Civitatis plurimi facio, ob egregias animi laudes, præsertim verò propter acre his de rebus existimandi judicium, quotidianis gravissimarum inter nos disputationum experimentis mihi perspectum & cognitum. Lege quæso illam diligenter, & ubi diù exactissimæ tuæ censuræ subjectam habueris, ecquid respondeat solitæ tuæ de meis hoc in genere cogitationibus, opinioni, pronuncia. Nam si hoc assequar, ut tibi cæterisque Amicis earundem Disciplinarum intelligentibus probetur, minus erit in posterum quam ob rem humanissimis tuis hortationibus obluenter, cùm autor mihi esse perseverabis edendi alia quæ tecum jampridem communicavi, de præceptis universæ Artis analyticæ, geometricæ methodo breviter & expeditè demonstratis, unà cum animadversione erratorum quæ in ipsis tradendis magni nominis Auctores errasse deprehendi; faciliusque obtinebis ne diutiùs premam apud me quæcumque de Geometria in genere disputata & literis consignata incertas propositiones redegi; & ex his illam præcipuè à Torricellio, & à

te quoque tantopere commendatam, quæ integram doctrinam triginta propositionum Archimedis, Lucæ Valerii, & aliorum, una complectitur; duasque præterea, quibus totam penè Jo. Caroli de la Faille de centro gravitatis partium circuli & ellipseos doctrinam [justo volumine ab ipso explicatam] absolvo. Statui autem pauca aliquot hujus scripti exemplaria typis imprimere, quò commodiùs possint ad peritos hujusmodi scientiarum Amicos, tum per Italiam, tum exteras apud gentes pervenire, accenso potius ea in re tuo studio obsecutus, quàm ingenio meo. Neque enim is ego sum, cui nomen famæ per ambitionem ingerere libeat; aut quem non magis indagatæ veritatis cognitio, quàm cognitæ ostentatio delectet. Interim hunc amicitiae nostræ jampridem institutæ, & literario præcipuè commercio nunquam coli intermissæ, fructum jucundissimum feram, ut quæ hac in re de me sentis amicè, hoc est [ut Euripidi placet] liberè, te loquentem audiam; eoque, quid cæteri & sentiant & loquantur, securus fiam. Vale. Romæ, octavo Idus Julii 1666.

MICHAELIS ANGELI RICCI

GEOMETRICA EXERCITATIO.

DEFINITIONES.

1 **P**OTESTATEM quamlibet, ejusque radicem, voco dignitatem.

2 Si dignitas in dignitatem ducatur, ut A^2 in B^3 , fiet productum A^2 in B^3 ; cui producto illud simile dicimus, quod gignitur ex dignitatibus graduum eorumdem. Ita, in facta hypothefi, productum E^2 in C^3 , ex quadrato & cubo, simile est producto A^2 in B^3 .

3 Homogenea producta sunt quæ ad eundem gradum pertinent; ut duo rectangula, quippe quæ ad secundum gradum pertinent; & duo solida, quæ ad tertium.

4 Terminos cùm dico, intelligi volo duos numeros seu æquales seu inæquales, vel numerum & unitatem, vel duas unitates. Terminos inæquales appello duos numeros inæquales, vel numerum & unitatem. Terminos autem æquales, duos æquales numeros, vel duas unitates.

5 Productum in linea fieri secundum terminos datos, aut positos, dicimus, quum illud fit ex duabus dignitatibus, quarum exponentes sunt ipsi termini dati vel positi; radices verò segmenta illius rectæ lineæ sectæ in proportionem terminorum eorumdem.

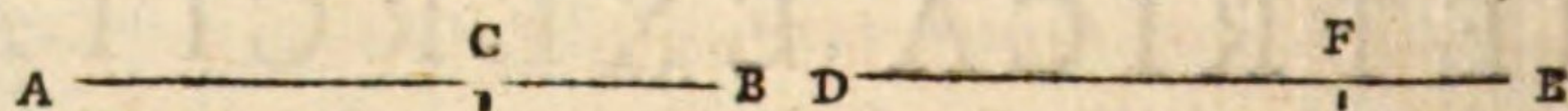
Sit, verbi causa, quæpiam recta linea, cujus majus segmentum ad minus sit in ratione 3 ad 2; productum ex cubo segmenti majoris in quadratum minoris erit factum in linea data secundum terminos positos 3, & 2; quia segmenta quæ sunt dignitatem radices habent rationem numeri 3 ad 2, & exponentes earumdem dignitatum sunt etiam 3, & 2.

Rursus esto, quemadmodum segmentum majus ad minus ejusdem lineæ, sic 3 ad 1, productum ex cubo majoris segmenti in segmentum ipsum minus, erit productum in linea factum secundum terminos positos, numerum & unitatem.

tem. Ita, A^3 in B^3 [si A vocetur majus segmentum, B verò minus] est productum factum in linea $A + B$ secundum terminos 3, & unitatem, quia radices A & B sic sunt, ut est numerus 3 ad unitatem; & dignitatis A^3 exponens est, 3, numerus datus; dignitatis B^3 exponens est, unitas, item data.

LEMMA PRIMUM.

Si duæ rectæ in eadem ratione secantur, producta similia facta ex segmentis tanquam ex radicibus, erunt proportionalia productis homogeneis quæ fient ex totis.



Sint AB , DE , rectæ, in punctis c , & F ita sectæ, ut quam rationem AC ad CB habet, eandem habeat DF ad FE , & fiant ex illarum segmentis producta AC^2 in CB^3 , & DF^2 in FE^3 , quæ sunt similia per secundam definitionem; iisque homogenea producta fiant ex totis AB , DE , nimirum AB^5 , DE^5 per tertiam definitionem. Dico AC^2 in CB^3 eandem rationem habere ad AB^5 , ac DF^2 in FE^3 ad DE^5 . Quia rationes ex quibus ratio producti AC^2 in CB^3 ad AB^5 componitur, eadem sunt, ac componentes rationem producti DF^2 in FE^3 ad DE^5 , ob sectionem linearum proportionalem, & inde proportionales dignitates ex quibus producta illa resultant. Quod, &c.

LEMMA SECUNDUM.

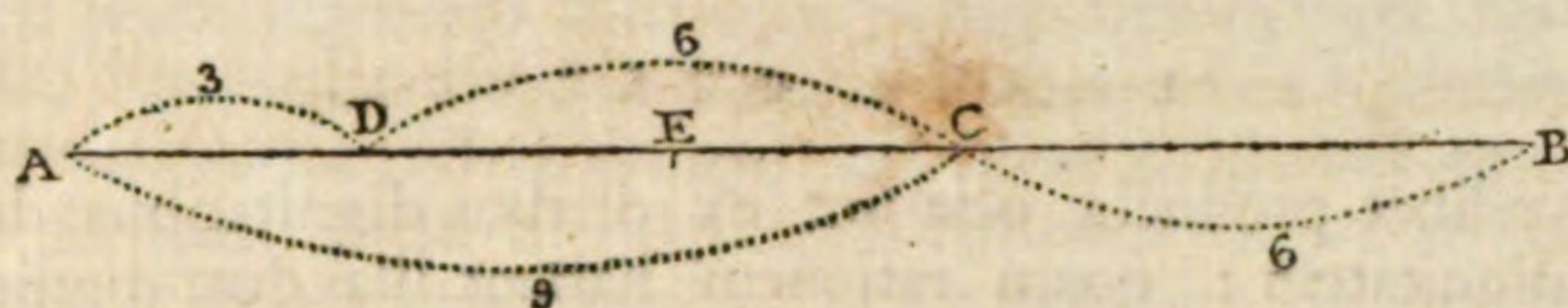
Iisdem positis, dico, AC^2 in CB^3 fuerit maximum omnium similium productorum ex binis segmentis rectæ AB , etiam DF^2 in FE^3 fore maximum productorum similium ex binis segmentis rectæ DE , tanquam ex radicibus.

Singulis enim productis ex segmentis rectæ DE alia respondent orta ex segmentis rectæ AB in eadem proportionem sectæ; & illa ad homogeneum suum DE^5 eandem rationem habent, atque ista ad suum AB^5 , ex primo lemmate. Ratio quidem AC^2 in CB^3 ad AB^5 ex hypothese est eadem ac ratio DF^2 in FE^3 ad DE^5 ; cæterorum verò productorum ex segmentis ipsius DE ad DE^5 , eadem est atque ratio productorum sibi respondentium, quæ fiunt ex segmentis rectæ AB , ad AB^5 . Cum igitur ratio AC^2 in CB^3 [quod maximum esse ponitur] ad AB^5 sit major, per octavam quinti Elem. ratione cæterorum productorum sibi similium ad AB^5 ; major etiam erit ratio DF^2 in FE^3 ad DE^5 , quam ratio cæterorum similium productorum ex segmentis rectæ DE ad DE^5 ; ac proinde ipsum DF^2 in FE^3 per decimam quinti Elem. est maximum. Quod, &c.

LEMMA TERTIUM.

Si data recta linea secetur in ratione terminorum inæqualium, & dividendo, fiat

fiat segmentorum differentia ad minus segmentum, ut differentia terminorum ad minorem terminum; hæc inventa proportionalitas vel ipsa erit proportionalitas æqualitatis, vel alia, in quam incidemus, iterum dividendo, & sic deinceps; & in eâ terminorum differentia æquabitur minori termino, & differentia segmentorum segmento minori.



Esto AC ad CB, ut 9 ad 6, & AD differentia segmentorum AC, CB: erit dividendo, 3 ad 6, ut AD ad CB, vel ad segmentum sibi æquale, DC: Quoniam verò hæc proportio non est proportio æqualitatis, fiat DE differentia segmentorum AD, & DC; 3, differentia numerorum 6 & 3; & dividendo, erit ut 3 ad 3, sic DE ad AD, proportio æqualitatis.



Rursus AC sit ad CB, ut 5 ad 3; & AD segmentorum differentia; dividendo erit, AD ad CB, seu ad sibi æquale segmentum DC, ut 2 ad 3. Et iterum dividendo [segmentorum AD & DC, esto, differentia, EC], 1 ad 2 ut EC ad AD seu DE; & tertio [facta FE terminorum DE & EC differentia] dividendo inveniemus, ut 1 ad 1, ita FE ad EC. Quod, &c.

Ratio lemmatis est, quod duorum quorumcumque numerorum differentia, vel differentia numeri & unitatis, semper est numerus aut unitas, ut per se patet: & nos dividendo, semel atque iterum, ac sæpius, demimus semper minorem terminum divisæ proportionalitatis qui est numerus vel unitas, de majori termino seu numero, utimurque deinceps residuo tantum [quod est eorum terminorum differentia] & comparamus illud cum minori termino proportionalitatis divisæ: at non possumus sic demendo progredi in infinitum, quia unitates in terminis sunt finitæ, sed exhauritur tandem omnis differentia, residuumque majoris termini proportionalitatis divisæ æquatur termino minori. Ita fit proportio æqualitatis, in qua unitas ad unitatem, vel numerus ad sibi æqualem numerum, est ut segmentum ad aliud æquale segmentum. Quod ostendere oportebat.

Quod si ab ea proportionem æqualitatis, in qua desitum est, rursus incipiamus, dico nos componendo gradatim, venturos per vestigia divisionis ad terminos primæ proportionalitatis, in qua segmenta datæ lineæ erant in ratione inæqualium terminorum. Cujus propositionis rationem facile intelliget Geometra, quem latere non potest, in Geometria omnia quæ dividendo concluduntur, ex

contrario converti posse, & componendo concludi illud ipsum, quod ponebatur ante divisionem, ut in quinto Elementorum ostenditur. Exempli gratia, sit majus segmentum datæ rectæ ad minus, ut 2 ad 1. Igitur dividendo 1 ad 1, est ut differentia segmentorum ad minus segmentum. Ex hac porrò æqualitatis proportionem componendo redimus ad primam proportionem, in qua segmenta erant in ratione 2 ad 1. Quod, &c.

LEMMA QUARTUM.

Si duo quælibet producta orta sint ex duabus dignitatibus ductis in aliam communem dignitatem; quam rationem habent illæ duæ dignitates inter se, eandem habent duo producta. Sic productum AB^3 in BC^3 eam rationem habet ad productum AB^3 in EF^3 , quam habet dignitas BC^3 ad dignitatem EF^3 ; in quas duas dignitates ducta communis dignitas AB^3 illa producta effecit.

Ex definitione multiplicationis probatur hoc lemma, quod alii in numeris demonstrarunt.

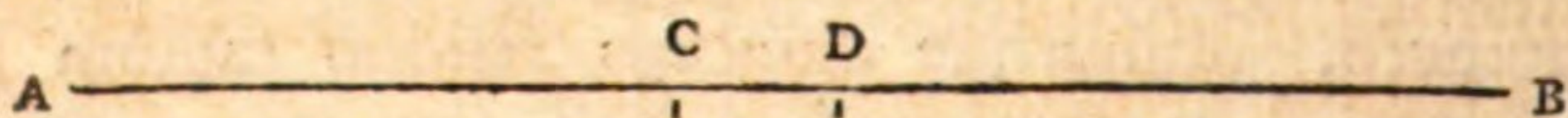
LEMMA QUINTUM.

Datis quatuor quantitibus, quarum prima ad secundam habeat minorem rationem, quam tertia ad quartam, productum quod gignitur ex duabus extremis est minus producto ex mediis.

Augeatur prima donec fiant quatuor geometricè proportionales; tunc prima in quartam ducta efficiet productum æquale producto ex mediis. Igitur productum quod efficiebat antequàm augetetur, erat productum minus eodem producto ex quantitibus mediis. Quod, &c.

THEOREMA PRIMUM.

Productum in aliqua recta linea factum secundum positos terminos æquales, maximum est omnium similium productorum, quæ fieri possunt ex binis lineæ datæ segmentis tanquam ex radicibus.

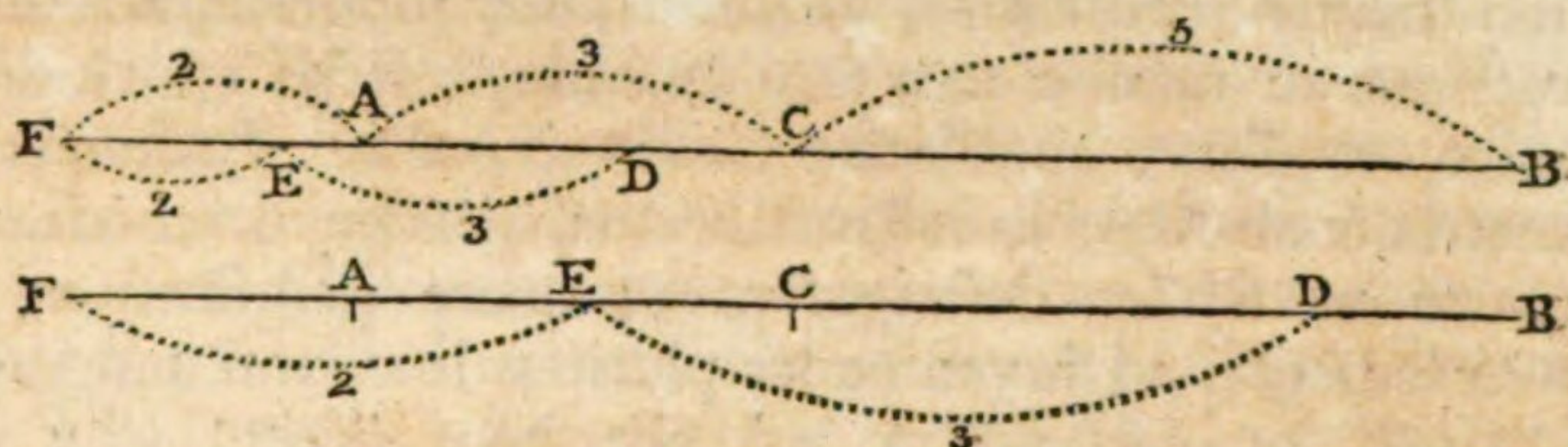


Recta linea AB secetur æqualiter in puncto c , & sit Ac ad cB ut 3 ad 3 [termini æquales positi] dico productum Ac^3 in cB^3 , quod sit in linea AB secundum positos terminos, esse omnium similium productorum maximum. Sumpto quolibet alio puncto D , faciamus aliud simile productum AD^3 in DB^3 . Cùm autem sint quatuor lineæ arithmeticè proportionales cum excessu CD , nimirum AD , Ac , cB , & BD , minor est ratio maximæ AD ad Ac , quàm cB ad BD ; & triplicata ratio ipsius AD ad Ac [seu ratio AD^3 ad Ac^3] minor est, quàm triplicata ipsius cB ad BD [seu cB^3 ad BD^3] & per quintum lemma, productum ex mediis quantitibus, Ac^3 , in cB^3 , majus est producto AD^3 in BD^3 facto

facto ex duabus extremis. Eodem pacto demonstratur AC^3 in CB^3 esse alio quocumque simili producto majus, & consequenter omnium similium maximum. Quod, &c.

THEOREMA SECUNDUM.

Si duo rectæ lineæ segmenta fuerint in ratione terminorum inæqualium, & per consequens, dividendo, sit differentia segmentorum ad minus segmentum, ut differentia terminorum ad minorem terminum; quoties ex dignitate differentiæ segmentorum ducta in dignitatem minoris segmenti fit productum maximum, toties fit etiam maximum ex eadem dignitate minoris segmenti ducta in dignitatem majoris; atque ita, si dignitates segmentorum pro exponentibus habeant terminos positos, & dignitas differentiæ, differentiam terminorum.



Sit AB recta linea inæqualiter secta in puncto c , & BC ad AC , ut 5 ad 3 , qui sint termini positi. Producatur BA in F , donec æquetur FC ipsi CB , & AF erit differentia segmentorum BC & AC . Quoniam vero segmentum majus BC sic est ad minus CA , ut 5 est ad 3 , erit dividendo AF ad CA , ut est 2 ad 3 . Nunc fiant duo producta qualia diximus, primum FA^2 in AC^3 , ex dignitate ipsius FA , differentiæ segmentorum, ducta in dignitatem minoris segmenti. AC . Secundum AC^3 in CB^3 , ortum ex eadem dignitate minoris segmenti ducta in dignitatem majoris. Prima dignitatis FA^2 habet pro exponente, 2 , differentiam datorum terminorum, reliquæ habent 3 & 5 , terminos positos, ut imperabatur. Dico, si productum primum est maximum omnium similium ex binis segmentis rectæ FC [esse autem ejusmodi supponamus], etiam secundum fore productum maximum omnium similium ex binis segmentis rectæ positæ AB .

Sumatur in AB alius punctus præter punctum c , & esto D ; qui accipi à nobis potest infra punctum c , vel supra. In utroque casu, FA nequit habere eam rationem ad AC , quam habet ad AD , sed majorem aut minorem habebit, atque adeò FD non est secta in puncto A secundum rationem ipsius FA ad CA : fiat porro FE ad ED , ut FA ad AC , & productum FE^2 in ED^3 , per secundum lemma, erit maximum [æque ac productum FA^2 in AC^3] & consequenter majus simili producto FA^2 in AD^3 , facto ex segmentis ejusdem rectæ FD . Quod maximum FE^2 in ED^3 habet eandem rationem ad FD^5 , dignitatem sibi homogeneam, quam FA^2 in AC^3 ad FC^5 , ut ex duobus primis lemmatibus colligitur; igitur FA^2 in AD^3 [quod diximus esse minus producto FE^2 in ED^3] minorem rationem habet ad FD^5 , quàm FE^2 in ED^3 ad idem FD^5 , seu minorem, quàm FA^2 in AC^3 ad FC^5 ; & permutando, FA^2 in AD^3 minorem habet rationem ad FA^2 in AC^3 [seu,

[seu, per lemma quartum, AD^3 minorem habet ratio ad AC^3] quàm FD^3 ad FC^3 , & longè minorem, quàm CB^3 ad DB^3 . Quippe sunt rectæ DB , CB , FC , & FD , arithmeticè proportionales cum excessu, DC ; ac propterea in primo casu, FD maxima, in secundo casu, FD minima, est ad FC in minori ratione quàm CB ad DB , & quintuplicata ratio FD ad FC , nempe ratio ipsius FD^3 ad FC^3 , est minor quintuplicatâ ratione CB ad DB , seu CB^3 ad DB^3 .

Igitur cùm quatuor quantitatum, AD^3 , AC^3 , CB^3 , & DB^3 , prima ad secundam habeat minorem rationem, quàm tertia ad quartam, per quintum lemma, productum AD^3 in DB^3 factum ex duabus extremis erit minus producto AC^3 in CB^3 ex mediis. Similiter ostendes, aliud quodcumque productum simile minus esse producto AC^3 in DB^3 , quia punctus D ad libitum sumitur. Ergo AC^3 in CB^3 productum est maximum omnium. Quod, &c.

Haecenus de recta linea AB inæqualiter secta, quum est segmentum majus ad minus, uti numerus ad numerum. Restaret altera pars theorematis, quum est quemadmodum majus segmentum ad minus, sic numerus ad unitatem. Hoc tamen constructione ac ratione tam similibus modò factis concluditur, ut id sibi quisque invenire, explicare ac dilatare facillimè possit. Lectoribus autem scribimus à Geometria & ab Algebra instructoribus, quos hujuscemodi rerum intellectu facilius explicatione frustra defatigaremus; quare pergimus ad reliqua usum præstantissimum habentia ad inveniendas plurimum linearum tangentes, figurarum centra gravitatis & quadraturas, & ad alia item multa, quæ justo servamus operi; ubi dabimus novam solidorum conicorum seriem, qui secti exhibent infinitas, uti vocant, hyperbolas, infinitas parabolas, infinitas ellipses, &, analogiam servando, circulos etiam infinitos. Unde lectoribus manifestè apparebit, de conicis me plus multò adinvenisse, quàm cæteros, eosque ingeniosissimos viros, qui communem tantum hyperbolen, parabolam, ellipsim, & circulum [figuras conici in nostra nova serie prædicta, secundi gradus] agnoverunt: alias tertii & quarti & cæterorum non item; nisi quod de parabolis infinitis per puncta in plano descriptis pauca, licet cognitione dignissima, tradidere nonnulli, quos inter, duo præcellentes ingenio viri, Fermatius, ac Torricellius, præceptor meus, inventorum præstantiâ & numero commendabiles, ac veteribus proximi; qui novum insuper excogitarunt hyperbolarum infinitarum genus. Neque prætereundum puto, quamplures Apollonii propositiones atque demonstrationes aptari sectionibus nostris & per omnia congruere, affectasque multipliciter æquationes harum sectionum ope resolveri facillimè, & determinari posse. Nunc revertor ad rem.

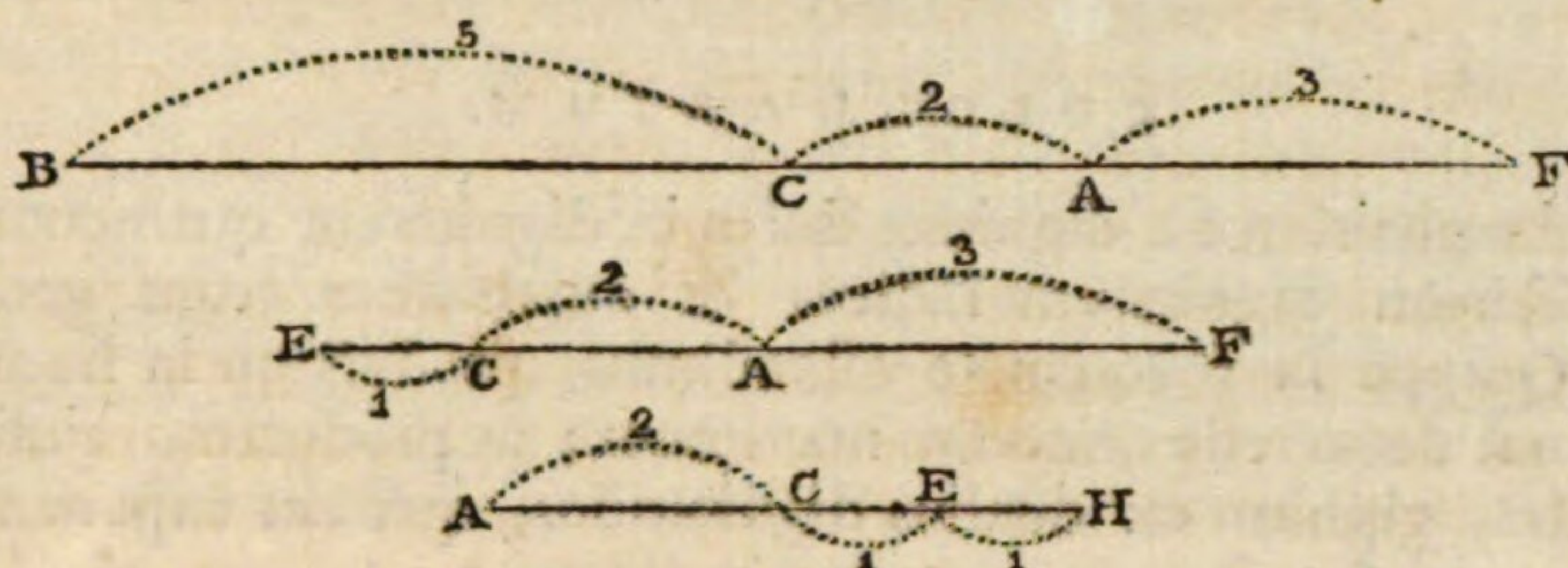
THEOREMA TERTIUM.

Datâ rectâ lineâ, & duobus terminis, secundum quos fiat in linea data productum: hoc erit maximum omnium similium productorum, quæ fieri possunt ex binis ejusdem rectæ segmentis, velut ex radicibus.

Propositionem feci in partes duas. Primum dico, productum, quale descripsimus, esse omnium similium maximum, quum dantur termini æquales; quod in primo theoremate demonstravimus.

Deinde

Deinde si dantur termini inæquales, sic rem ostendo.



Esto AB recta data, & termini dati 5 & 2. Secetur recta in puncto c fitque BC ad CA, ut 5 ad 2. Dico productum BC^5 in CA^2 factum in linea data secundum terminos datos esse maximum. Producatur BA in F, ut AF sit differentia segmentorum, & dividendo primam proportionalitatem, nempe BC ad CA, ut 5 ad 2 [sicut in tertio lemmate præscribitur] pergamus usque dum incidamus in proportionem æqualitatis. In nostra hypothesi, primum erit, dividendo, 3 ad 2, ut FA differentia segmentorum ad AC minus segmentum; quam secundam proportionalitatem exhibet secunda figura, in qua fiat CE differentia segmentorum CA, FA; per consequens erit, dividendo, 1 ad 2, ut CE ad AC; quam quidem proportionalitatem seorsim exhibet tertia figura. Fiat EH differentia segmentorum CE & AC, dividendo erit 1 ad 1, ut EH ad EC; quæ est demum proportio æqualitatis; semper autem minus segmentum producimus ut æquemus majori, & segmentorum differentiam constituamus.

At retrorsum vicissim, incipiendo à recta EA tertiæ figuræ cujus majus segmentum AC est 2, minus segmentum CE est 1, & illorum differentia HE itidem 1. Quoniam productum HE^1 in CE^1 est maximum in linea CH, per primum theorema nostrum, erit proinde, per secundum theorema, EC^1 in CA^2 maximum in recta EA.

Deinde in recta FC secundæ figuræ, majus segmentum AF est 3, minus AC est 2, & segmentorum differentia EC est 1; porro cum EC^1 in CA^2 sit maximum, erit per secundum theorema, etiam maximum in recta FC productum AF^3 in AC^2 .

Postremò in lineâ AB primæ figuræ, productum AF^3 in AC^2 est maximum, ut modò ostendimus, ergo per secundum theorema est etiam maximum AC^2 in BC^5 . Quod demonstrandum erat.

Si loco duorum numerorum detur numerus, & unitas, fit similis constructio, & demonstratio.

SCHOLIUM.

Id quod in secundo theoremate supponebamus; datâ rectâ lineâ, & datis numeris, 3 & 2, maximum fore productum in ea linea factum secundum numeros illos datos; nunc demonstravimus in theoremate hoc. Erat porro illius theorematis

rematis propositio conditionalis, ex posita illa hypothesi, non absoluta, ut patebit consideranti.

COROLLARIUM.

Si productum genitum ex dignitate ducta in dignitatem quamcumque, maximum fuerit, illarum dignitatum radices & exponentes erunt geometricè proportionales. Quippe in theoremate ostendimus, productum in linea factum secundum terminos datos esse omnium maximum; at productum ejusmodi, ex v definitione nostra, gignitur ex duabus dignitatibus, quarum exponentes rationem eam habent, quam dignitatum earundem radices.

PROBLEMA PRIMUM.

Datam lineam rectam ita secare, ut productum ex dignitatibus segmentorum sit omnium similium maximum.

Sumantur exponentes duarum illarum dignitatum, rectaque dividatur in ratione horum exponentium, & factum erit quod imperatur; quia productum erit in linea data factum secundum terminos positos, nimirum secundum exponentes; ac proinde erit maximum per theorema tertium.

PROBLEMA SECUNDUM.

Æquationem determinare, in qua potestas quæsitæ radice negatur de homogeneo sub radice data, & dignitate sua parodica, ut B in $A - A^2 || Z^2$; vel B in $A^3 - A^4 || Z^4$, &c.

Oritur hujusmodi æquatio ex dicta parodica dignitate potestatis negatæ ducta in $B - A$, differentiam datæ & quæsitæ radice, rem probo. Illa parodica dignitas affirmata, si primùm ducatur in A , radicem quæsitam negatam, gignet potestatem negatam uno gradu altiore, quàm sit ea parodica dignitas [ut patet ex natura multiplicationis] deinde in $+ B$ radicem datam affirmatam ducta, gignet homogeneum affirmatum, sub eadem dignitate parodica & radice data: quæ duo producta, sunt ipsa pars æquationis, de qua in problemate. Pars altera est homogeneum comparisonis.

Rursus, per lemma quartum, ratio homogenei ad potestatem negatam est eadem, ac radice datæ ad quæsitam; sed minor est potestas homogeneo, de quo ipsa negatur & demitur. Ergo etiam radix quæsitæ minor est datâ; in qua proinde radice datâ nos rectè sumimus segmentum æquale radice quæsitæ A , ut alterum segmentum sit $B - A$, differentia datæ ac quæsitæ radice.

Quoniam igitur prima pars æquationis oritur ex $B - A$ uno radice datæ segmento, ducto in altero segmentum A , vel in hujus potestatem, efficitur [per tertium theorema] ut inde resultans productum sit maximum omnium similium, quotiescunque A , & $B - A$, segmenta rationem habent eam, quam exponentes suarum dignitatum. Sic in æquatione B in $A^3 - A^4 || Z^4$; si A , & $B - A$ fuerint ut 3 ad 1, cubus segmenti A in BA ductus gignet partem æquationis B in $A^3 - A^4$; quæ est productum in linea data B , omnium similium maximum; cujus

cujus proinde magnitudinem non potest unquam excedere homogeneous comparisonis, quod semper æquari necesse est illi alteri æquationis parti. Unde canon pro determinanda problematis æquatione conficitur.

Fiat in radice maximum productum secundum terminos, qui sunt exponentes ejusdem radice & parodicæ dignitatis, sub quibus est homogeneous. Illius producti magnitudinem excedere non potest homogeneous comparisonis.

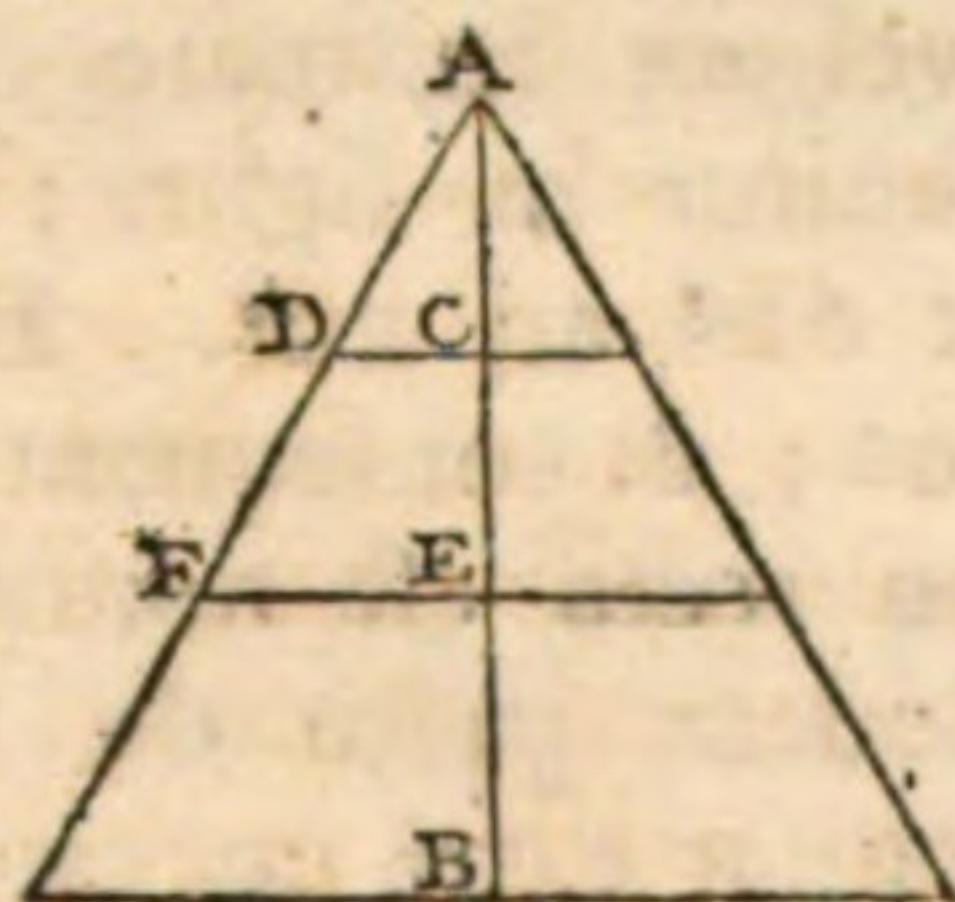
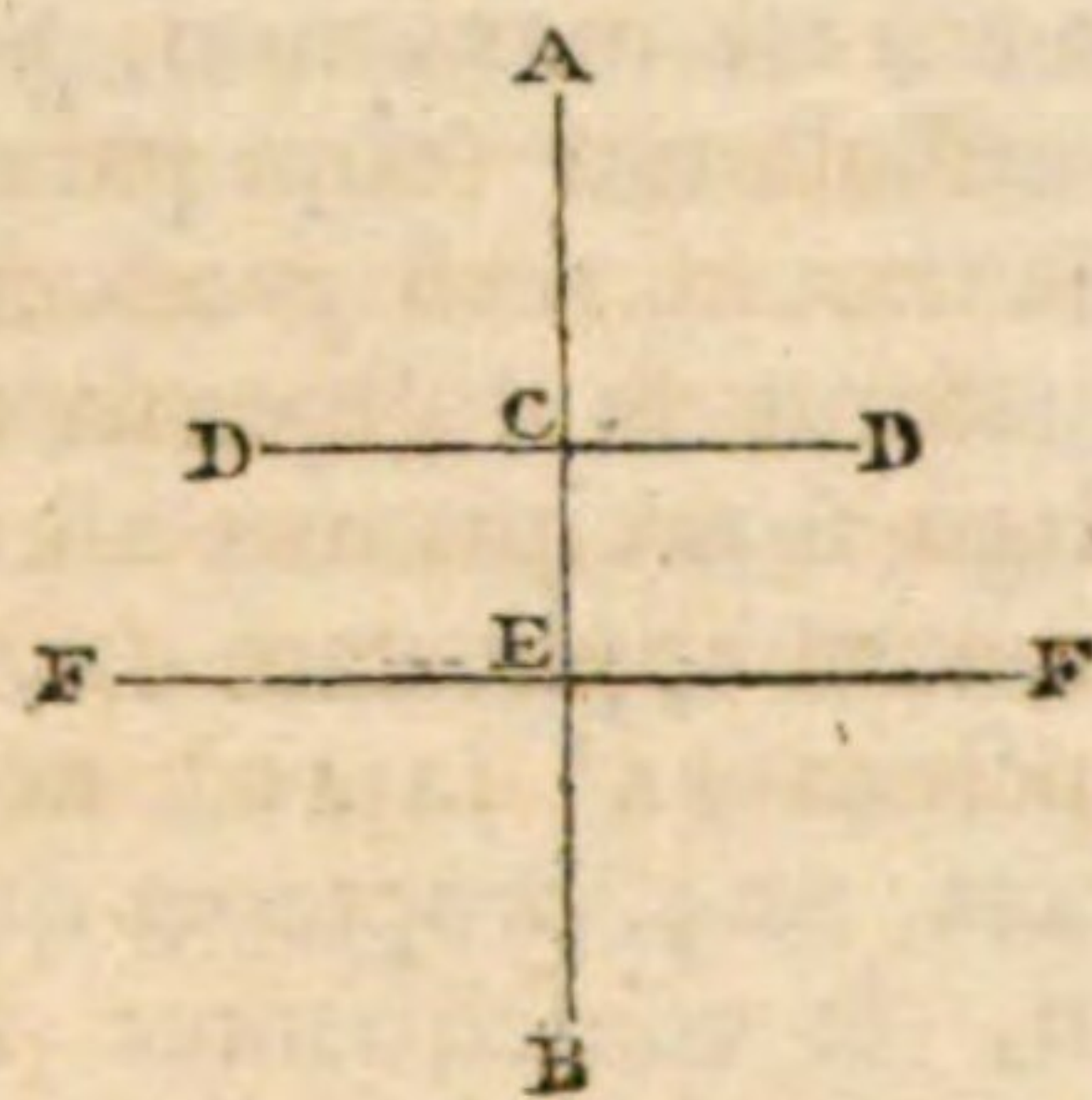
Idem procedit in alia æquatione orta ex multiplicatione ipsius, A , in $B - A$, vel in hujus potestatem; semper enim est idem casus tertii theorematis nostri, in quo productum factum in linea [seu datâ radice B] secundum terminos datos est maximum, termini verò sunt exponentes dignitatem segmenti A & alterius $B - A$.

Sed uno vel altero exemplo de geometricis nostris opusculis deprompto methodi facilitatem comprobemus.

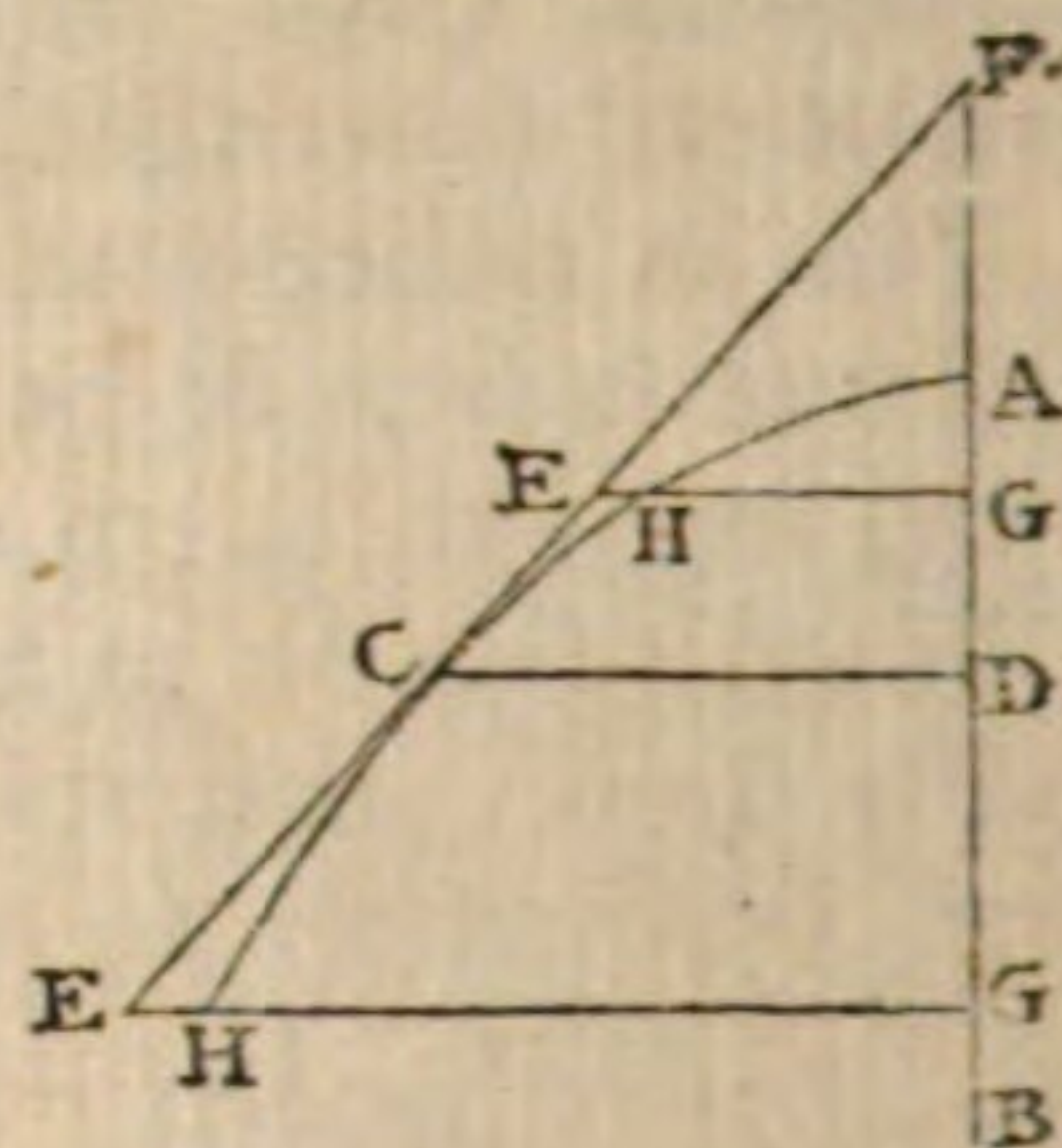
E singulis punctis datæ rectæ AB ducantur rectæ CD , EF , &c. rectæ inter se parallelæ, cum data AB angulum quemcumque efficientes. Sint autem harum parallelarum dignitates, & dignitates abscissarum AC , AE , &c. geometricè proportionales (id quod tripliciter contingere posse mox patebit) transibit per extrema parallelarum puncta, D , F , &c. perimeter figuræ, cujus diameter aut axis erit AB , vertex A , ordinatim verò ad diametrum applicatæ erunt ipsæ parallelæ.

Nam parallelarum abscissarumque dignitates si fuerint ejusdem gradus, exempli gratia, FE^2 ad DC^2 , ut AE^2 ad CA^2 : vel cubi parallelarum ut cubi abscissarum, figura erit triangulum, cujus proprietas notissima est, non parallelas modò & abscissas esse geometricè proportionales, sed parallelarum & abscissarum earundem potestates omnes homogeneas; quarum ratio æquè multiplex est rationis linearum seu radicum ita ut cubi, & quadrato quadrata, &c. abscissarum sint ut cubi, quadrato quadrata, &c. parallelarum; & illorum quoque radices geometricè proportionales.

Sin autem diversorum graduum fuerint dignitates parallelarum & abscissarum, linea descripta erit curva, habens suum axem, & ad illum ordinatim applicatas, quarum dignitas est gradu superior dignitate abscissarum: at contrâ dignitas applicatarum ordinatim ad rectam [quæ curvam in vertice contingit] sumptam pro axe, gradu inferior est dignitate abscissarum tangentis. De quo alibi latius dicam.



Est igitur $AGDB$ una ex præfatis figuris, ejusque axis AB , & vertex A ; in qua quidem gradus dignitatis parallelarum sit altior gradu dignitatis abscissarum; quærat autem linea recta contingens figuram in puncto dato C . Ducatur ex hoc puncto linea ad axem ordinatim applicata, ut CD , & ponantur exponentes dignitatum, 3 & 2. Erunt consequenter in figura parallelarum cubi ut quadrata abscissarum. Fiat abscissa AD , inter verticem & ordinatim applicatam, ad AF , axem productum, ut minor numerus 2 ad 1, differentiam exponentium, ductaque FC ; dico hanc esse tangentem quæsitam.



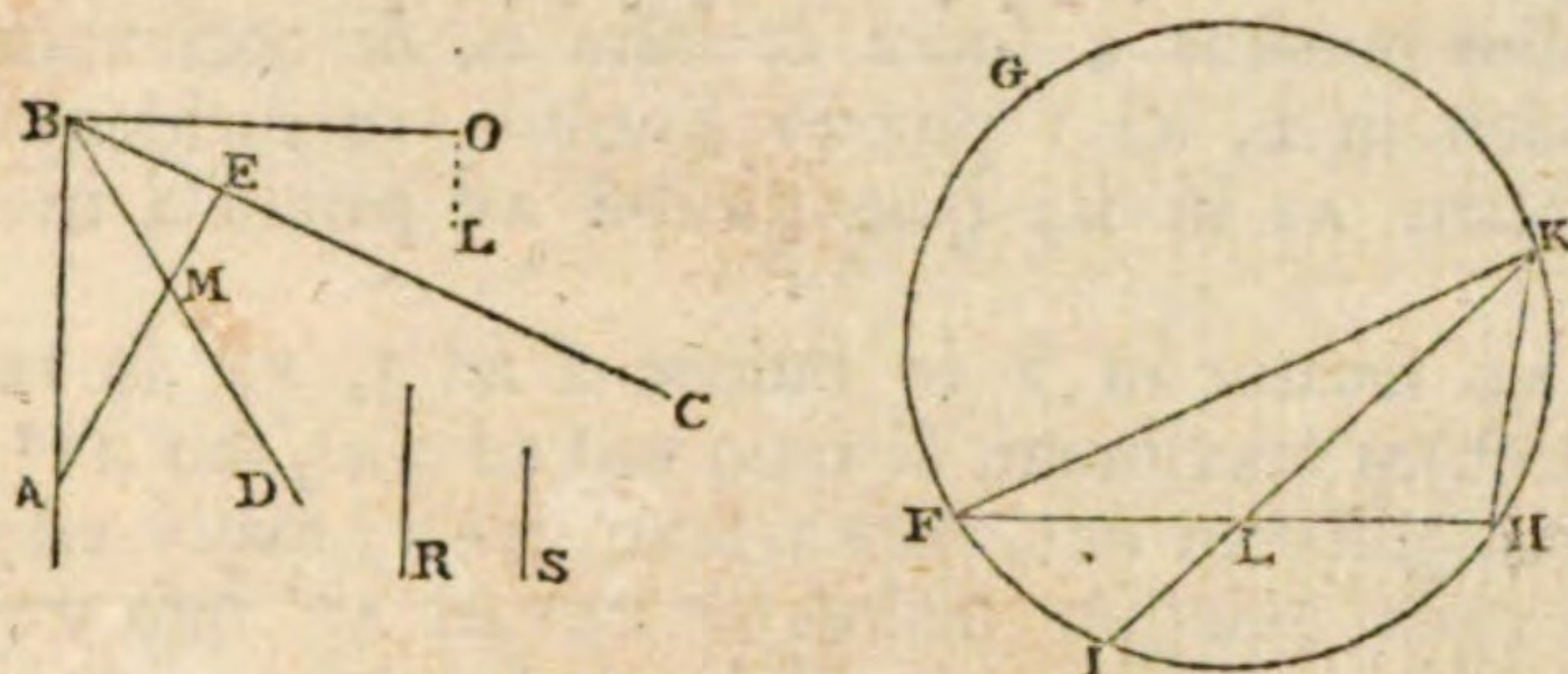
Productum enim FA^1 in AD^2 in linea DF , factum secundum terminos positos 1 & 2, est maximum, per theorema tertium; semperque homogeneum dignitati parallelarum (cùm parallelarum dignitatem exponat major datorum numerorum, maximum verò productum illud oriatur ex dignitatibus quas exponunt minor numerus & differentia numerorum, quæ duo simul efficiunt numerum majorem). Ergo si accipiamus alium punctum G in axe supra D , aut infra, & ducamus ordinatim applicatam GH , quæ secet E rectam FC (ubi opus fuerit productum), productum FA^1 in AG^2 non erit maximum in linea FG , quale est FA^1 in AD^2 in recta FD ; propterea quòd major est vel minor ratio ipsius FA ad AG quàm ad AD , & consequenter FG , FD non sunt proportionaliter divisæ. Ergo majorem rationem habet FA^1 in AD^2 ad FD^3 sibi homogeneum, quàm FA^1 in AG^2 ad FG^3 , & permutando, majorem rationem habet FA^1 in AD^2 ad FA^1 in AG^2 , (vel ex lemmate quarto, AD^2 ad AG^2) quàm FD^3 ad FG^3 . Sed AD^2 ad AG^2 ponitur in figura; ut CD^3 ad HG^3 : FD^3 ad FB^3 , ob similitudinem triangulorum, ut CD^3 ad EG^3 . Ergo majorem rationem habet CD^3 ad HG^3 , quàm CD^3 ad EG^3 ; & consequenter CD majorem rationem habet ad HG , quàm ad EG , ac proinde HG recta est minor quàm EG , & punctus E cadet extra datam curvam $AHCH$. Eodem pacto de singulis punctis ductæ lineæ FC demonstratur illos cadere semper extrà curvam. Ergo FC est illius tangens. Quod, &c.

Hæ sunt parabolæ, ut vocant, infinitæ, quarum contingentes lineæ, quo modo ad datum punctum duci possint, ostendimus. Nunc eandem methodum in hyperbolis quoque libet experiri. Præmittimus autem hoc necessarium lemma.

LEMMA SEXTUM.

Dato angulo ABC utcumque secto per rectam BD , & puncto E in alterutro laterum comprehendentium angulum datum; ex eo puncto ducere lineam rectam quæ angulum ABC subtendat, & à recta BD secetur in data ratione R ad s .

Fiat



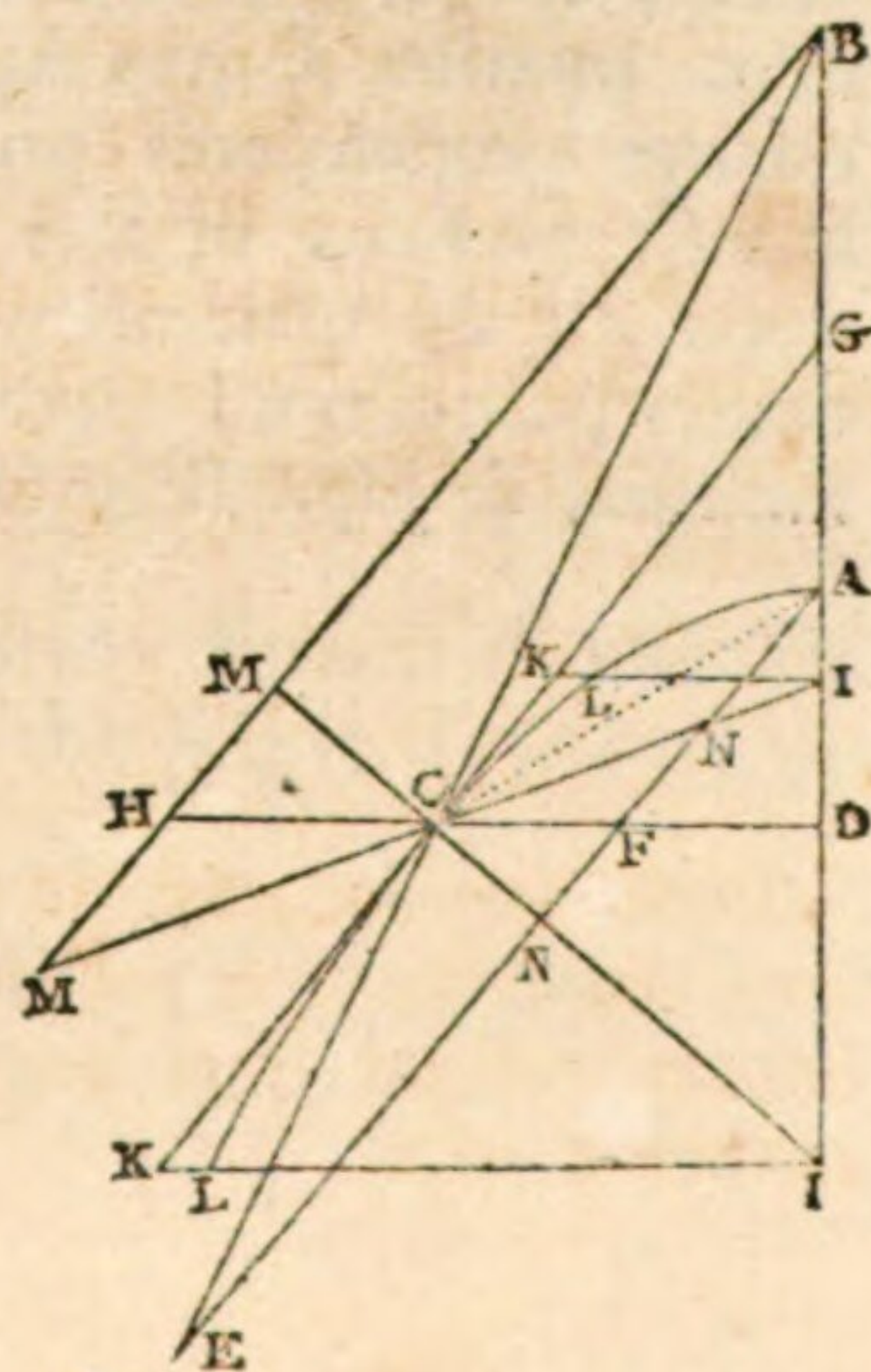
Fiat HGF segmentum circuli capiens angulum æqualem dato, & compleatur circulus; deinde ut R ad S, ita fiat FL ad LH; ut angulus ABD ad EBD, sic arcus FI ad IH; ductaque IL producaturs usque dum pertingat ad K in circumferentia circuli, & connectantur puncta F, K, H. Ad datum punctum E fiat angulus BEA æqualis KHL, & EA secet BD in M & BA in puncto A. Dico rectam EA esse quæsitam, quæ à BD in M dividitur in ratione data.

Siquidem anguli H & E : K & B sunt æquales, & hi facti proportionaliter [per trigessimam tertiam sexti Elementorum] à KLI , & BD . Ergo triacula FHK , ABE sunt æquiangula, & AE ad EB , ut HF ad HK . Rursus æquiangula fecimus triacula MBE , LKH , & consequenter EB est ad EM , ut HK ad HL , & ex æqualitate ordinata AE ad EM , ut HF ad HL , & dividendo FL ad LH [seu R ad s] ut AM ad ME . Quod, &c.

Quòd si punctus datus sit extrà, ut in o, ducemus BO rectam [punctus autem o sic detur oportet, ut OB recta cum AB angulum faciat, nec sit ad lineam posita] & faciemus angulum BOL æqualem differentiæ angulorum, H, & EBO; & OL producta satisfaciet quæsito.

Sit hyperbole ACL , cujus diameter AB , vertex A , & dignitates ordinatim applicatarum habentes eam proportionem quam producta illis dignitatibus homogenea, orta ex dignitate abscissæ ducta in dignitatem, abscissæ & diametri, ex quibus una recta conflata intelligatur. Exempli gratiâ, quadrato cubi ordinarum, hoc est LI^3 ad CD^3 , sint, ut producta BI^3 in AI^2 ad BD^3 in AD^2 , genita ex quadratis abscissarum AI , AD , & cubis rectarum BI , BD , quippe quas efficiunt eadem abscissæ & diameter.

Detur punctus c , ad quem ducenda sit tangens, & ordinatim applicetur cd . Porro ducatur bc , producta ad partes c , quoad oportuerit, & ex lemmate præcedenti, ae [secans cd in f , & in f item secta pro ratione 2 ad 3, qui numeri expouunt dignitates gignentes producta bi^3 in ai^2 , & bd^3 in ad^2 , subtendens angulum eca] & tandem gc parallela rectæ ae , occurrens ipsi ab in g . Dico tangentem quæsitam esse cg .



Sumatur in CG alius punctus K supra & infra C , & ordinatim applicatis KI fecantibus hyperbolen in L , ab I puncto ducatur IC incidens in rectam HB in puncto M , & secans AE in N ; quæ HB ipsi AE parallela in H occurrit DC productæ.

Quoniam verò AE fecatur in F in ratione 2 ad 3, FA^2 in FE^3 , per tertium theorema, est productum maximum, & ratio FE^3 ad NE^3 , seu HB^3 ad MB^3 [propter similitudinem triangulorum HCB , ECF : MBC , CEN] major est ratione NA^2 ad AF^2 . Ergo per lemma quintum majus est HB^3 in AF^2 ipso MB^3 in NA^2 ; quæ duo producta si comparentur cum CG^5 , primum habebit majorem rationem ad CG^5 , quàm secundum. Sed ratio primi, quod est HB^3 in AF^2 , ad CG^5 eadem est ac ratio BD^3 in AD^2 ad GD^5 [cum HB ad CG sit ut BD ad GD , ob similitudinem triangulorum HBD , CGD , eandemque proportionem habeant earum linearum cubi: tum CG^2 ad AF^2 , ut GD^2 ad AD^2] ratio secundi, seu MB^3 in NA^2 , ad CG^5 , est eadem ac ratio BI^3 in AI^2 ad IG^5 [quia similia sunt triangu-
la MBI , CGI ; & MB , CG , BI , IG rectæ earumque cubi proportionales: rursus ut GI^2 ad IA^2 , sic CG^2 ad AN^2] ergo majorem rationem habet BD^3 in AD^2 ad GD^5 , quàm BI^3 in IA^2 ad GI^5 , & permutando, BD^3 in AD^2 ad BI^3 in AI^2 [seu ex natura hyperboles, CD^5 ad LI^5] majorem rationem habet, quàm DG^5 ad GI^5 , seu [ob similitudinem triangulorum KGI , CGD] CD^5 ad IK^5 , & per decimam quinti Elem. dignitas, LI^5 minor est quàm KI^5 , & sua radix, LI recta, minor recta KI ; quare punctus K est extra curvam. Sic de ceteris punctis ostendetur cadere extra curvam, atque adeò CG hyperbolen tangere in solo C puncto. Quod, &c.

Hæc porrò demonstratio etiam ad ellipses, & circulos accommodari potest.

Jàm verò quàm latè pateat usus nostri theorematis tertii, ex propositis exemplis licet intelligere; nec ita multum dissimili aut difficiliore via centra gravitatis, & quadraturas, quorum problematum paulò ante meminimus, invenimus. Interim, si quis Apollonii constructionem atque demonstrationem trigesimæ quartæ propositionis primi Conicorum libri cum nostris comparabit, nonnihil fortasse proficiet in arte dilatandi propositiones & demonstrationes. Nam id quod ille de quadratica tantum hyperbole, ellipsi, & circulo statuit, nos ad omnes porrigimus hyperbolas, ellipses, circulosque infinitos. Quàm viam placuit indicare, & supradictò exemplo confirmare.

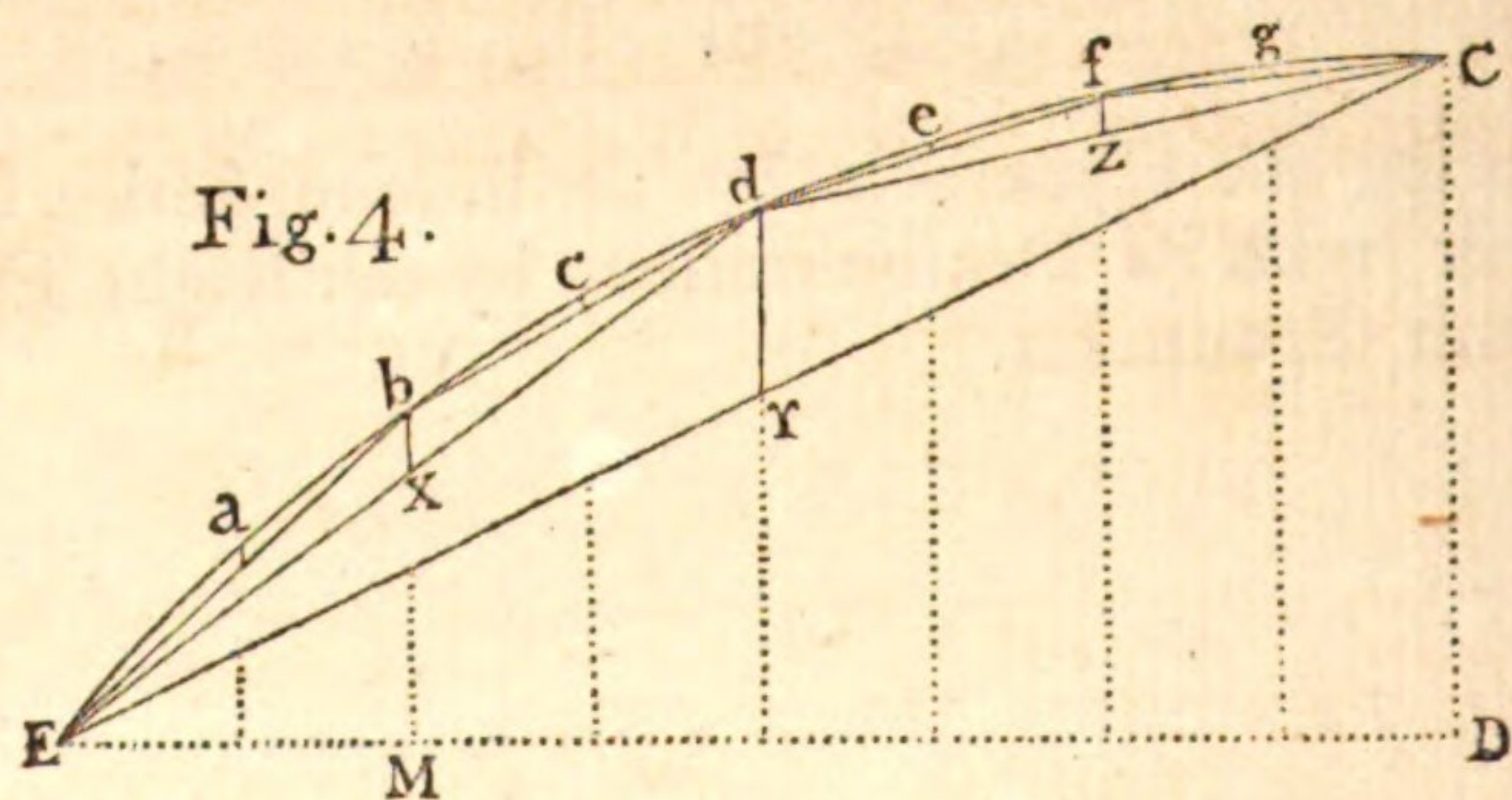
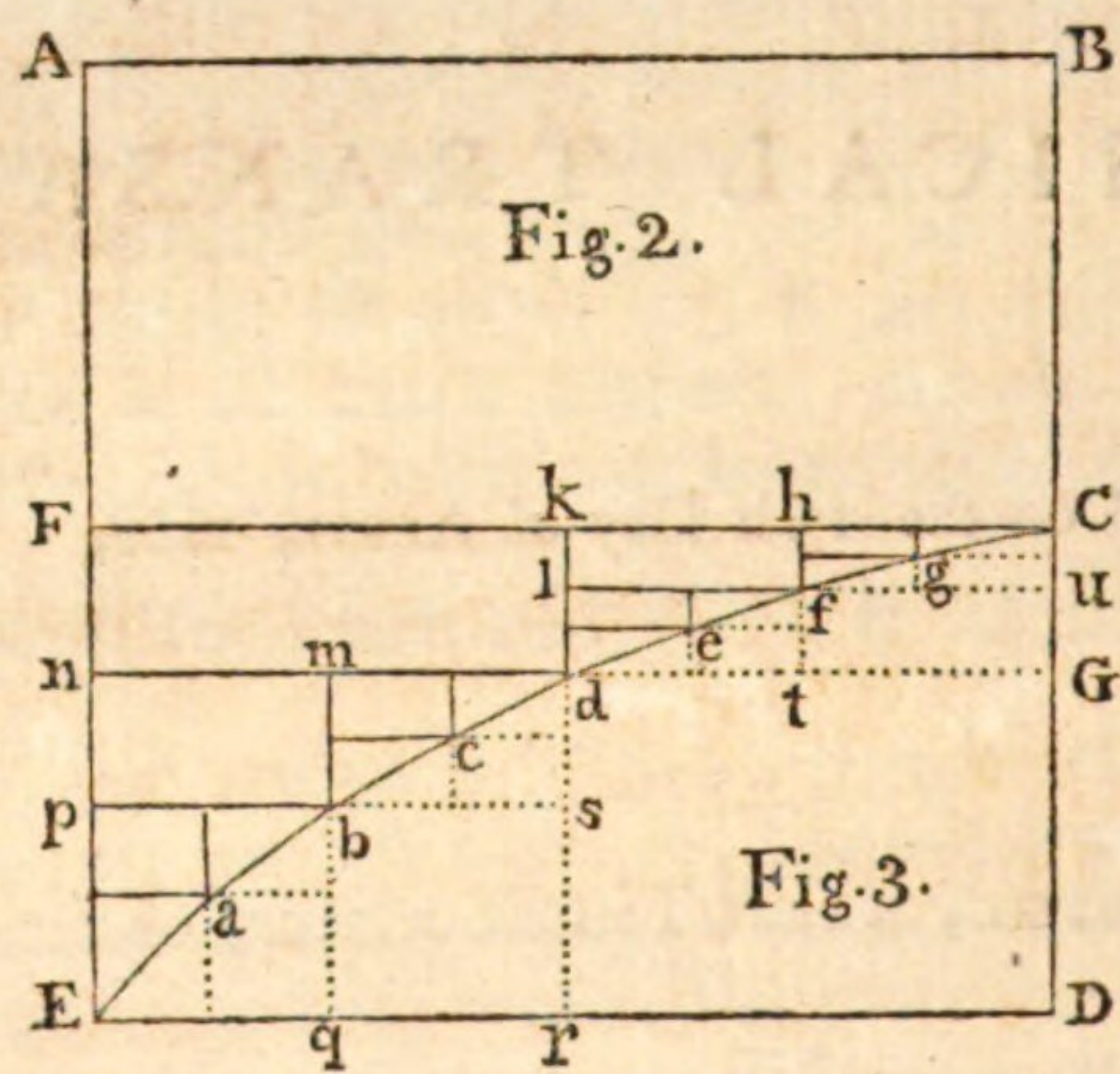
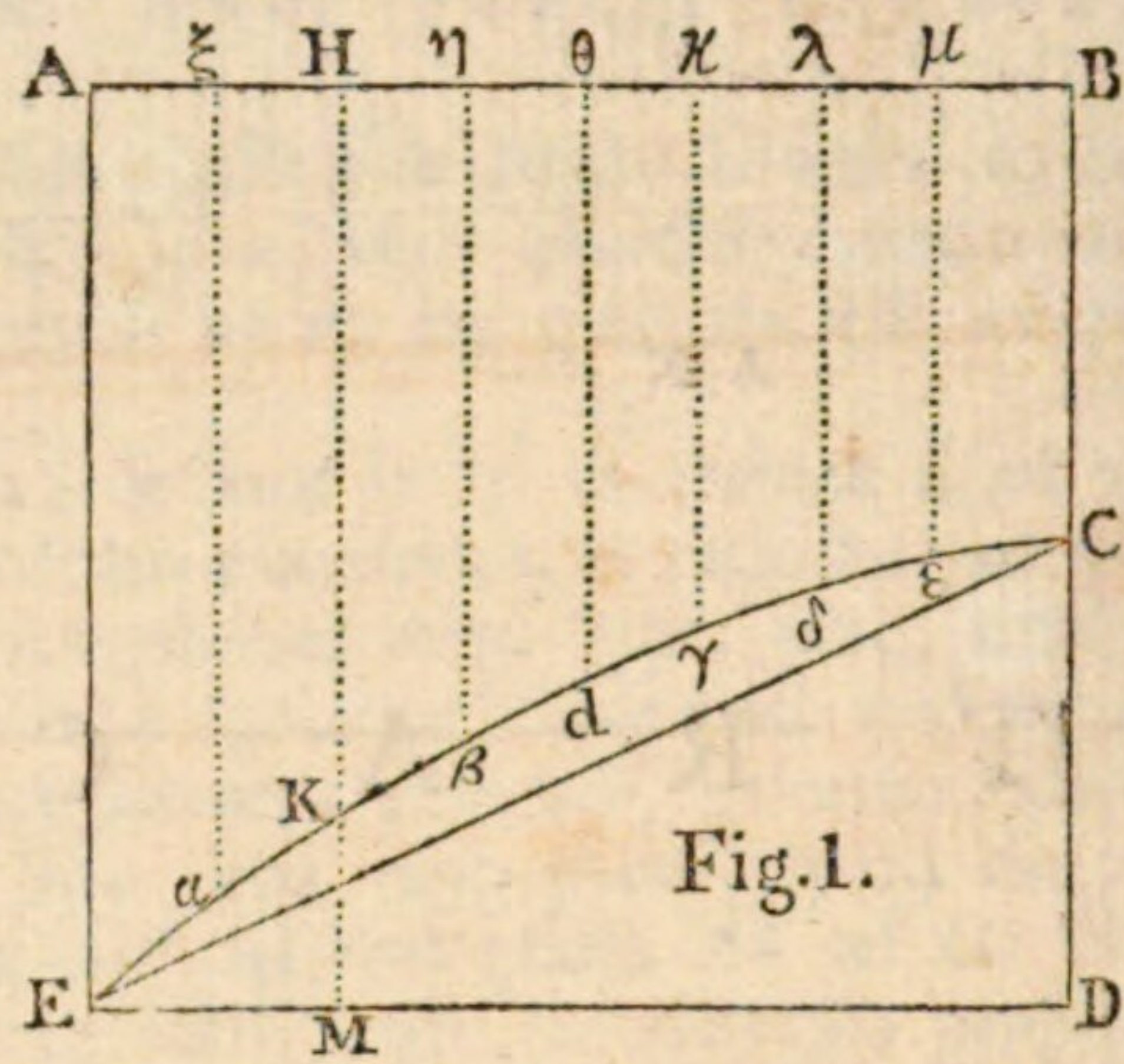
AN
EXTRACT
FROM THE THIRD VOLUME OF THE
PHILOSOPHICAL TRANSACTIONS,

Published on the 13th Day of APRIL, 1668,

By Mr. HENRY OLDENBURG, at that Time Secretary of the Royal Society ;

CONTAINING

A Method of squaring the Hyperbola by an infinite Series of rational Numbers, together with its Demonstration, by the Right Honourable the Lord Viscount Brouncker.



This Extract is contained in pages 645, 646, 647, 648, and 649 of the said Third Volume of the Philosophical Transactions, and is as follows :

The Squaring of the Hyperbola by an infinite Series of rational Numbers, together with its Demonstration, by that eminent Mathematician, the Right Honourable the Lord Viscount Brouncker.

WHAT the acute Dr. John Wallis had intimated, some years since, in the Dedication of his Answer to M. Meibomius de Proportionibus, to wit, "That the world one day would learn from the noble lord Brouncker, the quadrature of the hyperbole;" the ingenious reader may see performed in the subjoined operation, which its excellent author was now pleased to communicate, as followeth in his own words :

My method for squaring the hyperbola is this :

Let AB be one asymptote of the hyperbola edc ; and let AE and BC be parallel to the other : Let also AE be to BC as 2 to 1 ; and let the parallelogram ABDE equal 1. See Fig. 1.

Supposing the reader knows, that $EA, \alpha\zeta, KH, \epsilon\eta, d\theta, \gamma\kappa, \delta\lambda, \epsilon\mu, CB, \&c$ are in an harmonic series, or a series reciproca primanorum, seu arithmetice proportionalium (otherwise he is referred for satisfaction to the 87, 88, 89, 90, 91, 92, 93, 94, 95, prop. Arithm. Infinitor. Wallisii):

$$\begin{array}{lcl} \text{I say } ABCdEA = \frac{1}{1 \times 2} + \frac{1}{3 \times 4} + \frac{1}{5 \times 6} + \frac{1}{7 \times 8} + \frac{1}{9 \times 10} \&c & \\ EdCDE = \frac{1}{2 \times 3} + \frac{1}{4 \times 5} + \frac{1}{6 \times 7} + \frac{1}{8 \times 9} + \frac{1}{10 \times 11} \&c & \\ EdCyE = \frac{1}{2 \times 3 \times 4} + \frac{1}{4 \times 5 \times 6} + \frac{1}{6 \times 7 \times 8} + \frac{1}{8 \times 9 \times 10} \&c & \end{array} \left. \vphantom{\begin{array}{l} \\ \\ \end{array}} \right\} \begin{array}{l} \text{in} \\ \text{infi-} \\ \text{ni-} \\ \text{tum.} \end{array}$$

For (in Fig. 2 & 3) the Parallelog. And (in Fig. 4) the Triangl.

	$CA = \frac{1}{1 \times 2}$	$EdC = \frac{1}{2 \times 3 \times 4} = \frac{\square dD - \square dF}{2}$	Note.
$dD = \frac{1}{2 \times 3}$	$dF = \frac{1}{3 \times 4}$	$Ebd = \frac{1}{4 \times 5 \times 6} = \frac{\square br - \square bn}{2}$	
$br = \frac{1}{4 \times 5}$	$bn = \frac{1}{5 \times 6}$	$dfc = \frac{1}{6 \times 7 \times 8} = \frac{\square fG - \square fk}{2}$	
$fG = \frac{1}{6 \times 7}$	$fk = \frac{1}{7 \times 8}$	$Eab = \frac{1}{8 \times 9 \times 10} = \frac{\square aq - \square ap}{2}$	
$aq = \frac{1}{8 \times 9}$	$ap = \frac{1}{9 \times 10}$	$bcd = \frac{1}{10 \times 11 \times 12} = \frac{\square cs - \square cm}{2}$	
$cs = \frac{1}{10 \times 11}$	$cm = \frac{1}{11 \times 12}$	$def = \frac{1}{12 \times 13 \times 14} = \frac{\square et - \square el}{2}$	
$et = \frac{1}{12 \times 13}$	$el = \frac{1}{13 \times 14}$	$fgc = \frac{1}{14 \times 15 \times 16} = \frac{\square gu - \square gb}{2}$	
$gu = \frac{1}{14 \times 15}$	$gb = \frac{1}{15 \times 16}$	&c.	
&c.	&c.		

$$\begin{aligned}
 \frac{1}{2}CA &= dD + dF \\
 \frac{1}{2}dD &= br + bn \\
 \frac{1}{2}dF &= fG + fk \\
 \frac{1}{2}br &= aq + ap \\
 \frac{1}{2}bn &= cs + cm \\
 \frac{1}{2}fG &= et + el \\
 \frac{1}{2}fk &= gu + gb \\
 &\&c.
 \end{aligned}$$

And that therefore in the first series half the first term is greater than the sum of the two next, and half this sum of the second and third greater than the sum of the four next, and half the sum of those four greater than the sum of the next eight, &c in infinitum. For $\frac{1}{2}dD = br + bn$; but $bn > fG$, therefore $\frac{1}{2}dD > br + fG$, &c. And in the second series half the first term is less than the sum of the two next, and half this sum less than the sum of the four next, &c in infinitum.

That the first series are the even terms, viz. the 2^d, 4th, 6th, 8th, 10th, &c, and the second, the odd, viz. the 1st, 3^d, 5th, 7th, 9th, &c of the following series, viz. $\frac{1}{1 \times 2} \cdot \frac{1}{2 \times 3} \cdot \frac{1}{3 \times 4} \cdot \frac{1}{4 \times 5} \cdot \frac{1}{5 \times 6} \cdot \frac{1}{6 \times 7} \cdot \&c$ in infinitum = 1. Whereof

a being put for the number of terms taken at pleasure, $\frac{1}{a^2 + a}$ is the last, $\frac{a}{a+1}$ is the sum of all those terms from the beginning, and $\frac{1}{a+1}$ the sum of the rest to the end.

That $\frac{1}{4}$ of the first term in the third series is less than the sum of the two next, and a quarter of this sum, less than the sum of the four next, and one fourth of this last sum less than the next eight, I thus demonstrate.

Let $a =$ the 3^d or last number of any term of the first column, viz. of divisors,

$$\frac{1}{a \times a - 1 \times a - 2} = \frac{1}{a^3 - 3a^2 + 2a} = \frac{16a^3 - 48a^2 + 56a - 24}{16a^6 - 96a^5 + 232a^4 - 288a^3 + 184a^2 - 48a} = A$$

$$\frac{1}{2a \times 2a - 1 \times 2a - 2} = \frac{1}{8a^3 - 12a^2 + 4a}$$

$$\frac{1}{2a - 2 \times 2a - 3 \times 2a - 4} = \frac{1}{8a^3 - 36a^2 + 52a - 24}$$

$$\left. \begin{aligned} &\frac{1}{8a^3 - 12a^2 + 4a} \\ &\frac{1}{8a^3 - 36a^2 + 52a - 24} \end{aligned} \right\} = \frac{16a^3 - 48a^2 + 56a - 24}{64a^6 - 384a^5 + 880a^4 - 960a^3 + 496a^2 - 96a} = B$$

$$\frac{64a^6 - 384a^5 + 928a^4 - 1152a^3 + 736a^2 - 192a}{64a^6 - 384a^5 + 880a^4 - 960a^3 + 496a^2 - 96a} \times \frac{1}{4}A < B.$$

And $48a^4 - 192a^3 + 240a^2 - 96a =$ Excess of the numerator above denominator.

$$\left. \begin{array}{l} \text{But the affirm.} \\ \text{That is, } 48a^4 + 240a^2 > 192a^3 + 96a \\ \text{Because } a^4 + 5a^2 > 4a^3 + 2a \\ \qquad \qquad a^3 + 5a > 4a^2 + 2 \end{array} \right\} \text{if } a > 2.$$

Therefore $B > \frac{1}{4}A$.

Therefore $\frac{1}{4}$ of any number of A, or terms, is less than their so many respective B, that is, than twice so many of the next terms. Quod, &c.

By any one of which three series, it is not hard to calculate, as near as you please, these and the like hyperbolic spaces, whatever be the rational proportion of AE to BC. As for example, when AE is to BC, as 5 to 4 (whereof the calculation follows after that where the proportion is, as 2 to 1, and both by the third series.)

First then when (in Fig. 1) $AE : BC :: 2 : 1$.

$2 \times 3 \times 4$	I. (0,0416666666 —	0,0416666666
$4 \times 5 \times 6$	I. (0,0083333333 —	} 0,0113095237
$6 \times 7 \times 8$	I. (0,0029761904 —	
$8 \times 9 \times 10$	I. (0,0013888888 —	
$10 \times 11 \times 12$	I. (0,0007575757 —	} 0,0029019589
$12 \times 13 \times 14$	I. (0,0004578754 —	
$14 \times 15 \times 16$	I. (0,0002976190 —	
$16 \times 17 \times 18$	I. (0,0002042484 —	} 0,0007306482
$18 \times 19 \times 20$	I. (0,0001461988 —	
$20 \times 21 \times 22$	I. (0,0001082251 —	
$22 \times 23 \times 24$	I. (0,0000823452 —	} 0,0001829939
$24 \times 25 \times 26$	I. (0,0000641026 —	
$26 \times 27 \times 28$	I. (0,0000508751 —	
$28 \times 29 \times 30$	I. (0,0000410509 —	} 0,0001829939
$30 \times 31 \times 32$	I. (0,0000336021 —	
$32 \times 33 \times 34$	I. (0,0000278520 —	
$34 \times 35 \times 36$	I. (0,0000233426 —	} 0,0001829939
$36 \times 37 \times 38$	I. (0,0000197566 —	
$38 \times 39 \times 40$	I. (0,0000168691 —	
$40 \times 41 \times 42$	I. (0,0000145180 —	} 0,0001829939
$42 \times 43 \times 44$	I. (0,0000125843 —	
$44 \times 45 \times 46$	I. (0,0000109793 —	
$46 \times 47 \times 48$	I. (0,0000096361 —	} 0,0001829939
$48 \times 49 \times 50$	I. (0,0000085034 —	
$50 \times 51 \times 52$	I. (0,0000075415 —	
$52 \times 53 \times 54$	I. (0,0000067193 —	} 0,0001829939
$54 \times 55 \times 56$	I. (0,0000060125 —	
$56 \times 57 \times 58$	I. (0,0000054014 —	
$58 \times 59 \times 60$	I. (0,0000048704 —	} 0,0001829939
$60 \times 61 \times 62$	I. (0,0000044068 —	
$62 \times 63 \times 64$	I. (0,0000040002 —	

0,0416666666
0,0113095237
0,0029019589
0,0007306482
3) 0,0001829939 (0,0000609980
0,05679179
+ 0,00006100
0,05685279 < edcy
But 0,0007306482
0,0001829939
0,0000458315
Therefore 0,05679179
+ 0,00004583
+ 0,00001528
0,05685290 > edcy

ANOTHER
EXTRACT

FROM THE

PHILOSOPHICAL TRANSACTIONS, N^o XXXVIII.

Published on the 17th Day of AUGUST, 1668.

CONTAINING

First, An Account of Mr. NICHOLAS MERCATOR'S TRACT ON LOGARITHMS, called LOGARITHMO-TECHNIA, by Dr. JOHN WALLIS, Savilian Professor of Geometry in the University of Oxford, in a Letter to the Lord Viscount Brouncker; and 2dly, A Method of finding the Sums of Logarithms, by the said Dr. WALLIS, in another Letter to the same learned Lord; and, 3dly, An Illustration of the said Mr. Nicholas Mercator's Tract aforesaid, called LOGARITHMO-TECHNIA, by the said Mr. MERCATOR himself.

This Extract is contained in pages 753, 754, 755, 756, &c.—764, of the Third Volume of the Philosophical Transactions, and is as follows :

L O G A R I T H M O - T E C H N I A

N I C O L A I M E R C A T O R I S ;

Concerning which we shall here deliver the Account of the judicious Doctor JOHN WALLIS, given in a Letter to the Lord Viscount BROUNCKER, as follows :

INCIDEBAM heri, Illustrissime Domine, in D. Mercatoris Logarithmo-techniam, nupèr editam; quæ ita mihi placuit, ut non prius dimiserim quàm perlegissem totam. Et quamquam pauca quædam, phraseologiam quod spectat seu loquendi formulas nonnullas, mutata mallet; sunt tamen ipsa sensu suo sana: Eaque quæ superstruitur doctrina, logarithmos expeditè atque subtilitèr construendi, perspicuè fatiè atque ingeniosè traditur.

Quæ huic subjungitur Quadratura Hyperbolæ, elegans admodum est atque ingeniosa. Nempe ad hunc sensum. V. Fig. 1.

Postquam in hyperbolâ MBF, (cujus asymptotæ AN, AH, ad angulum rectum coeunt) ostenderat, prop. XIV, rectangula BIA, FHA, spA, &c. (ductis BI, FH, sp, &c, parallelis asymptotæ AN), invicem esse æqualia; adeoque latera habere reciprocè proportionalia (quæ nota est hyperbolæ proprietas): Positis AI =

BI = 1, & HI = a: ostendit, prop. XV, $FH = \frac{1}{1+a}$ (nempe propter analogiam AH . AI :: BI . FH, hoc est,

$1 + a . 1 :: 1 . \frac{1}{1+a}$. Sed & (quod di-

videndo 1 per $1 + a$ ostenditur), $\frac{1}{1+a} = 1 - a + a^2 - a^3 + a^4 \&c$ (continuatis deinceps, ipsius a potestatibus, alternatim negatis & affirmatis).

Cùmque hoc perinde obtineat, ubicunque ultra punctum I, ponatur H. Positis, ut prius, AI = 1; hujusque continuatione qualibet, ut IR = A; quæ intelligatur in

$$\begin{array}{r}
 \frac{1}{1+a} \\
 \hline
 -a \\
 \hline
 -a - a^2 \\
 \hline
 + a^2 \\
 \hline
 + a^2 + a^3 \\
 \hline
 - a^3 \\
 \hline
 - a^3 - a^4 \\
 \hline
 + a^4 \\
 \hline
 \&c.
 \end{array}$$

æquales

æquales partes innumeras dividi, quarum quælibet, ut ip , pq , &c, dicatur a ; adeoque ip , iq , &c, sint a , $2a$, $3a$, &c, usque ad A : Quæ his respondent rectæ ps , qt , &c, usque ad ru , (spatium $BIRU$ complentes) sunt,

$$1 - a + a^2 - a^3 + a^4 \text{ \&c.}$$

$$1 - 2a + 4a^2 - 8a^3 + 16a^4 \text{ \&c.}$$

$$1 - 3a + 9a^2 - 27a^3 + 81a^4 \text{ \&c.}$$

& sic deinceps usque ad

$$1 - A + A^2 - A^3 + A^4 \text{ \&c.}$$

$$\begin{array}{rcl} \text{Cum itaque sint } 1 + 1 + 1 \text{ \&c (usque ad ultimum)} & = & A \\ a + 2a + 3a \text{ \&c (usque ad } A) & = & \frac{1}{2}A^2 \\ a^2 + 4a^2 + 9a^2 \text{ \&c (usque ad } A^2) & = & \frac{1}{3}A^3 \\ a^3 + 8a^3 + 27a^3 \text{ \&c (usque ad } A^3) & = & \frac{1}{4}A^4 \\ & \text{\& sic deinceps :} & \end{array}$$

(quod ostendit ille prop. XVI, estque à me alibi demonstratum): Rectè colligit, prop. XVII. Expositum spatium hyperbolicum $BIRU = A - \frac{1}{2}A^2 + \frac{1}{3}A^3 - \frac{1}{4}A^4 + \frac{1}{5}A^5$, &c. Adeoque si (assignato, ipsi $A = 1$, valore suo in numeris, ut res postulaverit) distribuuntur in duas classes A , $\frac{1}{3}A^3$, $\frac{1}{5}A^5$, &c, (potestates affirmatæ), & $\frac{1}{2}A^2$, $\frac{1}{4}A^4$, &c, (potestates negatæ); harumque aggregatum, ex aggregato illarum, subducatur; residuum erit ipsum $BIRU$ spatium hyperbolicum.

Nequis autem operam lusum iri existimet, propter addendorum seriem in utrâque classe infinitam; adeoque non absolvendam: Huic incommodo medelam (tacitus) adhibet: ponendo $A = 0,1$, vel $A = 0,21$, aliæ fractioni decimali æqualem, adeoque minorem quam 1: (hoc est, sumptâ 1r minore quam $AI = 1$). Quo fit, ut posteriores ipsius A potestates tot gradibus infra integrorum sedem descendant, ut meritò negligi possint.

Exempli gratiâ; positis $AI = 1$, & $1r = 0,21$; erit

$A = 0,21$	
$\frac{1}{3}A^3 = 0,003087$	$\frac{1}{2}A^2 = 0,02205$
$\frac{1}{5}A^5 = 0,000081682$	$\frac{1}{4}A^4 = 0,000486202$
$\frac{1}{7}A^7 = 0,000002572$	$\frac{1}{6}A^6 = 0,000014294$
$\frac{1}{9}A^9 = 0,000000088$	$\frac{1}{8}A^8 = 0,000000472$
$\frac{1}{11}A^{11} = 0,000000003$	$\frac{1}{10}A^{10} = 0,000000016$

$$+ 0,213171345 \quad - \quad 0,022550984 = 0,190620361 = BIRU \text{ quæ est brevis synopsis quadraturæ suæ satis elegantis.}$$

Diffimulandum interim non est; si quis totius $BIRU$ spatii (cujus latus $1H$ longius intelligatur quàm AI) quadraturam postulet; rem non ita feliciter successuram: propter medelam, quam modò diximus, malo minùs sufficientem. Cum enim jam ponenda sit $A > 1$; manifestum est, posteriores ipsius potestates, altius in integrorum sedes penetraturas, adeoque minimè negligendas.

Huic autem incommodo, levi constructionis immutatione, facile subvenitur. Vid. Fig. 1.

Cæteris

Cæteris utique ut priùs constructis; quadrandum exponatur HFur spatium; (cujuscunque fuerit longitudinis AH; puta major minörve quàm AI, vel huic æqualis: sumptóque ubivis inter A & H, puncto r; puta ultra citràve punctum I, vel in ipso I puncto): ponantur autem (non, ut priùs, AI = I, & Ir = A. sed) AH = I; & Hr = A, quæ intelligatur in æquales partes innumeras dividi, quarum quælibet sit a. Erunt itaque, post AH = I, reliquæ deinceps decrefcentes I - a, I - 2a, I - 3a, &c. usque ad Ar = I - A. Item, propter æqualia rectangula FHA, uFA, BIA, &c. puta, = b²: Erit HF = $\frac{b^2}{1}$; reliquæque deinceps $\frac{b^2}{1-a}$, $\frac{b^2}{1-2a}$, $\frac{b^2}{1-3a}$, &c. usque ad ru = $\frac{b^2}{1-A}$ spatium HFur complentes. (Quæ omnia ostensa sunt, in meâ Arithmeticâ Infinitorum, prop. 88, 94, 95.)

Factâque divisione; reperietur

$$\frac{b^2}{1-a} = b^2 + b^2a + b^2a^2 + b^2a^3 + b^2a^4 \text{ \&c.}$$

Hoc est, b² in 1 + a + a² + a³ + a⁴, &c. (sumptis ipsius a potestatibus continuè sequentibus affirmatis omnibus). Cúmque de reliquis idem sit judicium; erunt rectæ omnes, ipsis HF & ru interjectæ,

$$\begin{array}{r} 1-a) \quad b^2(b^2 + b^2a + b^2a^2 + b^2a^3 \text{ \&c.} \\ \underline{b^2 - b^2a} \\ \quad + b^2a \\ \quad + b^2a - b^2a^2 \\ \quad \underline{\quad + b^2a^2} \\ \quad \quad + b^2a^2 - b^2a^3 \\ \quad \quad \underline{\quad + b^2a^3} \\ \quad \quad \quad + b^2a^3 - b^2a^4 \\ \quad \quad \quad \underline{\quad + b^2a^4} \\ \quad \quad \quad \quad \text{\&c.} \end{array}$$

$$\left. \begin{array}{l} 1 + a + a^2 + a^3 + a^4 \text{ \&c.} \\ 1 + 2a + 4a^2 + 8a^3 + 16a^4 \text{ \&c.} \\ 1 + 3a + 9a^2 + 27a^3 + 81a^4 \text{ \&c.} \\ \text{\& sic deinceps usque ad} \\ 1 + A + A^2 + A^3 + A^4 \text{ \&c.} \end{array} \right\} \text{ in } b^2$$

Omniumque aggregatum $A + \frac{1}{2}A^2 + \frac{1}{3}A^3 + \frac{1}{4}A^4 + \frac{1}{5}A^5 \text{ \&c.}$ in b² = FHru. (per Arithm. Infin. prop. 64).

Exempli gratiâ,
Positis AH = I.
Hr = A = 0,21
AI = b = 0,1
Adeóque b² = 0,01

$$\begin{array}{rcl} \text{Erunt } A & = & 0,21 \\ \frac{1}{2}A^2 & = & 0,02205 \\ \frac{1}{3}A^3 & = & 0,003087 \\ \frac{1}{4}A^4 & = & 0,00048623 \quad - \\ \frac{1}{5}A^5 & = & 0,000081682 \quad + \\ \frac{1}{6}A^6 & = & 0,000014294 \quad + \\ \frac{1}{7}A^7 & = & 0,000002573 \quad - \\ \frac{1}{8}A^8 & = & 0,000000473 \quad - \\ \frac{1}{9}A^9 & = & 0,000000088 \quad + \\ \frac{1}{10}A^{10} & = & 0,0000000017 \quad - \\ \frac{1}{11}A^{11} & = & 0,0000000003 \quad + \end{array}$$

Horum summa = 0,235722333
Ducta in b² = 0,01

Exhibet 0,00235722333 = FHru Qua-

figuræ & methodo quantum res ferebat accommodaveram) ad principia mea revocatam ab origine repetam. V. Fig. 2.

Ostenſum eſt, in mea^a Arithmetica Infinitorum, prop. xciv, ſpatium hyperbolicum $AD\beta\beta\delta$ (in infinitum continuatum à partè $\beta\delta$, ſed à parte $D\beta$ ubivis terminatum) figuram eſſe quam ex primariorum reciprocis conflatam appello, prop. LXXXVIII, definitam: Cujus nempe ordinatim — applicatæ $d\beta$, $d\beta$, ſint primariis (ſeu arithmetice proportionalibus) db , db , (triangulum complementibus) adeoque iſſis dA , dA , (ſuis à vertice diſtantiis) reciproce proportionales. Hoc eſt, (poſito $AD = d$; & rectangulo $AD\beta = b^2$; particulisque infinite exiguis, a , a , &c); ſi à vertice $A\delta$ incipias $\frac{b^2}{0}$, $\frac{b^2}{a}$, $\frac{b^2}{2a}$, $\frac{b^2}{3a}$, &c. uſque ad $\frac{b^2}{d} = D\beta$: vel, ſi à baſe $D\beta$ incipias $\frac{b^2}{d}$, $\frac{b^2}{d-a}$, $\frac{b^2}{d-2a}$, $\frac{b^2}{d-3a}$, &c. uſque ad $\frac{b^2}{d-d} = A\delta$ infinitæ (nempe, ſi ad verticem uſque proceſſum continuaveris); vel, uſque ad $\frac{b^2}{d-a} = c\beta$ (poſito $DC = A$), ſi continuaveris uſque ad $c\beta$, ubivis intra A & $D\beta$ ſumptam. (Adeoque omnium aggregatum, $\frac{b^2}{d} + \frac{b^2}{d-a} + \frac{b^2}{d-2a} + \frac{b^2}{d-3a}$, &c, eſt iſſum $DC\beta\beta$ planum.)

Maniſtum itaque eſt (& ibidem prop. xciv oſtenſum), ſi intelligantur ſingulæ $d\beta$, in ſuas à vertice diſtantias Ad , ductæ; hoc eſt, $\frac{b^2}{a}$ in a , $\frac{b^2}{2a}$ in $2a$ (& ſic de reliquis); erunt omnia rectangula $Ad\beta$; hoc eſt, rectarum $d\beta$ momenta reſpectu $A\delta$ (intellige, facta ex magnitudine in diſtantiâ ductâ); ſeu plana ſemiquadrantalem ungulam (cujus acies $A\delta$) complementia (eiſdem $d\beta$ rectis perpendiculariter inſiſtentia); invicem æqualia. Quippe ſingula $= b^2$. (Quorum cum unum ſit $AIv\delta$ quadratum, erit $AI = b$.)

Adeoque totius $AD\beta\beta\delta$ (plani infiniti) ſeu omnium $d\beta$ illud complementum, momentum reſpectu rectæ $A\delta$ (ut axis æquilibrii); ſeu ungula ſemiquadrantalſ eidem $AD\beta\beta\delta$ inſiſtens (aciem habens $A\delta$); ſunt totidem b^2 ; hoc eſt, db^2 . (Ungula magnitudinis finitæ plano infinitæ magnitudinis inſiſtens.) Ejuſque pars plano $Ac\beta\beta\delta$ inſiſtens (propter $AC = d - A$) ſimiliter oſtendetur æqualis iſſi $d - A$ in b^2 ductæ; hoc eſt, $db^2 - Ab^2$. Adeoque pars reliqua, iſſi $DC\beta\beta$ inſiſtens, æqualis iſſi Ab^2 . Quod itaque eſt ejuſdem $DC\beta\beta$ momentum reſpectu $A\delta$.

Atque hoc momentum per plani $DC\beta\beta$ magnitudinem, puta per pl , diviſum; exhibet plani diſtantiâ centri gravitatis ab $A\delta$, $\frac{ab^2}{pl}$: adeoque diſtantiâ ejusdem à $D\beta$, $d - \frac{ab^2}{pl}$.

Hæc itaque à $D\beta$ diſtantiâ, in pl (plani magnitudinem) ducta; exhibet $dpl - Ab$ ejuſdem $DC\beta\beta$ momentum reſpectu $D\beta$; ſeu ungulam eidem $DC\beta\beta$ inſiſtentem, cujus acies ſit $D\beta$.

Hæc denique ungula (cujus altitudo, in $D\beta$, nulla ſit, ſed, in $c\beta$, iſſi DC æqualis): ſi ex planis iſſi $DC\beta\beta$ parallelis conſtari intelligatur; erunt ea , $cd\beta\beta$, $cd\beta\beta$, & ſic deinceps; hoc eſt, aggregatum omnium $cd\beta\beta$, $cd\beta\beta$, uſque ad $cd\beta\beta$.

Sunt autem ea plana (ut ex Gregorii de Sancto Vincentio, aliorumque post illum, doctrina constat) tanquam logarithmi arithmetice proportionalium cd , cd , &c. usque ad cb ; (seu a , $2a$, $3a$, &c. usque ad A .) Ergo ungula ipsa, est eorundem aggregatum. Hoc est (posito $d = 1$), $dpl - Ab^2 = pl - Ab^2$. Quod ostendendum erat.

Porro; cum sit $\frac{b^2}{d-a} (= d\beta) = \frac{b^2}{d} + \frac{ab^2}{d^2} + \frac{a^2b^2}{d^3} + \frac{a^3b^2}{d^4} \&c.$ (Quod dividendo b^2 per $d - a$, patebit): vel, posito $d = 1$, (quò ipsius d potestates omnes deleantur), $b^2 + ab^2 + a^2b^2 + a^3b^2 \&c.$ seu $1 + a + a^2 + a^3 \&c.$ in b^2 ; & similiter $\frac{b^2}{d-2a} = \frac{b^2}{d} + \frac{2ab^2}{d^2} + \frac{4a^2b^2}{d^3} + \frac{8a^3b^2}{d^4} \&c. = b^2 + 2ab^2 + 4a^2b^2 + 8a^3b^2 \&c. = b^2$ in $1 + 2a + 4a^2 + 8a^3 \&c.$ & similiter in reliquis:

Erunt omnes $d\beta$ (spatium $dc\beta\beta$ complentes),

1	+	a	+	a ²	+	a ³	+	a ⁴	&c,
1	+	2a	+	4a ²	+	8a ³	+	16a ⁴	&c,
1	+	3a	+	9a ²	+	27a ³	+	81a ⁴	&c,
& sic deinceps usque ad									
1	+	A	+	A ²	+	A ³	+	A ⁴	&c,
—————									
A + $\frac{1}{2}A^2$ + $\frac{1}{3}A^3$ + $\frac{1}{4}A^4$ + $\frac{1}{5}A^5$ &c, in $b^2 = pl$.									

Adeoque (per Arithm. Infin. prop. LXIV), omnium aggregatum, seu ipsum $dc\beta\beta$ spatium, erit

Qualium $1 = ADE\delta$ quadrato vel rhombo.

Ideoque, plani $dc\beta\beta$ momentum respectu $d\beta$; seu semiquadrantalibus ungula eidem insistentis cujus acies sit $d\beta$; seu planorum aggregatum ipsam constituentium; seu logarithmorum summa quos ea representant, $dpl - Ab^2 = pl - Ab^2 = \frac{1}{2}A^2 + \frac{1}{3}A^3 + \frac{1}{4}A^4 + \frac{1}{5}A^5$ in b^2 :

Qualium cubus (seu rhombus solidus) $ADE\delta$ sit 1 .

Si vero non ponatur $d = 1$, sed cujuscunque magnitudinis: erit saltem

$$\frac{A}{d} + \frac{A^2}{2d^2} + \frac{A^3}{3d^3} + \frac{A^4}{4d^4} \&c \text{ in } b^2 = pl.$$

Vel (posito $\frac{A}{d} = e$) erit $e + \frac{1}{2}e^2 + \frac{1}{3}e^3 + \frac{1}{4}e^4 \&c$ in $b^2 = pl$ qualium $d^2 = ADE\delta$ quadrato vel rhombo.

Ungulaque (ut prius) $dpl - Ab^2$ qualium $d^3 = ADE\delta$ cubo, vel (si angulus A sit obliquus) rhombo solido.

Cumque A (posito $d = 1$) vel e (quicunque ponatur valor ipsius d) sit minor quam 1 (propter $A < d$): illius potestates posteriores ita continue decrescunt, ut tandem negligi possint; planique valor pl exhibeatur quantumlibet vero propinquus.

Atque hæc est, Illustrissime Domine, methodi, quam innuebam, ex meis principiis deductio, & demonstratio brevis. Vale. Oxon. d. 5. Aug. 1668.

Some Illustration of the Logarithmo-technia of M. Mercator, who communicated it to the Publisher, as follows :

Si quorum in manus incidit Logarithmo-technia mea, non inviti, opinor, adspicient paucula hæc exempla, miram istius methodi facilitatem cum summa præcissione conjunctam ostendentia.

Expo- nentes	Unitatis ordo		Binarii ordo	
1	1	1	2	2
2	1	0,5	4	2
3	1	0,333333	8	2,666666
4	1	0,25	16	4
5	1	0,2	32	6,4
6	1	0,166666	64	10,666666
7	1	0,142857	128	18,285714
8	1	0,125	256	32
9	1	0,111111	512	56,888888
10	1	0,1	1024	102,4

Duo sunt ordines tabellæ, prior unitatis, alter binarii propago, quorum uterque denorum numerorum primorum logarithmos producit, præter compositorum logarithmos, qui & ipsi requiruntur.

Ex primo ordine

$$\begin{array}{r}
 1 \\
 0,5 \\
 0,33333333 \\
 0,25 \\
 0,2 \\
 0,166666 \\
 0,1428 \\
 0,125 \\
 0,11 \\
 0,1 \\
 + 10033534772 \\
 - 502516792 \\
 \hline
 10536051564 \frac{9}{10} \\
 9531017980 \frac{1}{10}
 \end{array}$$

$$\begin{array}{r}
 1 \\
 0,5 \\
 0,333333 \\
 0,25 \\
 0,2 \\
 + 10000333353 \\
 - 500025 \\
 \hline
 10050335853 \frac{99}{100} \\
 9950330853 \frac{1}{100} \\
 \hline
 \text{Pari modo ex eodem} \\
 \text{ordine procedunt ra-} \\
 \text{tiones } \frac{999}{1000}, \frac{1000}{1001}, \\
 \frac{9999}{10000}, \frac{10000}{10001}, \frac{99999}{100000}, \\
 \frac{100000}{100001}
 \end{array}$$

Ex secundo ordine.

$$\begin{array}{r}
 \begin{array}{c} \cdot \\ 2 \end{array} \\
 \begin{array}{c} \cdot \\ 2 \end{array} \\
 \begin{array}{c} \cdot \\ 266666666 \end{array} \\
 \begin{array}{c} \cdot \\ 4 \end{array} \\
 \begin{array}{c} \cdot \\ 64 \end{array} \\
 \begin{array}{c} \cdot \\ 1066666 \end{array} \\
 \begin{array}{c} \cdot \\ 182857 \end{array} \\
 \begin{array}{c} \cdot \\ 32 \end{array} \\
 \begin{array}{c} \cdot \\ 5689 \end{array} \\
 \begin{array}{c} \cdot \\ 1024 \end{array} \\
 \begin{array}{c} \cdot \\ 186 \end{array} \\
 \begin{array}{c} \cdot \\ 341 \end{array} \\
 \begin{array}{c} \cdot \\ 630 \end{array} \\
 + 20273255404 \\
 - 2041099724
 \end{array}$$

$$\begin{array}{r}
 \begin{array}{c} \cdot \\ 2 \end{array} \\
 \begin{array}{c} \cdot \\ 2 \end{array} \\
 \begin{array}{c} \cdot \\ 2666666 \end{array} \\
 \begin{array}{c} \cdot \\ 4 \end{array} \\
 \begin{array}{c} \cdot \\ .64 \end{array} \\
 \begin{array}{c} \cdot \\ 10 \end{array} \\
 + 20002667306 \\
 - 200040010 \\
 \hline
 20202707316 \frac{98}{100} \\
 19802627296 \frac{100}{100} \\
 \hline
 \text{Haud fecus ex eo-} \\
 \text{dem ordine eliciuntur} \\
 \text{rationes } \frac{998}{1000}, \frac{1000}{1000}, \\
 \frac{9998}{10000}, \frac{10000}{10000}, \frac{99998}{100000}, \\
 \frac{100000}{100000}, \&c.
 \end{array}$$

1		22314355128 $\frac{8}{10}$
2		18232155680 $\frac{10}{12}$
3	$\begin{array}{c} \cdot \\ 1 + 2 \end{array}$	40546510808 $\frac{8}{12} = \frac{2}{3}$
4	ex pc. pag.	10536051564 $\frac{9}{10}$
5	$\begin{array}{c} \cdot \\ 2 + 4 \end{array}$	28768207244 $\frac{9}{12} = \frac{3}{4}$
6	$\begin{array}{c} \cdot \\ 3 + 5 \end{array}$	69314718052 $\frac{2}{4} = \frac{1}{2} = L2^{rii}$
7	$\begin{array}{c} \cdot \\ 6 \times 3 \end{array}$	207944154156 $\frac{1}{8} = L8^{rii}$
8	$\begin{array}{c} \cdot \\ 1 + 7 \end{array}$	230258509284 $\frac{1}{10} = L10^{rii}$
9	ex pc. pag.	9531017980 $\frac{10}{11}$
10	$\begin{array}{c} \cdot \\ 8 + 9 \end{array}$	239789527264 $\frac{1}{11} = L11^{rii}$
11	$\begin{array}{c} \cdot \\ 3 + 6 \end{array}$	109861228860 $\frac{1}{2} + \frac{2}{3} = L3^{rii}$

Similes ordines à 3^{rio} , 4^{rio} , & quovis alio numero derivari possunt, suas quifque rationes exhibiturus.

Acquisito logarithmo 10^{rii} , conficienda est statim tabella reducendorum logarithmorum naturalium ad tabulares, ut quævis ratio, simul ac inventa est, reducat ad mensuram tabularium; ita enim logarithmi compositorum, quorum ope

ope ad primorum logarithmos descenditur, simul fient tabulares absque reductione.

Fiat igitur, ut logarithmus 10^{rii} non-tabularis 2302585 ad tabularem 10000000, ita 1 ad 4,3429448. Hic numerus bis, ter, quater & pluries sumptus constituit tabellam reducendorum logarithmorum naturalium ad tabulares, quam hic subiectam vides.

1	043429448190
2	086858896380
3	130288344570
4	173717792761
5	217147240951
6	260576689141
7	304006137332
8	347435585522
9	390865033712

Hujus igitur ope tabellæ, rationis $\frac{98}{100}$ mensura naturalis 20202707316 reducitur ad tabularem hoc modo:

2	086858896381
0	0
2	0868588964
0	0
2	08685890
7	3040061
0	0
7	30401
3	1303
1	043
6	26

87739243069

Tum à logarithmo 100^{rii}	200000000000000
auferatur rationis $\frac{98}{100}$ mensura	87739243069
restat	19912260756031 = L 98
unde ablato logarithmo 2^{ii}	3010299956640
restat	16901960800291 = L 99
cujus femis	8450980400145 = L 7
Item rationis $\frac{100}{102}$ mensura naturalis	19802627296 reducta, fit 86001717619
Ergo jam logarithmo 100^{rii}	200000000000000
adde rationis $\frac{100}{102}$ mensuram	86001717619
fit	20086001717619 = L 102
unde ablato logarithmo 6^{rii}	7781512503836
restat	12304489213783 = L 17

Hic tabula numerorum primorum egregium usum præstare potest.

Sed & ejusdem primi 17 logarithmorum absque ambage invenire datur, dicendo: 20 . 17 :: 10 . 8|5; tum differentiae inter 10 & 8|5 (nimirum 1|5) sumendo quadrati semissem, cubi trientem, &c, tractandoque istum ordinem, ut supra, inveniemus simul logarithmos absolutorum 23, 197, 203, 1997, 2003, &c.

1	1,5	1,5	15
2	2,25	1,25	1125
3	3,375	1,125	1125
4	5,0625	1,265625	1265625
5	7,59375	1,51875	1518
6	11,390625	1,8984375	189
			22
			+ 15114040
			- 1137845
			16251885 $\frac{1}{2}$
			13976195 $\frac{2}{3}$

Cæterum isthæc omnia, & longè plura ex prop. XIII, XV & XVI Logarithmo-techniæ nostræ apertè derivantur, non magis considerando hyperbolam, quàm si ea nusquam in rerum natura extitisset. Quare frustra sunt, qui hyperbolam ad constructionem logarithmorum vel hilum conferre autumant; imo logarithmorum ope quadrare hyperbolam, verius est. Id quod exemplo ostendere haud pigebit. In diagrammate (Fig. 1) sit AH 74305816 parium, qualium AI est 1, & oporteat invenire aream BIHF.

Opus est ad eam rem tabella subiecta, quæ continet logarithmos naturales suprâ acquisitos, in priori columna ab 1 usque ad 9, in altera à 10 usque ad 1000000000.

1	0000000000	02,30258509299
2	69314718052	04,60517018599
3	109861228860	06,90775527898
4	138629436104	09,21034037198
5	160943791232	11,51292546497
6	179175946912	13,81551055796
7	194591014904	16,11809565096
8	207944154156	18,42068074395
9	219722457720	20,72326583695

Tum prima figura numeri dati semper distinguatur à sequentibus separatrice, hoc modò; 7,4305816, & ipsi primæ figuræ semper adjiciatur 1, ita constantur, hoc loco, 8. Quærenda est nunc rationis 8 ad 7,4305816 mensura naturalis. Id ut fiat commodius, dic: ut 8 ad 7,4305816; ita 1 ad 0,9288227, hunc quartum proportionalem aufer ab 1, reliquum 0,0711773 voco potestatem primam, quæ ducenda est in se ita, ut in facto idem numerus partium extet, qui erat in ipso 0,0711773; productum 0,0050662 est potestas secunda, quæ rursus ducatur in primam 0,0711773, ut idem numerus partium extet, prodit 0,0003606, quæ est tertia potestas; & eodem modo invenitur quarta 0,0000256, & quinta 0,0000018. Deinde

Potestas

Potestas prima	0,0711773	} addantur
Et secundæ semis	25331	
Et tertiæ triens	1202	
Et quartæ quadrans	64	
Et quintæ pars quinta	4	

summa — — — 0,0738374 est mensura rationis 8 ad 7,4305816, eadem scilicet cum ratione 80000000 ad 74305816. Porro logarithmus absoluti 80000000 facile acquiritur ex superiori tabella; cum enim index prima figuræ numeri 80000000 sit 7, è regione 7^{rii} ex secunda columna excerpo logarithmum absoluti 10000000 (hoc est unitatis septem cyphris affectæ)

qui reperitur	16,11809565	} addo
cui subscribo logarithmum 8 ^{rii}	2,07944154	

summa est logarithmus absoluti 80000000
 ablata mensura rationis 80000000 ad 743005816
 restat logarithmus absoluti 74305816
 tanta est area BIHF.

Mantissæ loco accipe modum facillimum quadrandi quamvis hyperbolæ partem per logarithmos tabulares. Dati numeri 74305816 logarithmus tabularis est 7,87102278, per superioris tabellæ columnam secundam reducendus ad naturalem, proditque eadem, quæ supra, area BIHF = 18,123699872.

Postremo, ne quis hæsitati locus restet, accipe, quo pacto ex prop. XIII, xv, xvi, Logarithmot. calculum superiorem derivem.

Differentia terminorum rationem quamvis exprimentium si concipiatur divisa in partes æquales innumeras; composita erit ratio tota extremorum terminorum ex innumeris ratiunculis terminorum à minimo ad maximum infinitissima parte ipsius differentiæ se mutuo excedentium. Sin iidem illi termini innumeri accipiantur pro mediis arithmeticis aliorum terminorum simili parte infinitissima distantium; summa omnium ratiuncularum posterioribus hisce terminis intercedentium deficiet à tota ratione extremorum, non nisi semisse primæ & ultimæ ratiuncularum à prioribus terminis contentarum, id est, ratiuncula minori, quam quæ ullis numeris exprimi possit. Quare posito maximo termino = 1, & parte infinitissima differentiæ = i, & mensura rationis minimæ itidem i; erit ut medium arithmeticum terminorum rationis minimam proximè præcedentis, ad medium arithmeticum terminorum ipsius minimæ; ita mensura minimæ, ad mensuram proxime maioris; hoc est:

$$\begin{array}{rcl}
 1 - i \cdot 1 :: i \cdot i + i^2 + i^3 + i^4 \&c \text{ mensuræ ultimæ} \\
 1 - 2i \cdot 1 :: i \cdot i + 2i^2 + 4i^3 + 8i^4 \&c \text{ penultimæ} \\
 1 - 3i \cdot 1 :: i \cdot i + 3i^2 + 9i^3 + 27i^4 \&c \text{ antepenultimæ}
 \end{array} \left. \vphantom{\begin{array}{l} 1 - i \cdot 1 \\ 1 - 2i \cdot 1 \\ 1 - 3i \cdot 1 \end{array}} \right\} \text{add.}$$

fit summa ratiuncul. = $3i + 6i^2 + 14i^3 + 36i^4 \&c$ = numero terminorum, plus summa eorundem terminorum, plus summa quadratorum ab iisdem, &c.

Sin minimus terminus ponatur $= 1$, manentibus cæteris ut upra; evadit summa ratiuncularum $= 3i - 6i^2 + 14i^3 - 36i^4$, &c.

Hinc data differentia terminorum $= 0.1$, erit numerus terminorum $= 0.1$, & per 16 Logarithmot. summa eorundem terminorum $= 0.005$, & summa quadratorum $= 0.000333$. At data differentia terminorum $= 0.01$, numerus terminorum est $= 0.01$, & summa eorundem $= 0.00005$, & summa quadratorum $= 0.00000333$, &c.

Nota. Prop. iv, Logarithmot. signa speciebus intercedentia debebant esse alternatim affirmata & negata: atque ubicunque lector offenderit *infinitissimam*, legat *infinitesimam*.

R E M A R K S

ON THE

T W O I N F I N I T E S E R I E S E S

$$A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c \text{ and}$$

$$A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \&c,$$

Which were found by Mr. Nicholas Mercator and Dr. John Wallis, in the foregoing Tracts, for the Purpose of squaring the Hyperbolick Spaces *Biru* and *FHru*.

By FRANCIS MASERES, Esq.

CURSITOR BARON OF THE EXCHEQUER.

R E M A R K S, &c.

A R T I C L E I.

THE former of these serieses, to wit, $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$, gives the logarithm of the ratio of $1 + A$ to 1 , whenever A is either equal to, or less than, 1 ; that is, if A be either equal to, or less than, 1 , so as to make the terms of the series $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$ converge, and B be any quantity different from A , and either equal to, or less than, 1 , so as to make the terms of the series $B - \frac{B^2}{2} + \frac{B^3}{3} - \frac{B^4}{4} + \frac{B^5}{5} - \frac{B^6}{6} + \&c$ converge, the series $B - \frac{B^2}{2} + \frac{B^3}{3} - \frac{B^4}{4} + \frac{B^5}{5} - \frac{B^6}{6} + \&c$ will be to the series $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$ in the same proportion as the ratio of $1 + B$ to 1 is to the ratio of $1 + A$ to 1 .

Mr. Mercator's series is the logarithm of the ratio of $1 + A$ to 1 .

2. In like manner the latter of these serieses, to wit, $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \&c$, gives the logarithm of the ratio of 1 to $1 - A$ when A is of any magnitude less than 1 ; that is, if A is of any magnitude less than 1 , and B be any quantity different from A , but also less than 1 , the series $B + \frac{B^2}{2} + \frac{B^3}{3} + \frac{B^4}{4} + \frac{B^5}{5} + \frac{B^6}{6} + \&c$ will be to the series $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \&c$ in the same proportion as the ratio of 1 to $1 - B$ to the ratio of 1 to $1 - A$.

Dr. Wallis's series is the logarithm of the ratio of 1 to $1 - A$.

3. By the former of these serieses (which was invented by Mr. Mercator) we may therefore find the logarithm of any ratio not greater than that of $1 + 1$ to 1 , or 2 to 1 , but not the logarithm of any greater ratio. I mean in a *direct manner*: for *indirectly* we may discover by it the logarithm of any ratio, how great soever; to wit, by dividing such greater ratio into several other ratios, that shall, each of them, be less than the ratio of 2 to 1 , and computing the logarithms of such lesser ratios by the said series, and then adding

Mercator's series will exhibit only the logarithms of ratios of majority that are not greater than the ratio of 2 to 1 .

adding them together into one sum. Thus, for example, the ratio of the number 1057 to 1 is equal to the sum of the several following ratios, to wit, the ratio of 1057 to 1050, the ratio of 1050 to 1000, the ratio of 1000 to 800, the ratio of 800 to 50, (which is equal to the ratio of 16 to 1, or to four times the ratio of 2 to 1, or to 8 times the ratio of $\sqrt{2}$ to 1) and the ratio of 50 to 48, the ratio of 48 to 3, (which is equal to the ratio of 16 to 1, or to 8 times the ratio of $\sqrt{2}$ to 1) the ratio of 3 to $2 + \frac{1}{2}$, the ratio of $2 + \frac{1}{2}$ to 2, the ratio of 2 to $\sqrt{2}$, and the ratio of $\sqrt{2}$ to 1. Therefore, if we were to compute by the said series $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$, the logarithms of the ratios of 1057 to 1050, of 1050 to 1000, of 1000 to 800, of $\sqrt{2}$ to 1, of 50 to 48, of 3 to $2 + \frac{1}{2}$, and of $2 + \frac{1}{2}$ to 2, (every one of which ratios is less than the ratio of 2 to 1), and then were to add together the logarithms of the ratios of 1057 to 1050, of 1050 to 1000, of 1000 to 800, and eighteen times the logarithm of the ratio of $\sqrt{2}$ to 1, and the logarithm of the ratio of 50 to 48, and the logarithm of the ratio of 3 to $2 + \frac{1}{2}$, and that of the ratio of $2 + \frac{1}{2}$ to 2, the sum would be the logarithm of the proposed ratio of 1057 to 1. And in this indirect manner, it is evident that the logarithm of any other ratio, how great soever, might be obtained by means of this series of Mercator. But by a direct, or immediate, application of it we cannot compute the logarithm of any ratio that is greater than that of 2 to 1.

But Dr. Wallis's serieses will exhibit the logarithm of any ratio of majority, how great soever.

4. But by the second series above-mentioned, to wit, the series $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \&c$ (which was invented by Dr. Wallis), it is possible to find at once the logarithm of any ratio, how great soever. For the ratio of 1 to $1 - A$, of which the said series exhibits the logarithm, may, by taking A of a sufficient magnitude, though always less than 1, be made to equal any ratio of majority how great soever. Thus for example, if A is $= \frac{999}{1000}$, we shall have $1 - A = 1 - \frac{999}{1000} = \frac{1000 - 999}{1000} = \frac{1}{1000}$; and consequently the ratio of 1 to $1 - A$ will in this case be equal to the ratio of 1 to $\frac{1}{1000}$, or of 1000 to 1, and therefore the series $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \&c$ will in this case be equal to the logarithm of the ratio of 1000 to 1. And in the same manner it may, by increasing the magnitude of A so as to make it approach still nearer to 1 (though it never can be absolutely equal to it), be made to exhibit the logarithm of any other and greater ratio, how great soever. It may therefore be justly considered as a complete supplement to the former series $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$, which had been invented by Mercator, and which was capable of exhibiting only the logarithms of such ratios as were not greater than the ratio of 2 to 1.

But neither of these serieses is of much use in practice, unless the ratio,

5. It must, however, be observed, that neither of these serieses is of much use in practice when A is but a little less than 1; as for example, when it is equal to $\frac{999}{1000}$, or $\frac{99}{100}$, or even $\frac{9}{10}$; because the convergency of their terms is in those cases so exceeding slow, that it would be necessary to compute a pro-

prodigious number of them in order to obtain the values of these serieses exact to a moderate number of decimal figures. And, when A is absolutely equal to 1, the series $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$, (which in this case becomes $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \&c$, and is equal to the logarithm of the ratio of 2 to 1) converges with such excessive slowness, that it is totally impossible to obtain its value exact to seven or eight places of figures, by the mere computation and addition and subtraction of its terms; though by certain compendious methods that have been invented for the purpose, its value may be found to that, or a still greater degree of accuracy. But, when A is equal only to $\frac{1}{9}$ or $\frac{1}{10}$, or any lesser fraction, the terms of these two serieses will converge with great swiftness, and they will be found exceedingly convenient for the purpose of computing logarithms. And by the computation of these serieses in these easy cases, in which their terms will decrease so swiftly, it will be possible, by a little address and management in the choice of the small ratios, the logarithms of which we compute by them, to find the logarithms of all other ratios, how great soever; of which I propose to give a few examples in the subsequent part of this discourse.

of which the logarithm is sought by it, is a small ratio, or differs but little from a ratio of equality.

But when that ratio is equal to the ratio of 10 to 9, or that of 11 to 10, or any lesser ratio, these serieses will be found very useful.

6. Some learned and ingenious mathematicians have, since the discovery of these two serieses by Mr. Mercator and Dr. Wallis, investigated some other infinite serieses, which converge with a still greater degree of swiftness than these do, in order to facilitate still further the computation of logarithms. But those serieses are less simple in the laws by which their terms are generated than these original ones invented by Mercator and Wallis; and there is usually need of a good deal of care and attention in the application of them to the computation of the logarithm of a given ratio: so that, upon the whole, I think these of Mercator and Wallis deserve to be preferred to them. Nor do I conceive that any more convenient methods of computing the logarithms of given ratios are likely ever to be found out, or indeed need be wished for, than those which are afforded us by these two valuable serieses.

7. These two serieses $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$, and $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \&c$ are nearly of equal use for the purpose of computing the logarithms of given ratios; inasmuch that I know not upon what ground to give one of them the preference to the other. For, though the latter will, in theory, exhibit the logarithm of any given ratio of majority how great soever; whereas the former can only exhibit the logarithm of a ratio that is not greater than the ratio of 2 to 1: yet, in practice, they are neither of them (as we have already observed) fit to exhibit the logarithms of any but small ratios, or such as differ but little from a ratio of equality, and arise from supposing A to be much smaller than 1, as, for example, to be equal to $\frac{1}{9}$ or $\frac{1}{10}$ or $\frac{1}{11}$, or some lesser fraction; and these logarithms they will enable us to compute with nearly the same degree of ease and expedition. It seems therefore to be almost a matter of indifference which of these two serieses we make use of in the business of computing logarithms; and perhaps it may be best in some cases to make use of one of them, and in other cases to make use of the other. Thus, for example, if we wanted

When the ratios, of which the logarithms are to be computed, are small, the two serieses are nearly equally useful.

to find the logarithm of the ratio of 11 to 10 by one of these serieses, it would be rather easier to find it by means of the former series $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$, which was invented by Mercator, than by the latter series $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \&c$, which was invented by Dr. Wallis. For, if we consider the ratio of 11 to 10 as being equal to the ratio of $\frac{11}{10}$ to $\frac{10}{10}$, or of $\frac{11}{10} + \frac{1}{10}$ to $\frac{10}{10}$, or of $1 + \frac{1}{10}$ to 1, its logarithm will be equal to the series $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$, upon a supposition that A is $= \frac{1}{10}$, that is to the series $\frac{1}{10} - \frac{1}{2 \times 10^2} + \frac{1}{3 \times 10^3} - \frac{1}{4 \times 10^4} + \frac{1}{5 \times 10^5} - \frac{1}{6 \times 10^6} + \&c$, or $0,100,000,000 - \frac{0,010,000,000}{2} + \frac{0,001,000,000}{3} - \frac{0,000,100,000}{4} + \frac{0,000,010,000}{5} - \frac{0,000,001,000}{6} + \&c$. But, if we consider the same ratio of 11 to 10 as being equal to the ratio of $\frac{11}{11}$ to $\frac{10}{11}$, or of $\frac{11}{11}$ to $\frac{11}{11} - \frac{1}{11}$, or of 1 to $1 - \frac{1}{11}$, its logarithm will be equal to the series $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \&c$, upon a supposition that A is $= \frac{1}{11}$, that is, to the series $\frac{1}{11} + \frac{1}{2 \times 11^2} + \frac{1}{3 \times 11^3} + \frac{1}{4 \times 11^4} + \frac{1}{5 \times 11^5} + \frac{1}{6 \times 11^6} + \&c$. And it is evident that this latter series, which proceeds by the powers of $\frac{1}{11}$, is not quite so easy to compute as the series $\frac{1}{10} - \frac{1}{2 \times 10^2} + \frac{1}{3 \times 10^3} - \frac{1}{4 \times 10^4} + \frac{1}{5 \times 10^5} - \frac{1}{6 \times 10^6} + \&c$, which proceeds by the powers of $\frac{1}{10}$; and therefore in computing the logarithm of this ratio of 11 to 10, it would be rather more convenient to make use of the series $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$ than of the series $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \&c$. And, on the other hand, in computing the logarithm of the ratio of 10 to 9, it would be rather more convenient to make use of the series $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \&c$ than of the series $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$. For, if we consider the ratio of 10 to 9 as being equal to the ratio of $\frac{10}{9}$ to $\frac{9}{9}$, or of $\frac{10}{9} + \frac{1}{9}$ to $\frac{9}{9}$, or of $1 + \frac{1}{9}$ to 1, its logarithm will be equal to the series $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$, upon a supposition that A is $= \frac{1}{9}$, that is, to the series $\frac{1}{9} - \frac{1}{2 \times 9^2} + \frac{1}{3 \times 9^3} - \frac{1}{4 \times 9^4} + \frac{1}{5 \times 9^5} - \frac{1}{6 \times 9^6} + \&c$, or $\frac{1}{9} - \frac{1}{2 \times 81} + \frac{1}{3 \times 729} - \frac{1}{4 \times 6561} + \frac{1}{5 \times 59,049} - \frac{1}{6 \times 531,441} + \&c$, which proceeds by the powers of $\frac{1}{9}$. But, if we consider the same ratio of 10 to 9 as being equal to the ratio of $\frac{10}{10}$ to $\frac{9}{10}$, or of $\frac{10}{10}$ to $\frac{10}{10} - \frac{1}{10}$, or of 1 to $1 - \frac{1}{10}$, its logarithm will be equal to the series $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \&c$, upon a supposition that A is $= \frac{1}{10}$, that is, to the series $\frac{1}{10} + \frac{1}{2 \times 10^2} + \frac{1}{3 \times 10^3} + \frac{1}{4 \times 10^4} + \frac{1}{5 \times 10^5} + \frac{1}{6 \times 10^6} + \&c$.

$$+ \frac{1}{4 \times 10^4} + \frac{1}{5 \times 10^5} + \frac{1}{6 \times 10^6} + \&c, \text{ or } 0,100,000,000 + \frac{0,010,000,000}{2} +$$

$$\frac{0,001,000,000}{3} + \frac{0,000,100,000}{4} + \frac{0,000,010,000}{5} + \frac{0,000,001,000}{6} + \&c, \text{ which pro-}$$

$$\frac{1}{9} - \frac{1}{2 \times 9^2} + \frac{1}{3 \times 9^3} - \frac{1}{4 \times 9^4} + \frac{1}{5 \times 9^5} - \frac{1}{6 \times 9^6} + \&c, \text{ which proceeds}$$

$$\frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \&c \text{ than of the series } A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} +$$

$$\frac{A^5}{5} - \frac{A^6}{6} + \&c. \text{ But the advantage of the one of these serieses over the other}$$

$$\frac{1}{9} - \frac{1}{2 \times 9^2} + \frac{1}{3 \times 9^3} - \frac{1}{4 \times 9^4} + \frac{1}{5 \times 9^5} - \frac{1}{6 \times 9^6} + \&c, \text{ which proceeds}$$

$$\frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \&c \text{ than of the series } A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} +$$

$$\frac{A^5}{5} - \frac{A^6}{6} + \&c. \text{ But the advantage of the one of these serieses over the other}$$

8. There is, however, one advantage in having both these serieses for the purpose of computing logarithms rather than only one of them, which may deserve to be taken notice of; which is, that they serve to confirm each other, or as proofs of the truth of the arithmetical operations, by which the logarithm of a given ratio has been obtained. Thus, for example, if we had computed the logarithm of the ratio of 11 to 10 by the series $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$ to eighteen or twenty places of figures, and were doubtful whether we had not made a slip in some of the arithmetical operations by which we had obtained it, it would be expedient to compute it also by means of the series $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \&c$ to the same number of places of figures. And, if the value obtained for it by this second series agreed exactly with the value found for it by the first series, we might conclude with confidence, that the arithmetical operations had been rightly performed, and that the value of the said logarithm was rightly assigned.

And they may be used as proofs and confirmations of each other.

9. As these two serieses $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$ and $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \&c$, are so beautiful from their simplicity, and at the same time so fit and convenient for the purpose of computing logarithms, it seems to be desirable that the truth of them should be established in the fullest and clearest manner possible, and by more than one demonstration. The foregoing demonstrations of them are derived from the contemplation of the hyperbola, and that of the latter series $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \&c$, which was invented by Dr. Wallis, is grounded likewise on some of the propositions of that learned author's ARITHMETICA INFINITORUM, which is a work that is now but little read, though formerly it had a great and deserved reputation. I shall therefore here subjoin other investigations of both these serieses, which have no relation to the hyperbola, or to any other curve-lined figure

As these two serieses are so highly useful, it is fit that their truth should be established by more than one demonstration.

Other demonstrations will therefore be given

of them in
the dis-
course.

whatsoever, but are grounded on principles that are purely arithmetical, and derived from the nature of ratios (of which logarithms are nothing more than the measures expressed in numbers), by the help of the celebrated theorem of Sir Isaac Newton for exhibiting, in an infinite series of rational quantities, the roots of a binomial quantity. They are, as I believe, the same in substance with the investigation given of these serieses by Dr. Edmund Halley in his discourse on this subject, in the Philosophical Transactions, N^o 216, and were suggested to my mind by reading that discourse; which, however, is written with so great a degree of obscurity that I am not sure that I rightly understand it. But, whether they express the ideas he meant to convey, and are therefore only a commentary upon his investigation, or whether they are something different from it, I have endeavoured to make them as clear and as easy as I could; and, in order to ensure that essential object in treating of these difficult subjects, have not scrupled to expose myself to the danger of being censured by some of my more acute and learned readers for the much less fault of too great prolixity. The investigation of Mercator's series $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$, which expresses the logarithm of the ratio of $1 + A$ to 1 , is as follows:

The propo-
sition which
is here in-
tended to be
demonstrat-
ed.

10. The proposition of which I here mean to establish the truth, is this:
If there be two different quantities, A and B , that are both of them less than 1 , and of which, consequently, the several powers $A^2, A^3, A^4, A^5, A^6, \&c$, and $B^2, B^3, B^4, B^5, B^6, \&c$, are decreasing quantities, the two infinite serieses $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$ and $B - \frac{B^2}{2} + \frac{B^3}{3} - \frac{B^4}{4} + \frac{B^5}{5} - \frac{B^6}{6} + \&c$, (which, it is evident, will be decreasing progressions) will be measures, or logarithms of the ratios of $1 + A$ to 1 and of $1 + B$ to 1 ; or the series $A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \frac{A^5}{5} - \frac{A^6}{6} + \&c$ will be to the series $B - \frac{B^2}{2} + \frac{B^3}{3} - \frac{B^4}{4} + \frac{B^5}{5} - \frac{B^6}{6} + \&c$ in the same proportion as the ratio of $1 + A$ to 1 is to the ratio of $1 + B$ to 1 .

Or, if we change the notation a little, and substitute the small letters k and q , instead of the capital letters A and B respectively (which will be more agreeable to the notation now most in use in treating of these kinds of serieses), the proposition which we are to demonstrate the truth of, will be as follows:

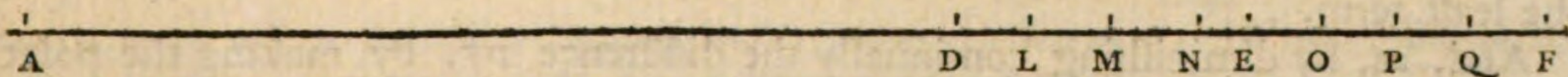
If there be two different quantities k and q , that are both of them less than 1 , and of which consequently the several powers $k^2, k^3, k^4, k^5, k^6, \&c$, and $q^2, q^3, q^4, q^5, q^6, \&c$, will be decreasing quantities, the two infinite serieses $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$ and $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c$, (which, it is evident, will be decreasing progressions), will be measures, or logarithms, of the ratios of $1 + k$ to 1 and of $1 + q$ to 1 ; or the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$ will be to the series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c$ in the same proportion as the ratio of $1 + k$ to 1 is to the ratio of $1 + q$ to 1 .

In

In order to demonstrate the truth of this proposition, it will be expedient to premise the two following propositions, as lemmas.

L E M M A I.

11. If there be three unequal quantities, as the right lines AD, AE, and AF in the figure hereunto annexed, whereof AD is the least, and AF is the greatest, and their differences DE, EF, are extremely small in comparison of the quantities themselves (as for example, less than a millionth part of either of them), the proportion of the ratio of AF (the greatest of the three quantities) to AD, (the least of them), to the ratio of AE (the middle quantity) to AD (the least quantity) will be very nearly equal to the proportion of the difference, DF, of the terms of the former ratio, to the difference, DE, of the terms of the latter ratio. And the differences DF, DE may be taken so very small in comparison of the least term AD, that the proportion of the said differences shall approach as near as we please to the proportion of the foresaid ratios, or shall differ from it by a quantity, or ratio, that is less than any assigned ratio whatsoever.



D E M O N S T R A T I O N.

Let the ratio of AF, the greatest of the three quantities, to AD, the least of them, be divided into a very great number of lesser ratios, all equal to each other, by the insertion of intermediate proportionals, as AL, AM, AN, &c, between AD and AF.

Then, since AM is to AL as AL is to AD, it will follow, *dividendo*, that LM is to AL as DL to AD, and, *permutando*, that LM is to DL as AL to AD. And, in like manner, because AN is to AM as AM is to AL, it will follow, *dividendo*, that MN is to AM as LM is to AL, and, *permutando*, that MN is to LM as AM is to AL. Therefore, when AD, AL, AM, and AN are nearly equal to each other, the differences DL, LM, and MN will also be nearly equal to each other. Therefore, when the two extreme terms AD and AF (and consequently, *à fortiori*, all the intermediate terms, AL, AM, AN, &c, between the said two extreme terms), are nearly equal to each other (as for example, when AF exceeds AD only in the proportion of 1000,001 to 1000,000), all the differences DL, LM, MN, &c contained between the points D and F, will also be nearly equal to each other, and more nearly than AD is to AF: so that, if AF is to AD as 1000,001 to 1000,000, the common ratio of MN to LM, and of LM to DL (being the same with the common ratio of AN to AM, and of AM to AL, and of AL to AD), will be less, or nearer to a ratio of equality, than the ratio of 1000,001 to 1000,000.

I i

Now,

Now, when two lines or other quantities, of unequal magnitudes, are divided into parts of the same size, the length or magnitude of the greater of the two will be to the length or magnitude of the lesser in the same proportion as the number of those equal parts contained in the greater to the number of them contained in the lesser. Therefore, when AF is to AD in a proportion that approaches very nearly to a ratio of equality (as for example, when it exceeds AD only in the proportion of 1000,001 to 1000,000, or the excess of AF above AD is equal to only one millionth part of AD), and consequently the differences DL, LM, and MN, &c, are nearly equal to each other, the line DF will be to the line DE very nearly in the same proportion as the number of the lesser differences DL, LM, MN, &c, contained between the points D and F, to the number of those differences contained between the points D and E, and consequently in the same proportion as the number of the equal ratios of AL to AD, AM to AL, AN to AM, &c, contained in the ratio of AF to AD, to the number of those ratios contained in the ratio of AE to AD, and therefore in the same proportion as the ratio of AF to AD to the ratio of AE to AD; that is, DF, the difference of the greatest and least terms AF and AD, is to DE, the difference of the middle term AE and the least term AD, in this case very nearly in the same proportion as the ratio of AF, the greatest term, to AD, the least term, is to the ratio of AE, the middle term, to AD, the least term.

And, as, by diminishing continually the difference DF, by making the point F approach nearer and nearer to the point D, the proportion of AF to AD may be made to approach as near as we please to a ratio of equality, it follows that that difference may be taken so small, in comparison of AD, that the common ratio of the differences DL, LM, MN, &c to each other shall approach as near as we please to a ratio of equality, and consequently that the proportion of DF to DE shall be as nearly equal as we please to the proportion of the ratio of AF to AD to the ratio of AE to AD. Q. E. D.

LEMMA II.

12. If x be any quantity not greater than 1, and n be any whole number

whatsoever, the quantity $1 + x^n$, or $\sqrt[n]{1 + x}$, that is, the n^{th} root of the binomial quantity $1 + x$, will be equal to the following infinite series of decreasing terms, to wit, $1 + \frac{1}{n} \times x - \frac{1}{n} \times \frac{n-1}{2n} \times x^2 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times x^3 - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times x^4 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n} \times x^5 - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n} \times \frac{5n-1}{6n} \times x^6 + \&c \text{ ad infinitum, or (if we put } A = 1, \text{ and } B = \frac{1}{n}, \text{ and } C = \frac{1}{n} \times \frac{n-1}{2n}, \text{ and } D = \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}, \text{ and } E, F, G, H, I, K, \&c \text{ for the several$

Sir Isaac
Newton's
binomial
theorem in
the case of
roots, ex-
pressed in the
plainest and
clearest man-
ner.

ral following co-efficients of the powers of x in this series), to the series $1 + \frac{1}{n} x$
 $Ax - \frac{n-1}{2n} \times Bx^2 + \frac{2n-1}{3n} \times Cx^3 - \frac{3n-1}{4n} \times Dx^4 + \frac{4n-1}{5n} \times Ex^5 - \frac{5n-1}{6n}$
 $\times Fx^6 + \frac{6n-1}{7n} \times Gx^7 - \frac{7n-1}{8n} \times Hx^8 + \frac{8n-1}{9n} \times Ix^9 - \frac{9n-1}{10n} \times Kx^{10}$
 $+ \&c$ ad infinitum; in which series the several co-efficients of $x, x^2, x^3, x^4,$
 $x^5, x^6, x^7, x^8, x^9, x^{10}, \&c,$ are derived, or generated, from the first term 1 by
the continual, or successive, multiplication of the fractions $\frac{1}{n}, \frac{n-1}{2n}, \frac{2n-1}{3n},$
 $\frac{3n-1}{4n}, \frac{4n-1}{5n}, \frac{5n-1}{6n}, \frac{6n-1}{7n}, \frac{7n-1}{8n}, \frac{8n-1}{9n}, \frac{9n-1}{10n}, \&c,$ which are there-
fore called their *generating fractions*.

The gene-
rating frac-
tions of the
terms.

13. In these generating fractions, it is evident that $n-1$, the numerator
of the second fraction $\frac{n-1}{2n}$, is formed from 1, the numerator of the first
fraction $\frac{1}{n}$ by the addition of $n-2$, or the addition of n and the subtraction of
2; for $1 + n - 2$ is $= n - 1$: And it is evident likewise, that the numera-
tors of all the following fractions after the second fraction $\frac{n-1}{2n}$, to wit, the
numerators $2n-1, 3n-1, 4n-1, 5n-1, 6n-1, 7n-1, 8n-1,$
 $9n-1, \&c,$ are formed from the numerator $n-1$ of the said second fraction,
and from each other by the continual addition of n to every preceding nume-
rator. And the denominators of the second fraction $\frac{n-1}{2n}$ and the third frac-
tion $\frac{2n-1}{3n}$ and all the following fractions, to wit, the denominators $2n, 3n,$
 $4n, 5n, 6n, 7n, 8n, 9n, 10n, \&c,$ are formed from n (the denominator of
the first fraction $\frac{1}{n}$), and from each other by the continual addition of n to
the preceding denominator.

The consti-
tution, or
law, of the
said gene-
rating frac-
tions.

14. This is Sir Isaac Newton's famous binomial theorem in the case of roots
or fractional powers, expressed in what I take to be the most intelligible and con-
venient manner possible; as by this way of expressing it we avoid all mention of
negative quantities in the co-efficients of the several terms of the series, from
the use of which negative quantities, I have observed that a good deal of per-
plexity often arises in other ways of expressing it. This expression of the value of
 $1 + x^{\frac{1}{n}}$ is derived from the series $1 + \frac{m}{1} Ax + \frac{m-1}{2} Bx^2 + \frac{m-2}{3} Cx^3 +$
 $\frac{m-3}{4} Dx^4 + \frac{m-4}{5} Ex^5 + \frac{m-5}{6} Fx^6 + \&c$ (which is $= 1 + x^m$) by sub-
stituting $\frac{1}{n}$ in its terms instead of m .

15. For a demonstration of this celebrated theorem in the case of roots, or
fractional powers, I refer the reader to a learned treatise of Algebra, lately
published at Dublin by Dr. Hales, a fellow of Trinity College in that city,
and intitled *Analysis Aequationum*. In this valuable work (which was published
in the year 1784) the reader will find, amongst many other curious matters,

Books in
which the
reader may
find demon-
strations of
this theorem.

a demonstration of this theorem of Sir Isaac Newton, that extends to the case of roots, or fractional powers, as well as to the case of integral powers. See that treatise, pages 33, 34, 35, &c, — 39. And he may likewise see another and very satisfactory demonstration of this theorem in the case of roots, or fractional powers, in the Mathematical Tracts of the very learned Dr. Charles Hutton, Professor of Mathematics at the Royal Military Academy at Woolwich, which were published at London in the year 1786. See the said Tracts, Tract vi, pages 65, 66, 67, &c, — 79. On the present occasion I shall take the truth of this theorem for granted.

16. Coroll. 1. It follows from the foregoing lemma, that $\sqrt[n]{1+x} - 1$, or $\sqrt[n]{1+x} - 1$, or the excess of the n^{th} root of the binomial quantity $1+x$ above 1, is equal to the infinite series $\frac{1}{n} Ax - \frac{n-1}{2n} Bx^2 + \frac{2n-1}{3n} Cx^3 - \frac{3n-1}{4n} Dx^4 + \frac{4n-1}{5n} Ex^5 - \frac{5n-1}{6n} Fx^6 + \&c$; in which A is, as before, equal to 1; and B is $= \frac{1}{n} A$, or the co-efficient of x ; and c is $= \frac{n-1}{2n} B$, or the co-efficient of x^2 ; and D is $= \frac{2n-1}{3n} c$, or the co-efficient of x^3 ; and the following capital letters E, F, G, &c, are equal to, or stand for, the co-efficients of x^4 , x^5 , x^6 , and the other following powers of x .

17. Coroll. 2. If n be a very large number, as for example, the ninth power of a million, or 1 with 54 cyphers annexed to it, or 1,000000,000000,000000,000000,000000,000000,000000,000000, (which Mr. Locke, in his

Essay on Human Understanding, calls a nonillion), the quantity $\sqrt[n]{1+x} - 1$, or the excess of the n^{th} root of the binomial quantity $1+x$ above 1, will be very nearly equal to the series $\frac{x}{n} - \frac{x^2}{2n} + \frac{x^3}{3n} - \frac{x^4}{4n} + \frac{x^5}{5n} - \frac{x^6}{6n} + \&c$, *ad infinitum*, or to $\frac{1}{n} \times$ the series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \&c$, *ad infinitum*: and n may be taken of so great a magnitude, that the ratio of $\frac{1}{n} \times$ the series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \&c$, to

$\sqrt[n]{1+x} - 1$ shall approach as near to a ratio of equality as we please.

For, when n is equal to a nonillion, or any such very great number, it is evident that $n-1$ and $2n-1$ and $3n-1$ and $4n-1$ and $5n-1$, &c, will be very nearly equal to n , $2n$, $3n$, $4n$, $5n$, &c, on account of the immense magnitudes of the numbers n , $2n$, $3n$, $4n$, $5n$, &c in comparison of the unit which is subtracted from them. Therefore the series $\frac{1}{n} Ax - \frac{n-1}{2n} Bx^2 + \frac{2n-1}{3n} Cx^3 - \frac{3n-1}{4n} Dx^4 + \frac{4n-1}{5n} Ex^5 - \frac{5n-1}{6n} Fx^6 + \&c$ (which is $=$

$\sqrt[n]{1+x} - 1$) will in this case be very nearly equal to $\frac{1}{n} Ax - \frac{n}{2n} Bx^2 + \frac{2n}{3n} Cx^3$

$ex^3 - \frac{3^n}{4^n} Dx^4 + \frac{4^n}{5^n} Ex^5 - \frac{5^n}{6^n} Fx^6 + \&c, = \frac{1}{n} Ax - \frac{1}{2} Bx^2 + \frac{2}{3}$
 $Cx^3 - \frac{3}{4} Dx^4 + \frac{4}{5} Ex^5 - \frac{5}{6} Fx^6 + \&c, = \frac{1}{n} \times 1 \times x - \frac{1}{2} \times \frac{1}{n} \times 1 \times$
 $x^2 + \frac{2}{3} \times \frac{1}{2} \times \frac{1}{n} \times 1 \times x^3 - \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{n} \times 1 \times x^4 + \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3}$
 $\times \frac{1}{2} \times \frac{1}{n} \times 1 \times x^5 - \frac{5}{6} \times \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{n} \times 1 \times x^6 + \&c, = \frac{1}{n}$
 $\times x - \frac{1}{n} \times \frac{x^2}{2} + \frac{1}{n} \times \frac{x^3}{3} - \frac{1}{n} \times \frac{x^4}{4} + \frac{1}{n} \times \frac{x^5}{5} - \frac{1}{n} \times \frac{x^6}{6} + \&c, =$
 $\frac{1}{n} \times \text{the series } x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \&c. \text{ Therefore } \frac{1}{1+x}^{\frac{1}{n}}$
 $- 1 \text{ will in this case be very nearly equal to } \frac{1}{n} \times \text{the series } x - \frac{x^2}{2} + \frac{x^3}{3}$
 $- \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \&c, \text{ ad infinitum. And it is evident, that if any ratio}$
 $\text{whatsoever be assigned, that differs very little from a ratio of equality, } n \text{ may}$
 $\text{be taken of so great a magnitude that the proportion of } \frac{1}{1+x}^{\frac{1}{n}} - 1 \text{ to } \frac{1}{n}$
 $\times \text{the series } x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \&c, \text{ shall approach still nearer}$
 $\text{to a ratio of equality than such assigned ratio. Q. E. D.}$

18. These lemmas being premised, the main proposition may be stated and demonstrated in the manner following.

THEOREM I.

If k and q are any two quantities less than 1, whereof we will suppose k to be the greater; the ratio of $1 + k$ to 1 will be to the ratio of $1 + q$ to 1 in the same proportion as the infinite series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$ to the infinite series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c.$

DEMONSTRATION.

Let n be put, as before, for any very large number, as, for example, for a nonillion, or the ninth power of a million.

Then, by the 2nd corollary of the 2nd of the foregoing lemmas, we shall have

$\frac{1}{1+k}^{\frac{1}{n}} - 1, \text{ very nearly, } = \frac{1}{n} \times \text{the series } k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} +$
 $\&c, \text{ ad infinitum, and } \frac{1}{1+q}^{\frac{1}{n}} - 1, \text{ very nearly, } = \frac{1}{n} \times \text{the series } q - \frac{q^2}{2} +$
 $\frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c. \text{ And, by increasing the magnitude of } n, \text{ each}$
 $\text{of these ratios may be made to come as near to a ratio of equality as we}$

please.

please. Therefore the proportion of $\sqrt[n]{1+k} - 1$ to $\sqrt[n]{1+q} - 1$ is in this case very nearly the same as that of $\frac{1}{n} \times$ the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$, to $\frac{1}{n} \times$ the series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c$, and consequently, as that of n times $\frac{1}{n} \times$ the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$, to n times $\frac{1}{n} \times$ the series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c$, or as that of the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$, to the series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c$; that is, the proportion of the excess of the n^{th} root of the binomial quantity $1+k$ above 1 to the excess of the n^{th} root of the binomial quantity $1+q$ above 1 is, in this case of the very great magnitude of n , very nearly the same as that of the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$ *ad infinitum* to the series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c$ *ad infinitum*.

Now the ratio of $1+k$ to 1 is to the ratio of $1+q$ to 1 in the same proportion as any given part of the former ratio, is to the like part of the latter ratio, and consequently, as the n^{th} , or nonillionth, part of the former ratio is to the n^{th} , or nonillionth, part of the latter ratio. But the ratio of the n^{th} root of $1+k$ to 1 is the n^{th} part of the ratio of $1+k$ to 1; and the ratio of the n^{th} root of $1+q$ to 1 is the n^{th} part of the ratio of $1+q$ to 1. Therefore the ratio of $1+k$ to 1 is to the ratio of $1+q$ to 1 in the same proportion as the very small ratio of the n^{th} root of $1+k$ to 1 is to the very small ratio of the n^{th} root of $1+q$ to 1.

But it has been shewn above in lemma 1, that when three quantities are very nearly equal to each other, the ratio of the greatest of the three to the least will be to the ratio of the middle quantity to the least in very nearly the same proportion as the excess of the greatest quantity above the least is to the excess of the middle quantity above the least: and by diminishing the greatest and middle quantities continually, so that they shall approach nearer and nearer to an equality with the least, we may make these two proportions, to wit, that of the said two ratios and that of the said two excesses, approach as near to each other as we please. Therefore in these three quantities, to wit, 1 and the n^{th} , or nonillionth, root of $1+q$ and the n^{th} , or nonillionth, root of $1+k$, (in which the excesses of the two latter quantities above the first quantity 1 are so extremely small that they will not appear before the 54^{th} place of decimal frac-

tions), we may conclude, that the ratio of the greatest, to wit, $\sqrt[n]{1+k}$, or the n^{th} root of $1+k$, to 1, the least, will be to the ratio of the second quantity,

to wit, $\sqrt[n]{1+q}$, or the n^{th} root of $1+q$, to 1, the least, very nearly in the same

same proportion as the excess of the greatest, or $\sqrt[n]{1+k}$, above the least, or

1, to the excess of the second quantity, or $\sqrt[n]{1+q}$, above the least, or 1.

But it has been before shewn, that the ratio of $1+k$ to 1 is to the ratio of $1+q$ to 1 in the same proportion as the very small ratio of the n^{th} root of $1+k$ to 1 is to the very small ratio of the n^{th} root of $1+q$ to 1.

Therefore the ratio of $1+k$ to 1 will be to the ratio of $1+q$ to 1 very nearly in the same proportion as the excess of the n^{th} root of $1+k$ above 1 to the excess of the n^{th} root of $1+q$ above 1.

But it has been shewn above, that the proportion of the excess of the n^{th} root of $1+k$ above 1 to the excess of the n^{th} root of $1+q$ above 1 is, in this case of the very great magnitude of n , very nearly the same as that of the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$ *ad infinitum* to the series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c$ *ad infinitum*.

Therefore the ratio of $1+k$ to 1 will be to the ratio of $1+q$ to 1, in the same proportion as the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$ *ad infinitum* to the series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c$ *ad infinitum*.

Q. E. D.

19. The foregoing demonstration may be expressed in fewer words, as follows :

Let n be put as before, for any large number ; as, for example, for a nonillion, or the ninth power of a million.

The foregoing demonstration expressed in fewer words.

Then it will follow from lemma 2, coroll. 2, that $\sqrt[n]{1+k} - 1$ will be very nearly equal to $\frac{1}{n} \times$ the infinite series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$, and that $\sqrt[n]{1+q} - 1$ will, in like manner, be, very nearly, equal to $\frac{1}{n} \times$ the series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c$. Therefore $\sqrt[n]{1+k} - 1$ will be to $\sqrt[n]{1+q} - 1$, very nearly, in the same proportion as $\frac{1}{n} \times$ the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$ to $\frac{1}{n} \times$ the series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c$, that is, in the same proportion as the n^{th} part of the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$ to the n^{th} part of the series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c$, and consequently in the same proportion as the whole series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$ to the whole series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c$.

Now

Now the ratio of $1 + k$ to 1 is to the ratio of $1 + q$ to 1 in the same proportion as the n^{th} part of the former ratio is to the n^{th} part of the latter ratio, and consequently in the same proportion as the ratio of $\sqrt[n]{1 + k}$ to 1 is to the ratio of $\sqrt[n]{1 + q}$ to 1 .

But, because $\sqrt[n]{1 + k}$ and $\sqrt[n]{1 + q}$ approach extremely near to an equality with 1 , it follows from lemma 1, that the ratio of $\sqrt[n]{1 + k}$ to 1 will be to the ratio of $\sqrt[n]{1 + q}$ to 1 , very nearly, in the same proportion as the excess of $\sqrt[n]{1 + k}$ above 1 to the excess of $\sqrt[n]{1 + q}$ above 1 , or as $\sqrt[n]{1 + k} - 1$ to $\sqrt[n]{1 + q} - 1$.

Therefore the ratio of $1 + k$ to 1 will be to the ratio of $1 + q$ to 1 , very nearly, in the same proportion as $\sqrt[n]{1 + k} - 1$ is to $\sqrt[n]{1 + q} - 1$.

But it has been shewn, that, in this case of the very great magnitude of n , the proportion of $\sqrt[n]{1 + k} - 1$ to $\sqrt[n]{1 + q} - 1$ is, very nearly, the same as that of the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$ *ad infinitum* to the series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c$ *ad infinitum*.

Therefore the ratio of $1 + k$ to 1 will be to the ratio of $1 + q$ to 1 in the same proportion as the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$ *ad infinitum* to the series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c$ *ad infinitum*. Q. E. D.

20. And, if still greater brevity of expression be desired, the foregoing demonstration may be expressed in a still conciser manner by the help of a little additional notation, as follows:

The same demonstration expressed with still greater brevity.

Let $R. \frac{1 + k}{1}$ be used to denote the ratio of $1 + k$ to 1 , and $R. \frac{1 + q}{1}$ to denote the ratio of $1 + q$ to 1 , and $R. \frac{\sqrt[n]{1 + k}}{1}$ to denote the ratio of $\sqrt[n]{1 + k}$ to 1 , and $R. \frac{\sqrt[n]{1 + q}}{1}$ to denote the ratio of $\sqrt[n]{1 + q}$ to 1 , and $\frac{1}{n} \times R. \frac{1 + k}{1}$ to denote the n^{th} part of the ratio of $1 + k$ to 1 , and $\frac{1}{n} \times R. \frac{1 + q}{1}$ to denote the n^{th} part of the ratio of $1 + q$ to 1 .

With this notation the demonstration will be as follows:

Let n be put, as before, for any very large number, as, for example, for a nonillion, or the ninth power of a million.

Then

Then will $R. \frac{1+k}{1}$ be to $R. \frac{1+q}{1}$ in the same proportion as $\frac{1}{n} \times R. \frac{1+k}{1}$ to $\frac{1}{n} \times R. \frac{1+q}{1}$, that is (by the nature of roots), as $R. \frac{1+k}{1}^{\frac{1}{n}}$ to $R. \frac{1+q}{1}^{\frac{1}{n}}$, that is, by lemma 1 (on account of the near approach of $\frac{1+k}{1}^{\frac{1}{n}}$ and $\frac{1+q}{1}^{\frac{1}{n}}$ to an equality with 1), as $\frac{1+k}{1}^{\frac{1}{n}} - 1$ to $\frac{1+q}{1}^{\frac{1}{n}} - 1$, that is, by lemma 2, coroll. 2 (on account of the very great magnitude of the number n), as $\frac{1}{n} \times$ the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c.$ to $\frac{1}{n} \times$ the series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \mathcal{E}c.$ and consequently (multiplying both sides by n), as the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c.$ to the series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \mathcal{E}c.$ Q. E. D.

SCHOLIUM.

21. This proposition is *accurately* true, though, when n is of any finite magnitude, how great soever, some of the intermediate propositions, by means of which it is proved, are only *very nearly* true. For, as these intermediate propositions are not limited in the degree in which they approach to truth or accuracy, but may be made to come as near to being accurately true as we please, by increasing the number n , the conclusion derived from them must be accurately true. For, if it were supposed to be not accurately true, but only to approach within certain limits of the truth, we might increase the magnitude of n till the said conclusion was made to approach nearer to being accurately true than the assigned limits; which would be contrary to the supposition of its having approached only within the said assigned limits of the truth, and consequently would prove that the said supposition was false. Therefore the said conclusion does not only approach within certain limits of the truth, but is accurately true.

22. We have now, I hope, sufficiently established the truth of Mr. Mercator's proposition, to wit, "that if q be made to represent successively several different fractions, or quantities less than 1, the corresponding values of the infinite series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \mathcal{E}c.$ will be proportional to the several ratios of the corresponding values of $1 + q$ to 1 respectively, or will be the logarithms of those ratios." It remains that we illustrate the use of this series in the business of computing logarithms by applying it to a few examples. I shall therefore now proceed to apply it to the computation of the logarithms of the nine following ratios, to wit, the ratio of 10 to 9, that of 11 to 10, that of 81 to 80, that of 121 to 120, that of 2401 to 2400, that of

169 to 168, that of 289 to 288, that of 361 to 360, and that of 529 to 528; by means of which we shall be able to discover the logarithms of the ratios of the 23 first natural numbers, to wit, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, and 24, to 1, or (according to the common abridged and imperfect way of speaking on this subject), the logarithms of those numbers. By these computations the great utility and excellency of this series will be most manifest. And as in making these computations in the instances above-mentioned the numerator of the fraction that is equal to q is always 1, and the denominator of it is a whole number, I think it will be convenient to substitute $\frac{1}{m}$ instead of q in the binomial quantity $1 + q$ and the infinite series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \mathcal{E}c.$ whereby the said binomial quantity will be changed into $1 + \frac{1}{m}$, and the said series will be changed into the series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \mathcal{E}c.$ or the logarithm of the ratio of $1 + \frac{1}{m}$ to 1 will be the series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \mathcal{E}c. ad infinitum.$

23. If we should have occasion to express this series in such a manner as to point out the generating fractions by the successive multiplication of which its second and third and other following terms are derived from the first term $\frac{1}{m}$ and from each other, it may be done by putting A for the whole first term $\frac{1}{m}$, and B for the whole second term $\frac{1}{2m^2}$, and C for the whole third term $\frac{1}{3m^3}$, and D for the whole fourth term $\frac{1}{4m^4}$, and E, F, G, H, I, K, L, M, and the following capital letters of the alphabet, for the whole fifth, sixth, seventh, eighth, ninth, tenth, eleventh, twelfth, and other following terms of the series; by which substitution the said series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \frac{1}{7m^7} - \frac{1}{8m^8} + \frac{1}{9m^9} - \frac{1}{10m^{10}} + \mathcal{E}c$ will be converted into the series $\frac{1}{m} - \frac{A}{2m} + \frac{2B}{3m} - \frac{3C}{4m} + \frac{4D}{5m} - \frac{5E}{6m} + \frac{6F}{7m} - \frac{7G}{8m} + \frac{8H}{9m} - \frac{9I}{10m} + \mathcal{E}c.$ But the terms of the series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \frac{1}{7m^7} - \frac{1}{8m^8} + \frac{1}{9m^9} - \frac{1}{10m^{10}} + \mathcal{E}c$ are so extremely simple and concise that there seems to be no occasion to change its form.

EXAMPLES

E X A M P L E S

OF THE

COMPUTATION OF THE LOGARITHMS

OF THE

RATIOS OF SEVERAL NUMBERS

Denoted by the binomial quantity $1 + \frac{1}{m}$

T O I.

By means of the infinite series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \frac{1}{7m^7}$
 $- \frac{1}{8m^8} + \frac{1}{9m^9} - \frac{1}{10m^{10}} + \frac{1}{11m^{11}} - \frac{1}{12m^{12}} + \frac{1}{13m^{13}} - \frac{1}{14m^{14}} + \frac{1}{15m^{15}} - \frac{1}{16m^{16}}$
 $+ \frac{1}{17m^{17}} - \frac{1}{18m^{18}} + \&c,$

INVENTED BY MR. NICHOLAS MERCATOR.

E X A M P L E I.

24. **L**ET it be required to find by means of the said series the logarithm
of the ratio of 10 to 9 (or of $9 + 1$ to 9, or of $\frac{9+1}{9}$ to $\frac{9}{9}$), or
of $1 + \frac{1}{9}$ to 1.

Here m is = 9, and $\frac{1}{m}$ is = $\frac{1}{9} = 0.111,111,111,111,111,111, \&c.$

We shall therefore have

$$\frac{1}{m^2} (= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{9} = \frac{0.111,111,111,111,111,111, \&c}{9}) = 0.012,345,679,012,345,679;$$

$$\text{And } \frac{1}{m^3} (= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{9} = \frac{0.012,345,679,012,345,679}{9}) = 0.001,371,742,112,482,853;$$

$$\text{And } \frac{1}{m^4} (= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{9} = \frac{0.001,371,742,112,482,853}{9}) = 0.000,152,415,790,275,872;$$

K k 2

And

$$\text{And } \frac{1}{m^5} (= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{9} = \frac{0.000,152,415,790,275,872}{9}) = 0.000,016,935,087,808,430;$$

$$\text{And } \frac{1}{m^6} (= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{9} = \frac{0.000,016,935,087,808,430}{9}) = 0.000,001,881,676,423,158;$$

$$\text{And } \frac{1}{m^7} (= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{9} = \frac{0.000,001,881,676,423,158}{9}) = 0.000,000,209,075,158,128;$$

$$\text{And } \frac{1}{m^8} (= \frac{1}{m^7} \times \frac{1}{m} = \frac{1}{m^7} \times \frac{1}{9} = \frac{0.000,000,209,075,158,128}{9}) = 0.000,000,023,230,573,125;$$

$$\text{And } \frac{1}{m^9} (= \frac{1}{m^8} \times \frac{1}{m} = \frac{1}{m^8} \times \frac{1}{9} = \frac{0.000,000,023,230,573,125}{9}) = 0.000,000,002,581,174,791;$$

$$\text{And } \frac{1}{m^{10}} (= \frac{1}{m^9} \times \frac{1}{m} = \frac{1}{m^9} \times \frac{1}{9} = \frac{0.000,000,002,581,174,791}{9}) = 0.000,000,000,286,797,199;$$

$$\text{And } \frac{1}{m^{11}} (= \frac{1}{m^{10}} \times \frac{1}{m} = \frac{1}{m^{10}} \times \frac{1}{9} = \frac{0.000,000,000,286,797,199}{9}) = 0.000,000,000,031,866,355;$$

$$\text{And } \frac{1}{m^{12}} (= \frac{1}{m^{11}} \times \frac{1}{m} = \frac{1}{m^{11}} \times \frac{1}{9} = \frac{0.000,000,000,031,866,355}{9}) = 0.000,000,000,003,540,706;$$

$$\text{And } \frac{1}{m^{13}} (= \frac{1}{m^{12}} \times \frac{1}{m} = \frac{1}{m^{12}} \times \frac{1}{9} = \frac{0.000,000,000,003,540,706}{9}) = 0.000,000,000,000,393,411;$$

$$\text{And } \frac{1}{m^{14}} (= \frac{1}{m^{13}} \times \frac{1}{m} = \frac{1}{m^{13}} \times \frac{1}{9} = \frac{0.000,000,000,000,393,411}{9}) = 0.000,000,000,000,043,712;$$

$$\text{And } \frac{1}{m^{15}} (= \frac{1}{m^{14}} \times \frac{1}{m} = \frac{1}{m^{14}} \times \frac{1}{9} = \frac{0.000,000,000,000,043,712}{9}) = 0.000,000,000,000,004,856;$$

$$\text{And } \frac{1}{m^{16}} (= \frac{1}{m^{15}} \times \frac{1}{m} = \frac{1}{m^{15}} \times \frac{1}{9} = \frac{0.000,000,000,000,004,856}{9}) = 0.000,000,000,000,000,539;$$

$$\text{And } \frac{1}{m^{17}} (= \frac{1}{m^{16}} \times \frac{1}{m} = \frac{1}{m^{16}} \times \frac{1}{9} = \frac{0.000,000,000,000,000,539}{9}) = 0.000,000,000,000,000,059;$$

$$\text{And } \frac{1}{m^{18}} (= \frac{1}{m^{17}} \times \frac{1}{m} = \frac{1}{m^{17}} \times \frac{1}{9} = \frac{0.000,000,000,000,000,059}{9}) = 0.000,000,000,000,000,006;$$

And consequently

$$\frac{1}{2m^2} (= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.012,345,679,012,345,679}{2}) = 0.006,172,839,506,172,839;$$

$$\text{And } \frac{1}{3m^3} (= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.001,371,742,112,482,853}{3}) = 0.000,457,247,370,827,617;$$

$$\text{And } \frac{1}{4m^4} (= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,152,415,790,275,872}{4}) = 0.000,038,103,947,568,968;$$

$$\text{And } \frac{1}{5m^5} (= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,016,935,087,808,430}{5}) = 0.000,003,387,017,561,686;$$

$$\text{And } \frac{1}{6m^6} (= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,001,881,676,423,158}{6}) = 0.000,000,313,612,737,193;$$

$$\text{And } \frac{1}{7m^7} (= \frac{1}{m^7} \times \frac{1}{7} = \frac{0.000,000,209,075,158,128}{7}) = 0.000,000,029,867,879,732;$$

And

$$\text{And } \frac{1}{8m^8} (= \frac{1}{m^8} \times \frac{1}{8} = \frac{0.000,000,023,230,573,125}{8}) = 0.000,000,002,903,821,640;$$

$$\text{And } \frac{1}{9m^9} (= \frac{1}{m^9} \times \frac{1}{9} = \frac{0.000,000,002,581,174,791}{9}) = 0.000,000,000,286,797,199;$$

$$\text{And } \frac{1}{10m^{10}} (= \frac{1}{m^{10}} \times \frac{1}{10} = \frac{0.000,000,000,286,797,199}{10}) = 0.000,000,000,028,679,719;$$

$$\text{And } \frac{1}{11m^{11}} (= \frac{1}{m^{11}} \times \frac{1}{11} = \frac{0.000,000,000,031,866,355}{11}) = 0.000,000,000,002,896,941;$$

$$\text{And } \frac{1}{12m^{12}} (= \frac{1}{m^{12}} \times \frac{1}{12} = \frac{0.000,000,000,003,540,706}{12}) = 0.000,000,000,000,295,058;$$

$$\text{And } \frac{1}{13m^{13}} (= \frac{1}{m^{13}} \times \frac{1}{13} = \frac{0.000,000,000,000,393,411}{13}) = 0.000,000,000,000,030,262;$$

$$\text{And } \frac{1}{14m^{14}} (= \frac{1}{m^{14}} \times \frac{1}{14} = \frac{0.000,000,000,000,043,712}{14}) = 0.000,000,000,000,003,122;$$

$$\text{And } \frac{1}{15m^{15}} (= \frac{1}{m^{15}} \times \frac{1}{15} = \frac{0.000,000,000,000,004,856}{15}) = 0.000,000,000,000,000,323;$$

$$\text{And } \frac{1}{16m^{16}} (= \frac{1}{m^{16}} \times \frac{1}{16} = \frac{0.000,000,000,000,000,539}{16}) = 0.000,000,000,000,000,033;$$

$$\text{And } \frac{1}{17m^{17}} (= \frac{1}{m^{17}} \times \frac{1}{17} = \frac{0.000,000,000,000,000,059}{17}) = 0.000,000,000,000,000,003;$$

$$\text{And } \frac{1}{18m^{18}} (= \frac{1}{m^{18}} \times \frac{1}{18} = \frac{0.000,000,000,000,000,006}{18}) = 0.000,000,000,000,000,000.$$

Therefore the series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \frac{1}{7m^7} - \frac{1}{8m^8} +$
 $\frac{1}{9m^9} - \frac{1}{10m^{10}} + \frac{1}{11m^{11}} - \frac{1}{12m^{12}} + \frac{1}{13m^{13}} - \frac{1}{14m^{14}} + \frac{1}{15m^{15}} - \frac{1}{16m^{16}} + \frac{1}{17m^{17}}$
 $- \frac{1}{18m^{18}} + \&c$ is =

0;III,III,III,III,III,III, — 0;006,172,839,506,172,839,
 + ;...457,247,370,827,617, — ;...38,103,947,568,968,
 + ;...3,387,017,561,686, — ;...313,612,737,193,
 + ;...29,867,879,732, — ;...2,903,821,640,
 + ;...286,797,199, — ;...28,679,719,
 + ;...2,896,941, — ;...295,058,
 + ;...30,262, — ;...3,122,
 + ;...323, — ;...33,
 + ;...3, — ;...;

= 0.III,571,775,657,104,874, — 0.006,211,259,999,278,572,
 = 0.105,360,515,657,826,302. Therefore this number 0.105,360,515,657,
 826,302 is the logarithm of the ratio of $1 + \frac{1}{9}$ to 1, or of 10 to 9.

N. B. This number is true in all the places of figures except the last, where there ought to be a unit instead of a 2, the more accurate value of this logarithm being 0.105,360,515,657,826,301,227.

EXAMPLE II.

25. Let it be required to find by means of the said series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \&c$ the logarithm of the ratio of 11 to 10 (or of 10 + 1 to 10, or of $\frac{10+1}{10}$ to $\frac{10}{10}$) or of $1 + \frac{1}{10}$ to 1.

Here m is = 10, and $\frac{1}{m}$ is = $\frac{1}{10} = 0.100,000,000,000,000,000$.

We shall therefore have

$$\frac{1}{m^2} (= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{10} = \frac{0.100,000,000,000,000,000}{10}) = 0.010,000,000,000,000,000;$$

$$\text{And } \frac{1}{m^3} (= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{10} = \frac{0.010,000,000,000,000,000}{10}) = 0.001,000,000,000,000,000;$$

$$\text{And } \frac{1}{m^4} (= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{10} = \frac{0.001,000,000,000,000,000}{10}) = 0.000,100,000,000,000,000;$$

$$\text{And } \frac{1}{m^5} (= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{10} = \frac{0.000,100,000,000,000,000}{10}) = 0.000,010,000,000,000,000;$$

$$\text{And } \frac{1}{m^6} (= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{10} = \frac{0.000,010,000,000,000,000}{10}) = 0.000,001,000,000,000,000;$$

$$\text{And } \frac{1}{m^7} (= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{10} = \frac{0.000,001,000,000,000,000}{10}) = 0.000,000,100,000,000,000;$$

$$\text{And } \frac{1}{m^8} (= \frac{1}{m^7} \times \frac{1}{m} = \frac{1}{m^7} \times \frac{1}{10} = \frac{0.000,000,100,000,000,000}{10}) = 0.000,000,010,000,000,000;$$

$$\text{And } \frac{1}{m^9} (= \frac{1}{m^8} \times \frac{1}{m} = \frac{1}{m^8} \times \frac{1}{10} = \frac{0.000,000,010,000,000,000}{10}) = 0.000,000,001,000,000,000;$$

$$\text{And } \frac{1}{m^{10}} (= \frac{1}{m^9} \times \frac{1}{m} = \frac{1}{m^9} \times \frac{1}{10} = \frac{0.000,000,001,000,000,000}{10}) = 0.000,000,000,100,000,000;$$

$$\text{And } \frac{1}{m^{11}} (= \frac{1}{m^{10}} \times \frac{1}{m} = \frac{1}{m^{10}} \times \frac{1}{10} = \frac{0.000,000,000,100,000,000}{10}) = 0.000,000,000,010,000,000;$$

$$\text{And } \frac{1}{m^{12}} (= \frac{1}{m^{11}} \times \frac{1}{m} = \frac{1}{m^{11}} \times \frac{1}{10} = \frac{0.000,000,000,010,000,000}{10}) = 0.000,000,000,001,000,000;$$

$$\text{And } \frac{1}{m^{13}} (= \frac{1}{m^{12}} \times \frac{1}{m} = \frac{1}{m^{12}} \times \frac{1}{10} = \frac{0.000,000,000,001,000,000}{10}) = 0.000,000,000,000,100,000;$$

$$\text{And } \frac{1}{m^{14}} (= \frac{1}{m^{13}} \times \frac{1}{m} = \frac{1}{m^{13}} \times \frac{1}{10} = \frac{0.000,000,000,000,100,000}{10}) = 0.000,000,000,000,010,000;$$

$$\text{And } \frac{1}{m^{14}} (= \frac{1}{m^{14}} \times \frac{1}{m} = \frac{1}{m^{14}} \times \frac{1}{10} = \frac{0.000,000,000,000,010,000}{10}) = 0.000,000,000,000,001,000;$$

$$\text{And } \frac{1}{m^{15}} (= \frac{1}{m^{15}} \times \frac{1}{m} = \frac{1}{m^{15}} \times \frac{1}{10} = \frac{0.000,000,000,000,001,000}{10}) = 0.000,000,000,000,000,100;$$

$$\text{And } \frac{1}{m^{17}} (= \frac{1}{m^{17}} \times \frac{1}{m} = \frac{1}{m^{17}} \times \frac{1}{10} = \frac{0.000,000,000,000,000,100}{10}) = 0.000,000,000,000,000,010;$$

And consequently

$$\frac{1}{2m^2} (= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.010,000,000,000,000,000}{2}) = 0.005,000,000,000,000,000;$$

$$\text{And } \frac{1}{3m^3} (= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.001,000,000,000,000,000}{3}) = 0.000,333,333,333,333,333;$$

$$\text{And } \frac{1}{4m^4} (= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,100,000,000,000,000}{4}) = 0.000,025,000,000,000,000;$$

$$\text{And } \frac{1}{5m^5} (= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,010,000,000,000,000}{5}) = 0.000,002,000,000,000,000;$$

$$\text{And } \frac{1}{6m^6} (= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,001,000,000,000,000}{6}) = 0.000,000,166,666,666,666;$$

$$\text{And } \frac{1}{7m^7} (= \frac{1}{m^7} \times \frac{1}{7} = \frac{0.000,000,100,000,000,000}{7}) = 0.000,000,014,285,714,285;$$

$$\text{And } \frac{1}{8m^8} (= \frac{1}{m^8} \times \frac{1}{8} = \frac{0.000,000,010,000,000,000}{8}) = 0.000,000,001,250,000,000;$$

$$\text{And } \frac{1}{9m^9} (= \frac{1}{m^9} \times \frac{1}{9} = \frac{0.000,000,001,000,000,000}{9}) = 0.000,000,000,111,111,111;$$

$$\text{And } \frac{1}{10m^{10}} (= \frac{1}{m^{10}} \times \frac{1}{10} = \frac{0.000,000,000,100,000,000}{10}) = 0.000,000,000,010,000,000;$$

$$\text{And } \frac{1}{11m^{11}} (= \frac{1}{m^{11}} \times \frac{1}{11} = \frac{0.000,000,000,010,000,000}{11}) = 0.000,000,000,000,909,090;$$

$$\text{And } \frac{1}{12m^{12}} (= \frac{1}{m^{12}} \times \frac{1}{12} = \frac{0.000,000,000,001,000,000}{12}) = 0.000,000,000,000,083,333;$$

$$\text{And } \frac{1}{13m^{13}} (= \frac{1}{m^{13}} \times \frac{1}{13} = \frac{0.000,000,000,000,100,000}{13}) = 0.000,000,000,000,007,692;$$

$$\text{And } \frac{1}{14m^{14}} (= \frac{1}{m^{14}} \times \frac{1}{14} = \frac{0.000,000,000,000,010,000}{14}) = 0.000,000,000,000,000,714;$$

$$\text{And } \frac{1}{15m^{15}} (= \frac{1}{m^{15}} \times \frac{1}{15} = \frac{0.000,000,000,000,001,000}{15}) = 0.000,000,000,000,000,066;$$

$$\text{And } \frac{1}{16m^{16}} (= \frac{1}{m^{16}} \times \frac{1}{16} = \frac{0.000,000,000,000,000,100}{16}) = 0.000,000,000,000,000,006;$$

$$\text{And } \frac{1}{17m^{17}} (= \frac{1}{m^{17}} \times \frac{1}{17} = \frac{0.000,000,000,000,000,010}{17}) = 0.000,000,000,000,000,000.$$

Therefore

Therefore the series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \frac{1}{7m^7} - \frac{1}{8m^8} +$
 $\frac{1}{9m^9} - \frac{1}{10m^{10}} + \frac{1}{11m^{11}} - \frac{1}{12m^{12}} + \frac{1}{13m^{13}} - \frac{1}{14m^{14}} + \frac{1}{15m^{15}} - \frac{1}{16m^{16}} + \dots$
 is =

0.100,000,000,000,000,000,	— 0.005,000,000,000,000,000,
+ ;...333,333,333,333,333,	;...25,000,000,000,000,
+ ;...2,000,000,000,000,	;...166,666,666,666,
+ ;...14,285,714,285,	;...1,250,000,000,
+ ;...111,111,111,	;...10,000,000,
+ ;...909,090,	;...83,333,
+ ;...7,692,	;...714,
+ ;...66,	;...6,
= 0.100,335,347,731,075,577,	— 0.005,025,167,926,750,719,
= 0.095,310,179,804,324,858.	Therefore this number 0.095,310,179,

804,324,858 is the logarithm of the ratio of $1 + \frac{1}{10}$ to 1, or of 11 to 10.

Q. E. I.

EXAMPLE III.

26. Let it be required to find, by means of the forefaid series, the logarithm of the ratio of 81 to 80, (or of $80 + 1$ to 80, or of $\frac{80+1}{80}$ to $\frac{80}{80}$) or of $1 + \frac{1}{80}$ to 1.

Here m is = 80, and $\frac{1}{m}$ is $\frac{1}{80} = 0.012,500,000,000,000,000$.

We shall, therefore, in this case have

$\frac{1}{m^2} (= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{80} = \frac{0.012,500,000,000,000,000}{80}) = 0.000,156,250,000,000,000$;
 And $\frac{1}{m^3} (= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{80} = \frac{0.000,156,250,000,000,000}{80}) = 0.000,001,953,125,000,000$;
 And $\frac{1}{m^4} (= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{80} = \frac{0.000,001,953,125,000,000}{80}) = 0.000,000,024,414,062,500$;
 And $\frac{1}{m^5} (= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{80} = \frac{0.000,000,024,414,062,500}{80}) = 0.000,000,000,305,175,781$;
 And $\frac{1}{m^6} (= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{80} = \frac{0.000,000,000,305,175,781}{80}) = 0.000,000,000,003,814,697$;
 And $\frac{1}{m^7} (= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{80} = \frac{0.000,000,000,003,814,697}{80}) = 0.000,000,000,000,047,683$;
 And $\frac{1}{m^8} (= \frac{1}{m^7} \times \frac{1}{m} = \frac{1}{m^7} \times \frac{1}{80} = \frac{0.000,000,000,000,047,683}{80}) = 0.000,000,000,000,000,596$;

And

$$\text{And } \frac{1}{m^9} (= \frac{1}{m^8} \times \frac{1}{m} = \frac{1}{m^8} \times \frac{1}{80} = \frac{0.000,000,000,000,000,596}{80}) = 0.000,000,000,000,000,007;$$

$$\text{And } \frac{1}{m^{10}} (= \frac{1}{m^9} \times \frac{1}{m} = \frac{1}{m^9} \times \frac{1}{80} = \frac{0.000,000,000,000,000,007}{80}) = 0.000,000,000,000,000,000;$$

And consequently

$$\frac{1}{2m^2} (= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,156,250,000,000,000}{2}) = 0.000,078,125,000,000,000;$$

$$\text{And } \frac{1}{3m^3} (= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,001,953,125,000,000}{3}) = 0.000,000,651,041,666,666;$$

$$\text{And } \frac{1}{4m^4} (= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,000,024,414,062,500}{4}) = 0.000,000,006,103,515,625;$$

$$\text{And } \frac{1}{5m^5} (= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,305,175,781}{5}) = 0.000,000,000,061,035,156;$$

$$\text{And } \frac{1}{6m^6} (= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,003,814,697}{6}) = 0.000,000,000,000,635,782;$$

$$\text{And } \frac{1}{7m^7} (= \frac{1}{m^7} \times \frac{1}{7} = \frac{0.000,000,000,000,047,683}{7}) = 0.000,000,000,000,006,811;$$

$$\text{And } \frac{1}{8m^8} (= \frac{1}{m^8} \times \frac{1}{8} = \frac{0.000,000,000,000,000,596}{8}) = 0.000,000,000,000,000,074;$$

$$\text{And } \frac{1}{9m^9} (= \frac{1}{m^9} \times \frac{1}{9} = \frac{0.000,000,000,000,000,007}{9}) = 0.000,000,000,000,000,000.$$

Therefore the series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \frac{1}{7m^7} - \frac{1}{8m^8} + \mathcal{E}c.$

$$\begin{aligned} \text{is} &= 0;012,500,000,000,000,000, - 0;000,078,125,000,000,000, \\ &+ ;\dots\dots\dots,651,041,666,666, - ;\dots\dots\dots,6,103,515,625, \\ &+ ;\dots\dots\dots,61,035,156, - ;\dots\dots\dots,635,782, \\ &+ ;\dots\dots\dots,6,811, - ;\dots\dots\dots,74, \\ &= 0.012,500,651,102,708,633, - 0.000,078,131,104,151,481, \\ &= 0.012,422,519,998,557,152. \end{aligned}$$

Therefore this number 0.012,422,519,998,557,152 is the logarithm of the ratio of $1 + \frac{1}{80}$ to 1, or of 81 to 80.

Q. E. I.

EXAMPLE IV.

27. Let it be required to find by means of the same series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \mathcal{E}c$ the logarithm of the ratio of 121 to 120 (or of 120 to 120, or of $\frac{120+1}{120}$ to $\frac{120}{120}$), or of $1 + \frac{1}{120}$ to 1.

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Here

Here m is $= 120$, and $\frac{1}{m} = \frac{1}{120} = 0.008,333,333,333,333, \&c.$

We shall therefore have,

$$\frac{1}{m^2} (= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{120} = \frac{0.008,333,333,333,333}{120}) = 0.000,069,444,444,444;$$

$$\text{And } \frac{1}{m^3} (= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{120} = \frac{0.000,069,444,444,444}{120}) = 0.000,000,578,703,703,703;$$

$$\text{And } \frac{1}{m^4} (= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{120} = \frac{0.000,000,578,703,703,703}{120}) = 0.000,000,004,822,530,864;$$

$$\text{And } \frac{1}{m^5} (= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{120} = \frac{0.000,000,004,822,530,864}{120}) = 0.000,000,000,040,187,757$$

$$\text{And } \frac{1}{m^6} (= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{120} = \frac{0.000,000,000,040,187,757}{120}) = 0.000,000,000,000,334,897$$

$$\text{And } \frac{1}{m^7} (= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{120} = \frac{0.000,000,000,000,334,897}{120}) = 0.000,000,000,000,002,790;$$

$$\text{And } \frac{1}{m^8} (= \frac{1}{m^7} \times \frac{1}{m} = \frac{1}{m^7} \times \frac{1}{120} = \frac{0.000,000,000,000,002,790}{120}) = 0.000,000,000,000,000,023;$$

$$\text{And } \frac{1}{m^9} (= \frac{1}{m^8} \times \frac{1}{m} = \frac{1}{m^8} \times \frac{1}{120} = \frac{0.000,000,000,000,000,023}{120}) = 0.000,000,000,000,000,000;$$

And consequently

$$\frac{1}{2m^2} (= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,069,444,444,444}{2}) = 0.000,034,722,222,222,222;$$

$$\text{And } \frac{1}{3m^3} (= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,000,578,703,703,703}{3}) = 0.000,000,192,901,234,567;$$

$$\text{And } \frac{1}{4m^4} (= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,000,004,822,530,864}{4}) = 0.000,000,001,205,632,716;$$

$$\text{And } \frac{1}{5m^5} (= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,040,187,757}{5}) = 0.000,000,000,008,037,551;$$

$$\text{And } \frac{1}{6m^6} (= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,000,334,897}{6}) = 0.000,000,000,000,055,816;$$

$$\text{And } \frac{1}{7m^7} (= \frac{1}{m^7} \times \frac{1}{7} = \frac{0.000,000,000,000,002,790}{7}) = 0.000,000,000,000,000,398;$$

$$\text{And } \frac{1}{8m^8} (= \frac{1}{m^8} \times \frac{1}{8} = \frac{0.000,000,000,000,000,023}{8}) = 0.000,000,000,000,000,002.$$

Therefore

Therefore the series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \frac{1}{7m^7} - \frac{1}{8m^8} +$
 $\&c$ is =

$$\begin{array}{r} 0.008,333,333,333,333,333, \quad - \quad 0.000,034,722,222,222,222, \\ + \quad ; \dots, \dots, 192,901,234,567, \quad - \quad ; \dots, \dots, \dots, 1,205,632,716, \\ + \quad ; \dots, \dots, \dots, \dots, 8,037,551, \quad - \quad ; \dots, \dots, \dots, \dots, 55,816, \\ + \quad ; \dots, \dots, \dots, \dots, \dots, 398, \quad - \quad ; \dots, \dots, \dots, \dots, \dots, 2, \\ \hline = 0.008,333,526,242,605,849, \quad - \quad 0.000,034,723,427,910,756, \\ = 0.008,298,802,814,695,093. \end{array}$$

Therefore this number 0.008,298,802,814,695,093 is the logarithm of the ratio of $1 + \frac{1}{120}$ to 1, or of 121 to 120.

Q. E. I.

EXAMPLE V.

28. Let it be required to find by means of the said series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} -$
 $\frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \&c$ the logarithm of the ratio of 2401 (which is the
fourth power of 7) to 2400, (or of $2400 + 1$ to 2400 or of $\frac{2400+1}{2400}$ to $\frac{2400}{2400}$),
or of $1 + \frac{1}{2400}$ to 1.

Here m is = 2400, and $\frac{1}{m}$ is = $\frac{1}{2400} = 0.000,416,666,666,666,666$.

We shall therefore have

$$\begin{array}{l} \frac{1}{m^2} (= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{2400} = \frac{0.000,416,666,666,666,666}{2400}) = 0.000,000,173,611,111,111; \\ \text{And } \frac{1}{m^3} (= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{2400} = \frac{0.000,000,173,611,111,111}{2400}) = 0.000,000,000,072,337,962; \\ \text{And } \frac{1}{m^4} (= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{2400} = \frac{0.000,000,000,072,337,962}{2400}) = 0.000,000,000,000,030,140; \\ \text{And } \frac{1}{m^5} (= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{2400} = \frac{0.000,000,000,000,030,140}{2400}) = 0.000,000,000,000,000,012; \\ \text{And } \frac{1}{m^6} (= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{2400} = \frac{0.000,000,000,000,000,012}{2400}) = 0.000,000,000,000,000,000; \end{array}$$

And consequently

$$\begin{array}{l} \frac{1}{2m^2} (= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,000,173,611,111,111}{2}) = 0.000,000,086,805,555,555; \\ \text{And } \frac{1}{3m^3} (= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,000,000,072,337,962}{3}) = 0.000,000,000,024,112,654; \end{array}$$

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And

$$\text{And } \frac{1}{4m^4} (= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,000,000,000,030,140}{4}) = 0.000,000,000,000,007,535;$$

$$\text{And } \frac{1}{5m^5} (= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,000,000,012}{5}) = 0.000,000,000,000,000,002.$$

Therefore the series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \&c$ is in this case

0.000,416,666,666,666,666,	—	0.000,000,086,805,555,555,
+ ;...;...;...; 24,112,654,	—	;...;...;...;...; 7,535,
+ ;...;...;...;...;...; 2,	—	;...;...;...;...;...;
0.000,416,666,690,779,322,	—	0.000,000,086,805,563,090,

$= 0.000,416,579,885,216,232.$ Therefore this number 0.000,416,579,885,216,232 is the logarithm of the ratio of $1 + \frac{1}{2400}$ to 1, or of 2401 to 2400.

Q. E. I.

EXAMPLE VI.

29. Let it be required to find, by means of the same series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \&c$ the logarithm of the ratio of 169 (which is the square of 13) to 168 (or of 168 + 1 to 168, or of $\frac{168+1}{168}$ to $\frac{168}{168}$), or of $1 + \frac{1}{168}$ to 1.

Here m is = 168, and $\frac{1}{m}$ is = $\frac{1}{168} = 0.005,952,380,952,380,952.$

We shall therefore have

$$\frac{1}{m^2} (= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{168} = \frac{0.005,952,380,952,380,952}{168}) = 0.000,035,430,839,002,267;$$

$$\text{And } \frac{1}{m^3} (= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{168} = \frac{0.000,035,430,839,002,267}{168}) = 0.000,000,210,897,851,203;$$

$$\text{And } \frac{1}{m^4} (= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{168} = \frac{0.000,000,210,897,851,203}{168}) = 0.000,000,001,255,344,352;$$

$$\text{And } \frac{1}{m^5} (= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{168} = \frac{0.000,000,001,255,344,352}{168}) = 0.000,000,000,007,472,287;$$

$$\text{And } \frac{1}{m^6} (= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{168} = \frac{0.000,000,000,007,472,287}{168}) = 0.000,000,000,000,044,477;$$

$$\text{And } \frac{1}{m^7} (= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{168} = \frac{0.000,000,000,000,044,477}{168}) = 0.000,000,000,000,000,264;$$

And

And $\frac{1}{m^8} (= \frac{1}{m^7} \times \frac{1}{m} = \frac{1}{m^7} \times \frac{1}{168} = \frac{0.000,000,000,000,000,264}{168}) = 0.000,000,000,000,000,001;$

And consequently

$\frac{1}{2m^2} (= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,035,430,839,002,267}{2}) = 0.000,017,715,419,501,133;$

And $\frac{1}{3m^3} (= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,000,210,897,851,203}{3}) = 0.000,000,070,299,283,734;$

And $\frac{1}{4m^4} (= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,000,001,255,344,352}{4}) = 0.000,000,000,313,836,088;$

And $\frac{1}{5m^5} (= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,007,472,287}{5}) = 0.000,000,000,001,494,457;$

And $\frac{1}{6m^6} (= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,000,044,477}{6}) = 0.000,000,000,000,007,412;$

And $\frac{1}{7m^7} (= \frac{1}{m^7} \times \frac{1}{7} = \frac{0.000,000,000,000,000,264}{7}) = 0.000,000,000,000,000,037;$

And $\frac{1}{8m^8} (= \frac{1}{m^8} \times \frac{1}{8} = \frac{0.000,000,000,000,000,001}{8}) = 0.000,000,000,000,000,000.$

Therefore the series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \frac{1}{7m^7} - \frac{1}{8m^8} +$

&c is = 0.005,952,380,952,380,952, — 0.000,017,715,419,501,133,
 + ;...,,...,, 70,299,283,734, — ;...,,...,,...,, 313,836,088,
 + ;...,,...,,...,, 1,494,457, — ;...,,...,,...,,...,, 7,412,
 + ;...,,...,,...,,...,,...,, 37, — ;...,,...,,...,,...,,...,,
 = 0.005,952,451,253,159,180, — 0.000,017,715,733,344,633,
 = 0.005,934,735,519,814,547. Therefore this number 0.005,934,735,
 519,814,547, is the logarithm of the ratio of $1 + \frac{1}{168}$ to 1, or of 169 to 168.

Q. E. I.

EXAMPLE VII.

30. Let it be required to find by means of the same series the logarithm of the ratio of 289 (which is the square of 17,) to 288 (or of $288 + 1$ to 288, or of $\frac{288+1}{288}$ to $\frac{288}{288}$), or of $1 + \frac{1}{288}$ to 1.

Here m is = 288, and $\frac{1}{m}$ is = $\frac{1}{288} = 0.003,472,222,222,222, \&c.$

We

We shall therefore have

$$\frac{1}{m^2} (= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{288} = \frac{0.003,472,222,222,222,222}{288}) = 0.000,012,056,327,160,493;$$

$$\text{And } \frac{1}{m^3} (= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{288} = \frac{0.000,012,056,327,160,493}{288}) = 0.000,000,041,862,247,085;$$

$$\text{And } \frac{1}{m^4} (= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{288} = \frac{0.000,000,041,862,247,085}{288}) = 0.000,000,000,145,355,024;$$

$$\text{And } \frac{1}{m^5} (= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{288} = \frac{0.000,000,000,145,355,024}{288}) = 0.000,000,000,000,504,704;$$

$$\text{And } \frac{1}{m^6} (= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{288} = \frac{0.000,000,000,000,504,704}{288}) = 0.000,000,000,000,001,752;$$

$$\text{And } \frac{1}{m^7} (= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{288} = \frac{0.000,000,000,000,001,752}{288}) = 0.000,000,000,000,000,006;$$

And consequently

$$\frac{1}{2m^2} (= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,012,056,327,160,493}{2}) = 0.000,006,028,163,580,246;$$

$$\text{And } \frac{1}{3m^3} (= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,000,041,862,247,085}{3}) = 0.000,000,013,954,082,361;$$

$$\text{And } \frac{1}{4m^4} (= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,000,000,145,355,024}{4}) = 0.000,000,000,036,338,756;$$

$$\text{And } \frac{1}{5m^5} (= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,000,504,704}{5}) = 0.000,000,000,000,100,940;$$

$$\text{And } \frac{1}{6m^6} (= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,000,001,752}{6}) = 0.000,000,000,000,000,292;$$

$$\text{And } \frac{1}{7m^7} (= \frac{1}{m^7} \times \frac{1}{7} = \frac{0.000,000,000,000,000,006}{7}) = 0.000,000,000,000,000,000.$$

Therefore the series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \frac{1}{7m^7} - \frac{1}{8m^8}$
 $+ \&c$ is in this case =

$$\begin{array}{r} 0.003,472,222,222,222,222, \quad - \quad 0.000,006,028,163,580,246, \\ + \quad ; \dots, \dots, 13,954,082,361, \quad - \quad ; \dots, \dots, \dots, 36,338,756, \\ + \quad ; \dots, \dots, \dots, 100,940, \quad - \quad ; \dots, \dots, \dots, \dots, 292, \\ + \quad ; \dots, \dots, \dots, \dots, \dots, \quad - \quad ; \dots, \dots, \dots, \dots, \dots, \\ \hline = 0.003,472,236,176,405,523, \quad - \quad 0.000,006,028,199,919,294, \\ = 0.003,466,207,976,486,229. \end{array}$$

Therefore this number 0.003,466,207,976,486,229 is the logarithm of the ratio of $1 + \frac{1}{288}$ to 1, or of 289 to 288.

Q. E. I.

2

EXAMPLE

EXAMPLE VIII.

31. Let it be required to find by means of the same series the logarithm of the ratio of 361 (which is the square of 19,) to 360 (or of $360 + 1$ to 360, or of $\frac{360+1}{360}$ to $\frac{360}{360}$ or of $1 + \frac{1}{360}$ to 1.

Here m is = 360, and consequently $\frac{1}{m}$ is = $\frac{1}{360} = 0.002,777,777,777,777$.

We shall therefore have,

$$\frac{1}{m^2} (= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{360} = \frac{0.002,777,777,777,777}{360}) = 0.000,007,716,049,382,716;$$

$$\text{And } \frac{1}{m^3} (= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{360} = \frac{0.000,007,716,049,382,716}{360}) = 0.000,000,021,433,470,507;$$

$$\text{And } \frac{1}{m^4} (= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{360} = \frac{0.000,000,021,433,470,507}{360}) = 0.000,000,000,059,537,418;$$

$$\text{And } \frac{1}{m^5} (= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{360} = \frac{0.000,000,000,059,537,418}{360}) = 0.000,000,000,000,165,381;$$

$$\text{And } \frac{1}{m^6} (= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{360} = \frac{0.000,000,000,000,165,381}{360}) = 0.000,000,000,000,000,459;$$

$$\text{And } \frac{1}{m^7} (= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{360} = \frac{0.000,000,000,000,000,459}{360}) = 0.000,000,000,000,000,001;$$

And consequently

$$\frac{1}{2m^2} (= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,007,716,049,382,716}{2}) = 0.000,003,858,024,691,358;$$

$$\text{And } \frac{1}{3m^3} (= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,000,021,433,470,507}{3}) = 0.000,000,007,144,490,169;$$

$$\text{And } \frac{1}{4m^4} (= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,000,000,059,537,418}{4}) = 0.000,000,000,014,884,354;$$

$$\text{And } \frac{1}{5m^5} (= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,000,165,381}{5}) = 0.000,000,000,000,033,076;$$

$$\text{And } \frac{1}{6m^6} (= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,000,000,459}{6}) = 0.000,000,000,000,000,076;$$

$$\text{And } \frac{1}{7m^7} (= \frac{1}{m^7} \times \frac{1}{7} = \frac{0.000,000,000,000,000,001}{7}) = 0.000,000,000,000,000,000.$$

Therefore

Therefore the series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \text{Ec}$ is in this case =

$$\begin{aligned} & 0.002,777,777,777,777,777, - 0.000,003,858,024,691,358, \\ & + \dots, \dots, \dots, 7,144,490,169, - \dots, \dots, \dots, 14,884,354, \\ & + \dots, \dots, \dots, \dots, \dots, 33,076, - \dots, \dots, \dots, \dots, \dots, 76, \\ & = 0.002,777,784,922,301,022, - 0.000,003,858,039,575,788, \\ & = 0.002,773,926,882,725,234. \end{aligned}$$

Therefore this number 0.002,773,926,882,725,234, is the logarithm of the ratio of $1 + \frac{1}{360}$ to 1, or of 361 to 360.

Q. E. I.

EXAMPLE IX.

32. Let it be required to find by the same series the logarithm of the ratio of 529 (which is the square of 23,) to 528 (or of $528 + 1$ to 528, or of $\frac{528+1}{528}$ to $\frac{528}{528}$), or of $1 + \frac{1}{528}$ to 1.

Here m is = 528, and consequently $\frac{1}{m}$ is = $\frac{1}{528} = 0.001,893,939,393,939,393, \text{Ec}$.

We shall therefore, in this case, have

$$\frac{1}{m^2} (= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{528} = \frac{0.001,893,939,393,939,393}{528}) = 0.000,003,587,006,427,915;$$

$$\text{And } \frac{1}{m^3} (= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{528} = \frac{0.000,003,587,006,427,915}{528}) = 0.000,000,006,793,572,780;$$

$$\text{And } \frac{1}{m^4} (= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{528} = \frac{0.000,000,006,793,572,780}{528}) = 0.000,000,000,012,866,615;$$

$$\text{And } \frac{1}{m^5} (= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{528} = \frac{0.000,000,000,012,866,615}{528}) = 0.000,000,000,000,024,368;$$

$$\text{And } \frac{1}{m^6} (= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{528} = \frac{0.000,000,000,000,024,368}{528}) = 0.000,000,000,000,000,046;$$

$$\text{And } \frac{1}{m^7} (= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{528} = \frac{0.000,000,000,000,000,046}{528}) = 0.000,000,000,000,000,000;$$

And consequently

$$\frac{1}{2m^2} (= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,003,587,006,427,915}{2}) = 0.000,001,793,503,213,957;$$

$$\text{And } \frac{1}{3m^3} (= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,000,006,793,572,780}{3}) = 0.000,000,002,264,524,260;$$

And

$$\text{And } \frac{1}{4m^4} (= \frac{1}{m^4} \times \frac{1}{528} = \frac{0.000,000,000,012,866,615}{4}) = 0.000,000,000,003,216,653;$$

$$\text{And } \frac{1}{5m^5} (= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,000,024,368}{5}) = 0.000,000,000,000,004,873;$$

$$\text{And } \frac{1}{6m^6} (= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,000,000,046}{6}) = 0.000,000,000,000,000,007.$$

Therefore the series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \text{\&c}$ is =

$$\begin{array}{r} 0.001,893,939,393,939,393, - 0.000,001,793,503,213,957, \\ + \quad ; \dots, \dots, \dots, 2,264,524,260, - \quad ; \dots, \dots, \dots, 3,216,653, \\ + \quad ; \dots, \dots, \dots, \dots, 4,873, - \quad ; \dots, \dots, \dots, \dots, 7, \\ \hline = 0.001,893,941,658,468,526, - 0.000,001,793,506,430,617, \end{array}$$

= 0.001,892,148,152,037,909. Therefore this number 0.001,892,148,152,037,909, is the logarithm of the ratio of $1 + \frac{1}{528}$ to 1, or of 529 to 528.

Q. E. I.

An application of the foregoing nine logarithms of small ratios to the investigation of the logarithms of the ratios of the first 23 natural numbers, (2, 3, 4, 5, &c. to 24 inclusively,) to 1.

33. Having thus found, with no great labour, the logarithms of the nine small ratios of 10 to 9, of 11 to 10, of 81 to 80, of 121 to 120, of 2401 to 2400, of 169 to 168, of 289 to 288, of 361 to 360, and of 529 to 528, by means of the series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \frac{1}{7m^7} - \frac{1}{8m^8} + \text{\&c}$, we may, by combining these few logarithms with each other by addition, subtraction, multiplication, and division, easily discover the logarithms of the greater ratios of 2 to 1, 3 to 1, 4 to 1, 5 to 1, 6 to 1, 7 to 1, 8 to 1, 9 to 1, 10 to 1, 11 to 1, 12 to 1, 13 to 1, 14 to 1, 15 to 1, 16 to 1, 17 to 1, 18 to 1, 19 to 1, 20 to 1, 21 to 1, 22 to 1, 23 to 1, and 24 to 1, or (as, for the sake of brevity, they are more frequently called) the logarithms of the numbers 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, and 24. This may be done in the manner following.

34. The ratio of 11 to 9 is equal to the sum of the ratios of 11 to 10 and 10 to 9. Therefore the logarithm of the ratio of 11 to 9 is equal to the sum of the logarithms of the two latter ratios, that is, (as appears by Examples first and second,) to the sum of 0.105,360,515,657,826,302, and 0.095,310,179,804,324,858, or to 0.200,670,695,462,151,160.

The ratio of 121 (which is the square of 11) to 81 (which is the square of 9) is double of the ratio of 11 to 9. Therefore the logarithm of the ratio of 121 to 81 is double of the logarithm of 11 to 9, and consequently is equal to $2 \times 0.200,670,695,462,151,160$, or to 0.401,341,390,924,302,320.

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The ratio of 121 to 80 is equal to the sum of the ratios of 121 to 81, and of 81 to 80. Therefore the logarithm of the ratio of 121 to 80 is equal to the sum of the logarithms of the ratios of 121 to 81, and of 81 to 80. But the logarithm of the ratio of 121 to 81 has just been shewn to be $= 0.401,341,390,924,302,320$; and the logarithm of the ratio of 81 to 80 has been shewn in Example third to be $= 0.012,422,519,998,557,152$. Therefore the logarithm of the ratio of 121 to 80 is

$$\begin{aligned} &= 0.401,341,390,924,302,320 \\ &+ 0.012,422,519,998,557,152 \end{aligned} \Bigg\} = 0.413,763,910,922,859,472.$$

The ratio of 120 to 80 is equal to the excess of the ratio of 121 to 80 above the ratio of 121 to 120. Therefore the logarithm of the ratio of 120 to 80 is equal to the excess of the logarithm of the ratio of 121 to 80 above the logarithm of the ratio of 121 to 120. But the logarithm of the ratio of 121 to 80 has just now been shewn to be $0.413,763,910,922,859,472$; and the logarithm of the ratio of 121 to 120 is shewn in Example 4 to be $0.008,298,802,814,695,093$. Therefore the logarithm of the ratio of 120 to 80 is equal to the excess of $0.413,763,910,922,859,472$ above $0.008,298,802,814,695,093$, or to $0.405,465,108,108,164,379$.

The ratio of 3 to 2 is the same with the ratio of 120 to 80. Therefore the logarithm of the ratio of 3 to 2 is $= 0.405,465,108,108,164,379$.

The ratio of 9 (which is the square of three) to 4 (which is the square of 2) is double of the ratio of 3 to 2. Therefore the logarithm of the ratio of 9 to 4 is double of the logarithm of the ratio of 3 to 2, and consequently is equal to $2 \times 0.405,465,108,108,164,379$, or to $0.810,930,216,216,328,758$.

The ratio of 81 (which is the square of 9) to 16 (which is the square of 4) is double of the ratio of 9 to 4. Therefore the logarithm of the ratio of 81 to 16 is double of the logarithm of the ratio of 9 to 4, and consequently is equal to $2 \times 0.810,930,216,216,328,758$, or to $1.621,860,432,432,657,516$.

The ratio of 80 to 16 is equal to the excess of the ratio of 81 to 16 above the ratio of 81 to 80. Therefore the logarithm of the ratio of 80 to 16 is equal to the excess of the logarithm of the ratio of 81 to 16 above the logarithm of 81 to 80, that is, to the excess of $1.621,860,432,432,657,516$ above $0.012,422,519,998,557,152$, or to $1.609,437,912,434,100,364$.

The ratio of 5 to 1 is equal to the ratio of 80 to 16. Therefore the logarithm of the ratio of 5 to 1 is equal to the logarithm of the ratio of 80 to 16, and consequently is $1.609,437,912,434,100,364$; or (in the usual abridged but inaccurate phrase), the logarithm of the number 5 is $1.609,437,912,434,100,364$.

Q. E. I.

35. The ratio of 10 to 2 is the same with the ratio of 5 to 1. Therefore the logarithm of the ratio of 10 to 2 is $1.609,437,912,434,100,364$.

The ratio of 10 to 4 is equal to the sum of the ratios of 10 to 9 and of 9 to 4. Therefore the logarithm of the ratio of 10 to 4 is equal to the sum of the logarithms of the ratios of 10 to 9 and of 9 to 4, that is, to the sum of the logarithm $0.105,360,$

360,515,657,826,302, and the logarithm 0.810,930,216,216,328,758, or to 0.916,290,731,874,155,060.

The ratio of 4 to 2 is equal to the excess of the ratio of 10 to 2 above the ratio of 10 to 4. Therefore the logarithm of the ratio of 4 to 2 is equal to the excess of the logarithm of the ratio of 10 to 2 above the logarithm of the ratio of 10 to 4. But we have just now seen that the logarithm of the ratio of 10 to 2 is 1.609,437,912,434,100,364, and the logarithm of the ratio of 10 to 4 is 0.916,290,731,874,155,060. Therefore the logarithm of the ratio of 4 to 2 will be equal to the excess of 1.609,437,912,434,100,364 above 0.916,290,731,874,155,060, that is, to 0.693,147,180,559,945,304.

The ratio of 2 to 1 is equal to the ratio of 4 to 2. Therefore the logarithm of the ratio of 2 to 1 is equal to the logarithm of the ratio of 4 to 2, and consequently is = 0.693,147,180,559,945,304, or (in the common abridged, but inaccurate, language upon this subject), the logarithm of the number 2 is = 0.693,147,180,559,945,304. Q. E. I.

36. The ratio of 4 to 1 is double of the ratio of 2 to 1. Therefore the logarithm of the ratio of 4 to 1 is double of the logarithm of the ratio of 2 to 1; and consequently is equal to $2 \times 0.693,147,180,559,945,304$, or to 1.386,294,361,119,890,608; or, in other words, the logarithm of the number 4 is = 1.386,294,361,119,890,608. Q. E. I.

37. The ratio of 8 to 1 is triple of the ratio of 2 to 1. Therefore the logarithm of the ratio of 8 to 1 is triple of the logarithm of the ratio of 2 to 1; and consequently is equal to $3 \times 0.693,147,180,559,945,304$, or to 2.079,441,541,679,835,912; or, in other words, the logarithm of 8 is = 2.079,441,541,679,835,912. Q. E. I.

38. The ratio of 9 to 1 is equal to the sum of the ratios of 9 to 4 and of 4 to 1. Therefore the logarithm of the ratio of 9 to 1 is equal to the sum of the logarithms of the ratios of 9 to 4 and of 4 to 1. But it has been shewn that the logarithm of the ratio of 9 to 4 is = 0.810,930,216,216,328,758, and the logarithm of the ratio of 4 to 1 is = 1.386,294,361,119,890,608. Therefore the logarithm of the ratio of 9 to 1 is

$$\begin{aligned} &= 0.810,930,216,216,328,758, \\ &+ 1.386,294,361,119,890,608, \end{aligned} \Bigg\} = 2.197,224,577,336,219,366;$$
 or the logarithm of the number 9 is 2.197,224,577,336,219,366. Q. E. I.

39. The ratio of 3 to 1 is equal to half the ratio of 9 to 1. Therefore the logarithm of the ratio of 3 to 1 is equal to half the logarithm of the ratio of 9 to 1, and consequently is = $\frac{2.197,224,577,336,219,366}{2}$, or 1.098,612,288,668,109,683; or the logarithm of the number 3 is = 1.098,612,288,668,109,683. Q. E. I.

40. The ratio of 6 to 1 is equal to the sum of the ratios of 6 to 3 and of 3 to 1, and consequently to the sum of the ratios of 2 to 1 (which is equal to the ratio of 6 to 3,) and of 3 to 1. Therefore the logarithm of the ratio of 6 to 1 is equal to the sum of the logarithms of the ratios of 2 to 1 and of 3 to 1, that is, to the sum of 0.693,147,180,559,945,304, and 1.098,612,288,668,109,683, or to 1.791,759,469,228,054,987; or (according to the common way of speaking) the logarithm of 6 is = 1.791,759,469,228,054,987. Q. E. I.

41. The ratio of 10 to 1 is equal to the sum of the ratios of 10 to 9 and of 9 to 1. Therefore the logarithm of the ratio of 10 to 1 is equal to the sum of the logarithms of the ratios of 10 to 9 and of 9 to 1. But it has been shewn that the logarithm of the ratio of 10 to 9 is 0.105,360,515,657,826,302, and that the logarithm of the ratio of 9 to 1 is 2.197,224,577,336,219,366. Therefore the logarithm of the ratio of 10 to 1 is equal to the sum of 0.105,360,515,657,826,302 and 2.197,224,577,336,219,366, or to 2.302,585,092,994,045,668, or (in the usual way of speaking on this subject) the logarithm of 10 is 2.302,585,092,994,045,668. Q. E. I.

N. B. This number is true to seventeen places of figures, or in all the figures but the two last, the more accurate value of the logarithm of 10 being 2.302,585,092,994,045,684,017.

42. The ratio of 11 to 1 is equal to the sum of the ratios of 11 to 10, and of 10 to 1. Therefore the logarithm of the ratio of 11 to 1 is equal to the sum of the logarithms of the ratios of 11 to 10 and of 10 to 1, that is, to the sum of 0.095,310,179,804,324,858, and 2.302,585,092,994,045,668, or to 2.397,895,272,798,370,526; or 2.397,895,272,798,370,526 is the logarithm of the number 11. Q. E. I.

43. The ratio of 24 to 1 is equal to the sum of the ratios of 24 to 8, and of 8 to 1, that is, to the sum of the ratios of 3 to 1 (which is equal to the ratio of 24 to 8) and of 8 to 1. Therefore the logarithm of the ratio of 24 to 1 is equal to the sum of the logarithms of the ratios of 3 to 1 and of 8 to 1, that is, to the sum of 1.098,612,288,668,109,683 and 2.079,441,541,679,835,912, or to 3.178,053,830,347,945,595.

The ratio of 2400 to 100 is equal to the ratio of 24 to 1. Therefore the logarithm of the ratio of 2400 to 100 is equal to the logarithm of the ratio of 24 to 1, or to 3.178,053,830,347,945,595.

The ratio of 100 to 1 is double of the ratio of 10 to 1. Therefore the logarithm of the ratio of 100 to 1 is double of the logarithm of the ratio of 10 to 1, and consequently is equal to $2 \times 2.302,585,092,994,045,668$, or to 4.605,170,185,988,091,336.

The ratio of 2400 to 1 is equal to the sum of the ratios of 2400 to 100 and of 100 to 1. Therefore the logarithm of the ratio of 2400 to 1 is equal to the sum of the logarithms of the ratios of 2400 to 100 and of 100 to 1, that is, to the sum of
of

of 3.178,053,830,347,945,595 and 4.605,170,185,988,091,336, or to 7.783,224,016,336,036,931.

The ratio of 2401 to 1 is equal to the sum of the ratios of 2401 to 2400 and of 2400 to 1. Therefore the logarithm of the ratio of 2401 to 1 is equal to the sum of the logarithms of the ratios of 2401 to 2400 and of 2400 to 1. But it has been shewn in Example 5th, that the logarithm of the ratio of 2401 to 2400 is 0.000,416,579,885,216,232; and we have just now seen that the logarithm of the ratio of 2400 to 1 is 7.783,224,016,336,036,931. Therefore the logarithm of the ratio of 2401 to 1 is equal to 0.000,416,579,885,216,232 + 7.783,224,016,336,036,931 = 7.783,640,596,221,253,163.

The ratio of 7 to 1 is one fourth part of the ratio of 2401 to 1. Therefore the logarithm of the ratio of 7 to 1 is one fourth part of the logarithm of the ratio of 2401 to 1, and consequently is $= \frac{7.783,640,596,221,253,163}{4} = 1.945,910,149,055,313,290$; or, (according to the usual way of speaking on this subject,) the logarithm of 7 is 1.945,910,149,055,313,290. Q. E. I.

44. The ratio of 14 to 1 is equal to the sum of the ratios of 14 to 7 and of 7 to 1, or (because the ratio of 14 to 7 is equal to the ratio of 2 to 1) to the sum of the ratios of 2 to 1 and of 7 to 1. Therefore the logarithm of the ratio of 14 to 1 is equal to the sum of the logarithms of the ratios of 2 to 1 and of 7 to 1; that is, to the sum of 0.693,147,180,559,945,304 and 1.945,910,149,055,313,290, or to 2.639,057,329,615,258,594; or 2.639,057,329,615,258,594 is the logarithm of 14. Q. E. I.

45. The ratio of 21 to 1 is equal to the sum of the ratios of 21 to 7 and of 7 to 1, or (because the ratio of 21 to 7 is equal to the ratio of 3 to 1) to the sum of the ratios of 3 to 1 and of 7 to 1. Therefore the logarithm of the ratio of 21 to 1 is equal to the sum of the logarithms of the ratios of 3 to 1 and of 7 to 1; that is, to the sum of 1.098,612,288,668,109,683 and 1.945,910,149,055,313,290, or to 3.044,522,437,723,422,973; or, in other words, the logarithm of 21 is 3.044,522,437,723,422,973. Q. E. I.

46. The ratio of 16 to 1 is equal to four times the ratio of 2 to 1. Therefore the logarithm of the ratio of 16 to 1 will be equal to four times the logarithm of the ratio of 2 to 1; that is, to four times 0.693,147,180,559,945,304, or to 2.772,588,722,239,781,216. Therefore 2.772,588,722,239,781,216 is the logarithm of 16. Q. E. I.

47. The ratio of 18 to 1 is equal to the sum of the ratios of 18 to 9 and of 9 to 1, or (because the ratio of 18 to 9 is equal to the ratio of 2 to 1,) to the sum of the ratios of 2 to 1 and of 9 to 1. Therefore the logarithm of the ratio of 18 to 1 is equal to the sum of the logarithms of the ratios of 2 to 1 and of 9 to 1; that is, to the sum of 0.693,147,180,559,945,304 and 2.197,224,577,336,219,366, or to

to 2.890,371,757,896,164,670; or the logarithm of 18 is 2.890,371,757,896,164,670. Q. E. I.

48. The ratio of 12 to 1 is equal to the sum of the ratios of 12 to 6 and of 6 to 1, or (because the ratio of 12 to 6 is equal to the ratio of 2 to 1,) to the sum of the ratios of 2 to 1 and of 6 to 1. Therefore the logarithm of the ratio of 12 to 1 is equal to the sum of the logarithms of the ratios of 2 to 1 and of 6 to 1; that is, to the sum of 0.693,147,180,559,945,304 and 1.791,759,469,228,054,987, or to 2.484,906,649,788,000,294; or the logarithm of 12 is 2.484,906,649,788,000,294. Q. E. I.

49. The ratio of 24 to 1 is equal to the sum of the ratios of 24 to 12 and of 12 to 1, or (because the ratio of 24 to 12 is equal to the ratio of 2 to 1,) to the sum of the ratios of 2 to 1 and of 12 to 1. Therefore the logarithm of the ratio of 24 to 1 is equal to the sum of the logarithms of the ratios of 2 to 1 and of 12 to 1; that is, to the sum of 0.693,147,180,559,945,304 and 2.484,906,649,788,000,294, or to 3.178,053,830,347,945,598; or the logarithm of 24 is 3.178,053,830,347,945,598. Q. E. I.

50. The ratio of 15 to 1 is equal to the sum of the ratios of 15 to 5 and of 5 to 1, or (because the ratio of 15 to 5 is equal to the ratio of 3 to 1,) to the sum of the ratios of 3 to 1 and of 5 to 1. Therefore the logarithm of the ratio of 15 to 1 is equal to the sum of the logarithms of the ratios of 3 to 1 and of 5 to 1; that is, to the sum of 1.098,612,288,668,109,683 and 1.609,437,912,434,100,364, or to 2.708,050,201,102,210,047; or, in other words, the logarithm of 15 is 2.708,050,201,102,210,047. Q. E. I.

51. The ratio of 20 to 1 is equal to the sum of the ratios of 20 to 10 and of 10 to 1, or (because the ratio of 20 to 10 is equal to the ratio of 2 to 1,) to the sum of the ratios of 2 to 1 and of 10 to 1. Therefore the logarithm of the ratio of 20 to 1 is equal to the sum of the logarithms of the ratios of 2 to 1 and of 10 to 1; that is to the sum of 0.693,147,180,559,945,304 and 2.302,585,092,994,045,668, or to 2.995,732,273,553,990,972; or the logarithm of 20 is 2.995,732,273,553,990,972. Q. E. I.

52. The ratio of 22 to 1 is equal to the sum of the ratios of 22 to 11 and of 11 to 1, or (because the ratio of 22 to 11 is equal to the ratio of 2 to 1,) to the sum of the ratios of 2 to 1 and of 11 to 1. Therefore the logarithm of the ratio of 22 to 1 is equal to the sum of the logarithms of the ratios of 2 to 1 and of 11 to 1; that is, to the sum of 0.693,147,180,559,945,304 and 2.397,895,272,798,370,526, or to 3.091,042,453,358,315,830; or the logarithm of 22 is 3.091,042,453,358,315,830. Q. E. I.

53. To find the logarithm of the ratio of 13 to 1 we must proceed as follows.

The ratio of 168 to 1 is equal to the sum of the ratios of 168 to 21 and of 21 to 1, or (because the ratio of 168 to 21 is equal to the ratio of 8 to 1,) to the sum of the ratios of 8 to 1 and of 21 to 1. Therefore the logarithm of the ratio of 168 to 1 is equal to the sum of the logarithms of the ratios of 8 to 1 and of 21 to 1; that is, to the sum of 2.079,441,541,679,835,912 and 3.044,522,437,723,422,973, or to 5.123,963,979,403,258,885.

Further, the ratio of 169 to 1 is equal to the sum of the ratios of 169 to 168 and of 168 to 1. Therefore the logarithm of the ratio of 169 to 1 is equal to the sum of the logarithms of the ratios of 169 to 168 and of 168 to 1. But the logarithm of the ratio of 168 to 1 has just now been shewn to be $= 5.123,963,979,403,258,885$; and the logarithm of the ratio of 169 to 168 has been found in Example 6th to be $= 0.005,934,735,519,814,547$. Therefore the logarithm of the ratio of 169 to 1 will be equal to the sum of 0.005,934,735,519,814,547 and 5.123,963,979,403,258,885, or to 5.129,898,714,923,073,432. But the ratio of 13 to 1 is equal to half the ratio of 169 to 1, because 169 is the square of 13. Therefore the logarithm of the ratio of 13 to 1 will be equal to half the logarithm of the ratio of 169 to 1, and consequently will be equal to half of 5.129,898,714,923,073,432, or to 2.564,949,357,461,536,716; or, in other words, the logarithm of the number 13 will be 2.564,949,357,461,536,716. Q. E. I.

54. To find the logarithm of the ratio of 17 to 1 we must proceed as follows.

The ratio of 288 to 1 is equal to the sum of the ratios of 288 to 18 and of 18 to 1, or (because the ratio of 288 to 18 is equal to the ratio of 16 to 1, 288 being equal to 16×18 ,) to the sum of the ratios of 16 to 1 and of 18 to 1. Therefore the logarithm of the ratio of 288 to 1 is equal to the sum of the logarithms of the ratios of 16 to 1 and of 18 to 1; that is, to the sum of 2.772,588,722,239,781,216 and 2.890,371,757,896,164,670, or to 5.662,960,480,135,945,886.

Further, the ratio of 289 to 1 is equal to the sum of the ratios of 289 to 288 and of 288 to 1. Therefore the logarithm of the ratio of 289 to 1 is equal to the sum of the logarithms of the ratios of 289 to 288 and of 288 to 1. But the logarithm of the ratio of 289 to 288 has been found, in Example 7th, to be 0.003,466,207,976,486,229; and the logarithm of the ratio of 288 to 1 has been just now shewn to be 5.662,960,480,135,945,886. Therefore the logarithm of the ratio of 289 to 1 will be $= 0.003,466,207,976,486,229 + 5.662,960,480,135,945,886 = 5.666,426,688,112,432,115$. Therefore the logarithm of the ratio of 17 (which is the square-root of 289) to 1 will be $= \frac{5.666,426,688,112,432,115}{2} = 2.833,213,344,056,216,057$; or the logarithm of the number 17 will be 2.833,213,344,056,216,057. Q. E. I.

55. To find the logarithm of the ratio of 19 to 1 we must proceed as follows.

The ratio of 360 to 1 is equal to the sum of the ratios of 360 to 20 and of 20 to 1, or (because the ratio of 360 to 20 is equal to the ratio of 18 to 1,) to the sum of the

the ratios of 18 to 1 and of 20 to 1. Therefore the logarithm of the ratio of 360 to 1 is equal to the sum of the logarithms of the ratios of 18 to 1 and of 20 to 1; that is, to the sum of 2.890,371,757,896,164,670 and 2.905,732,273,553,990,972, or to 5.886,104,031,450,155,642.

Further, the ratio of 361 to 1 is equal to the sum of the ratios of 361 to 360 and of 360 to 1. Therefore the logarithm of the ratio of 361 to 1 is equal to the sum of the logarithms of the ratios of 361 to 360 and of 360 to 1. But the logarithm of the ratio of 361 to 360 has been found in Example 8th to be 0.002,773,926,882,725,234; and the logarithm of the ratio of 360 to 1 has been just now shewn to be 5.886,104,031,450,155,642. Therefore the logarithm of the ratio of 361 to 1 will be $= 0.002,773,926,882,725,234 + 5.886,104,031,450,155,642 = 5.888,877,958,332,880,876$; and consequently the logarithm of the ratio of 19 (which is the square-root of 361) to 1 will be $= \frac{5.888,877,958,332,880,876}{2} = 2.944,438,979,166,440,438$; or, in other words, the logarithm of the number 19 will be 2.944,438,979,166,440,438. Q. E. I.

56. It remains that we find the logarithm of the ratio of 23 to 1. Now this may be done as follows.

The ratio of 528 to 1 is equal to the sum of the ratios of 528 to 22 and of 22 to 1, or (because the ratio of 528 to 22 is equal to the ratio of 24 to 1, 528 being $= 22 \times 24$) to the sum of the ratios of 24 to 1 and of 22 to 1. Therefore the logarithm of the ratio of 528 to 1 is equal to the sum of the logarithms of the ratios of 24 to 1 and of 22 to 1, that is, to the sum of 3.178,053,830,347,945,598 and 3.091,042,453,358,315,830, or to 6.269,096,283,706,261,428.

Further, the ratio of 529 to 1 is equal to the sum of the ratios of 529 to 528 and of 528 to 1. Therefore the logarithm of the ratio of 529 to 1 is equal to the sum of the logarithms of the ratios of 529 to 528 and of 528 to 1. But the logarithm of the ratio of 529 to 528 has been found in Example 9th to be $= 0.001,892,148,152,037,909$; and we have just now seen that the logarithm of the ratio of 528 to 1 is $= 6.269,096,283,706,261,428$. It follows therefore that the logarithm of the ratio of 529 to 1 will be $= 0.001,892,148,152,037,909 + 6.269,096,283,706,261,428 = 6.270,988,431,858,299,337$; and consequently the logarithm of the ratio of 23 (which is the square-root of 529) to 1 will be equal to half the logarithm 6.270,988,431,858,299,337, or to 3.135,494,215,929,149,668; or, in other words, the logarithm of the number 23 is $= 3.135,494,215,929,149,668$. Q. E. I.

57. It appears therefore that the logarithms of the ratios of the first 23 natural numbers, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, and 24, to 1, or (according to the common way of speaking on this subject) the logarithms of those numbers are as follows; to wit,

$$\text{Log. 2.} = 0.693,147,180,559,945,304;$$

$$\text{Log. 3.} = 1.098,612,288,668,109,683;$$

Log.

Log. 4.	=	1.386,294,361,119,890,608;
Log. 5.	=	1.609,437,912,434,100,364;
Log. 6.	=	1.791,759,469,228,054,985;
Log. 7.	=	1.945,910,149,055,313,290;
Log. 8.	=	2.079,441,541,679,835,912;
Log. 9.	=	2.197,224,577,336,219,366;
Log. 10.	=	2.302,585,092,994,045,668;
Log. 11.	=	2.397,895,272,798,370,526;
Log. 12.	=	2.484,906,649,788,000,294;
Log. 13.	=	2.564,949,357,461,536,716;
Log. 14.	=	2.639,057,329,615,258,594;
Log. 15.	=	2.708,050,201,102,210,047;
Log. 16.	=	2.772,588,722,239,781,216;
Log. 17.	=	2.833,213,344,056,216,057;
Log. 18.	=	2.890,371,757,896,164,670;
Log. 19.	=	2.944,438,979,166,440,438;
Log. 20.	=	2.995,732,273,553,990,972;
Log. 21.	=	3.044,522,437,723,422,973;
Log. 22.	=	3.091,042,453,358,315,830;
Log. 23.	=	3.135,494,215,929,149,668;
Log. 24.	=	3.178,053,830,347,945,598.

These logarithms (if no mistakes have been made in my computing them) are exact to the sixteenth place of decimal figures, or in all but the two last places of figures. The first nine of them have been computed to a few more places of figures, by the learned Mr. Leonard Euler, and are inserted in his *Introductio in Analysin Infinitorum*, in two volumes quarto, published at Lausanne, in the year 1748. They are as follows,

Log. 2.	=	0.693,147,180,559,945,309,417,232,1;
Log. 3.	=	1.098,612,288,668,109,691,395,245,2;
Log. 4.	=	1.386,294,361,119,890,618,834,464,2;
Log. 5.	=	1.609,437,912,434,100,374,600,759,3;
Log. 6.	=	1.791,759,469,228,055,000,812,477,3;
Log. 7.	=	1.945,910,149,055,313,305,105,463,9;
Log. 8.	=	2.079,441,541,679,835,928,251,696,4;

N. n

Log.

$$\text{Log. } 9. = 2.197,224,577,336,219,382,790,490,5;$$

$$\text{Log. } 10. = 2.302,585,092,994,045,684,017,991,4.$$

See Euler's *Introductio in Analysin Infinitorum*, tom. i. page 91.

A recapitulation of the foregoing deductions of the logarithms of the ratios of 2 to 1, 3 to 1, 4 to 1, 5 to 1, and the other following numbers, up to 24, to 1, from the logarithms of the nine smaller ratios before computed, expressed in a more concise manner.

58. The foregoing deductions of the logarithms of the ratios of 2 to 1, 3 to 1, 4 to 1, and the other following numbers, up to 24, to 1, from the logarithms of the nine smaller ratios of 10 to 9, 11 to 10, 81 to 80, 121 to 120, 2401 to 2400, 169 to 168, 289 to 288, 361 to 360, and 529 to 528 (which were computed by means of the infinite series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \mathcal{E}c.$) may be expressed more concisely as follows:

Let the logarithm of any ratio be denoted by the capital letter L. prefixed to a fraction of which the antecedent of the ratio is the numerator, and the consequent of it is the denominator; so that, for example, the logarithm of the ratio of 10 to 9 shall be denoted by $L. \frac{10}{9}$.

Then we shall have in the 1st place,

$$L. \frac{10}{9} = 0.105,360,515,657,826,302;$$

$$2\text{dly, } L. \frac{11}{10} = 0.095,310,179,804,324,858;$$

$$3\text{dly, } L. \frac{81}{80} = 0.012,422,519,998,557,152;$$

$$4\text{thly, } L. \frac{121}{120} = 0.008,298,802,814,695,093;$$

$$5\text{thly, } L. \frac{2401}{2400} = 0.000,416,579,885,216,232;$$

$$6\text{thly, } L. \frac{169}{168} = 0.005,934,735,519,814,547;$$

$$7\text{thly, } L. \frac{289}{288} = 0.003,466,207,976,486,229;$$

$$8\text{thly, } L. \frac{361}{360} = 0.002,773,926,882,725,234;$$

$$\text{And } 9\text{thly, } L. \frac{529}{528} = 0.001,892,148,152,037,909.$$

These logarithms were all computed by means of the series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \mathcal{E}c.$ in the above nine examples. And from these original logarithms

logarithms all the others were derived, by reasonings which may be more briefly expressed in the following manner:

$$59. L. \frac{11}{9} \text{ is } = L. \frac{11}{10} + L. \frac{10}{9} = \left\{ + \begin{array}{l} 0.095,310,179,804,324,858 \\ 0.105,360,515,657,826,302 \end{array} \right\} = 0.200,670,695,462,151,160.$$

$$L. \frac{121}{81} \text{ is } = L. \frac{11^2}{9^2} = 2 L. \frac{11}{9} = 2 \times 0.200,670,695,462,151,160 = 0.401,341,390,924,302,320.$$

$$L. \frac{121}{80} \text{ is } = L. \frac{121}{81} + L. \frac{81}{80} = \left\{ + \begin{array}{l} 0.401,341,390,924,302,320 \\ 0.012,422,519,998,557,152 \end{array} \right\} = 0.413,763,910,922,859,472.$$

$$L. \frac{120}{80} \text{ is } = L. \frac{121}{80} - L. \frac{121}{120} = \left\{ - \begin{array}{l} 0.413,763,910,922,859,472 \\ 0.008,298,802,814,695,093 \end{array} \right\} = 0.405,465,108,108,164,379.$$

$$L. \frac{3}{2} \text{ is } = L. \frac{120}{80} = 0.405,465,108,108,164,379.$$

$$L. \frac{9}{4} \text{ is } = L. \frac{3^2}{2^2} = 2 \times L. \frac{3}{2} = 2 \times 0.405,465,108,108,164,379 = 0.810,930,216,216,328,758.$$

$$L. \frac{81}{16} \text{ is } = L. \frac{9^2}{4^2} = 2 \times L. \frac{9}{4} = 2 \times 0.810,930,216,216,328,758 = 1.621,860,432,432,657,516.$$

$$L. \frac{80}{16} \text{ is } = L. \frac{81}{16} - L. \frac{81}{80} = \left\{ - \begin{array}{l} 1.621,860,432,432,657,516 \\ 0.012,422,519,998,557,152 \end{array} \right\} = 1.609,437,912,434,100,364.$$

$$L. \frac{5}{1} \text{ is } = L. \frac{80}{16} = 1.609,437,912,434,100,364; \text{ or, in the usual way of expressing it, the logarithm of the number 5 is } = 1.609,437,912,434,100,364. \quad Q. E. I.$$

$$L. \frac{10}{2} \text{ is } = L. \frac{5}{1} = 1.609,437,912,434,100,364.$$

$$L. \frac{10}{4} \text{ is } = L. \frac{10}{9} + L. \frac{9}{4} = \left\{ + \begin{array}{l} 0.105,360,515,657,826,302 \\ 0.810,930,216,216,328,758 \end{array} \right\} = 0.916,290,731,874,155,060.$$

$$L. \frac{4}{2} \text{ is } = L. \frac{10}{2} - L. \frac{10}{4} = \left\{ - \begin{array}{l} 1.609,437,912,434,100,364 \\ 0.916,290,731,874,155,060 \end{array} \right\} = 0.693,147,180,559,945,304.$$

$$L. \frac{2}{1} \text{ is } = L. \frac{4}{2} = 0.693,147,180,559,945,304; \text{ or the logarithm of the number 2 is } = 0.693,147,180,559,945,304. \quad Q. E. I.$$

$$L. \frac{4}{1} \text{ is } = 2 \times L. \frac{2}{1} = 2 \times 0.693,147,180,559,945,304 = 1.386,294,361,119,890,608; \text{ or the logarithm of the number 4 is } 1.386,294,361,119,890,608. \quad Q. E. I.$$

$L. \frac{8}{1}$ is $= 3 \times L. \frac{2}{1} = 3 \times 0.693,147,180,559,945,304 = 2.079,441,541,679,835,912$; or the logarithm of the number 8 is 2.079,441,541,679,835,912. Q. E. I.

$L. \frac{9}{1}$ is $= L. \frac{9}{4} + L. \frac{4}{1} = \left\{ + \begin{smallmatrix} 0.810,930,216,216,328,758 \\ 1.386,294,361,119,890,608 \end{smallmatrix} \right\} = 2.197,224,577,336,219,366$; or the logarithm of the number 9 is 2.197,224,577,336,219,366. Q. E. I.

$L. \frac{3}{1}$ is $= \frac{1}{2} \times L. \frac{9}{1} = \frac{2.197,224,577,336,219,366}{2} = 1.098,612,288,668,109,683$; or the logarithm of the number 3 is 1.098,612,288,668,109,683. Q. E. I.

$L. \frac{6}{1}$ is $= L. \frac{6}{3} + L. \frac{3}{1} = L. \frac{2}{1} + L. \frac{3}{1} = \left\{ + \begin{smallmatrix} 0.693,147,180,559,945,304 \\ 1.098,612,288,668,109,683 \end{smallmatrix} \right\} = 1.791,759,469,228,054,987$; or the logarithm of the number 6 is 1.791,759,469,228,054,987. Q. E. I.

$L. \frac{10}{1}$ is $= L. \frac{10}{9} + L. \frac{9}{1} = \left\{ + \begin{smallmatrix} 0.105,360,515,657,826,302 \\ 2.197,224,577,336,219,366 \end{smallmatrix} \right\} = 2.302,585,092,994,045,668$; or the logarithm of the number 10 is 2.302,585,092,994,045,668. Q. E. I.

The more exact value of this logarithm is 2.302,585,092,994,045,684,017.

$L. \frac{11}{1}$ is $= L. \frac{11}{10} + L. \frac{10}{1} = \left\{ + \begin{smallmatrix} 0.095,310,179,804,324,858 \\ 2.302,585,092,994,045,668 \end{smallmatrix} \right\} = 2.397,895,272,798,370,526$; or the logarithm of the number 11 is 2.397,895,272,798,370,526. Q. E. I.

$L. \frac{24}{1}$ is $= L. \frac{24}{8} + L. \frac{8}{1} = L. \frac{3}{1} + L. \frac{8}{1} = \left\{ + \begin{smallmatrix} 1.098,612,288,668,109,683 \\ 2.079,441,541,679,835,912 \end{smallmatrix} \right\} = 3.178,053,830,347,945,595$.

$L. \frac{2400}{100}$ is $= L. \frac{24}{1} = 3.178,053,830,347,945,595$.

$L. \frac{100}{1}$ is $= 2 \times L. \frac{10}{1} = 2 \times 2.302,585,092,994,045,668 = 4.605,170,185,988,091,336$.

$L. \frac{2400}{1}$ is $= L. \frac{2400}{100} + L. \frac{100}{1} = L. \frac{24}{1} + L. \frac{100}{1} = \left\{ + \begin{smallmatrix} 3.178,053,830,347,945,595 \\ 4.605,170,185,988,091,336 \end{smallmatrix} \right\} = 7.783,224,016,336,036,931$.

$L. \frac{2401}{1}$ is $= L. \frac{2401}{2400} + L. \frac{2400}{1} = \left\{ + \begin{smallmatrix} 0.000,416,579,885,216,232 \\ 7.783,224,016,336,036,931 \end{smallmatrix} \right\} = 7.783,640,596,221,253,163$.

$L. \frac{7}{1}$ is $= \frac{1}{4} L. \frac{2401}{1} = \frac{7.783,640,596,221,253,163}{4} = 1.945,910,149,055,313,290$; or the logarithm of the number 7 is 1.945,910,149,055,313,290. Q. E. I.

$L. \frac{14}{1} \text{ is } = L. \frac{14}{7} + L. \frac{7}{1} = L. \frac{2}{1} + L. \frac{7}{1} = \left\{ \begin{array}{l} 0.693,147,180,559,945,304 \\ + 1.945,910,149,055,313,290 \end{array} \right\} = 2.639,057,329,615,258,594;$
 or the logarithm of the number 14 is $= 2.639,057,329,615,258,594.$ Q. E. I.

$L. \frac{21}{1} \text{ is } = L. \frac{21}{7} + L. \frac{7}{1} = L. \frac{3}{1} + L. \frac{7}{1} = \left\{ \begin{array}{l} 1.098,612,288,668,109,683 \\ + 1.945,910,149,055,313,290 \end{array} \right\} = 3.044,522,437,723,422,973;$
 or the logarithm of the number 21 is $= 3.044,522,437,723,422,973.$ Q. E. I.

$L. \frac{16}{1} \text{ is } = 4 \times L. \frac{2}{1} = 4 \times 0.693,147,180,559,945,304 = 2.772,588,722,239,781,216;$ or
 the logarithm of 16 is $2.772,588,722,239,781,216.$ Q. E. I.

$L. \frac{18}{1} \text{ is } = L. \frac{18}{9} + L. \frac{9}{1} = L. \frac{2}{1} + L. \frac{9}{1} = \left\{ \begin{array}{l} 0.693,147,180,559,945,304 \\ + 2.197,224,577,336,219,366 \end{array} \right\} = 2.890,371,757,896,164,670;$
 or the logarithm of 18 is $2.890,371,757,896,164,670.$ Q. E. I.

$L. \frac{12}{1} \text{ is } = L. \frac{12}{6} + L. \frac{6}{1} = L. \frac{2}{1} + L. \frac{6}{1} = \left\{ \begin{array}{l} 0.693,147,180,559,945,304 \\ + 1.791,759,469,228,054,987 \end{array} \right\} = 2.484,906,649,788,000,294;$
 or the logarithm of 12 is $2.484,906,649,788,000,294.$ Q. E. I.

$L. \frac{24}{1} \text{ is } = L. \frac{24}{12} + L. \frac{12}{1} = L. \frac{2}{1} + L. \frac{12}{1} = \left\{ \begin{array}{l} 0.693,147,180,559,945,304 \\ + 2.484,906,649,788,000,294 \end{array} \right\} = 3.178,053,830,347,945,598;$
 or the logarithm of 24 is $3.178,053,830,347,945,598.$ Q. E. I.

$L. \frac{15}{1} \text{ is } = L. \frac{15}{5} + L. \frac{5}{1} = L. \frac{3}{1} + L. \frac{5}{1} = \left\{ \begin{array}{l} 1.098,612,288,668,109,683 \\ + 1.609,437,912,434,100,364 \end{array} \right\} = 2.708,050,201,102,210,047;$
 or the logarithm of 15 is $2.708,050,201,102,210,047.$ Q. E. I.

$L. \frac{20}{1} \text{ is } = L. \frac{20}{10} + L. \frac{10}{1} = L. \frac{2}{1} + L. \frac{10}{1} = \left\{ \begin{array}{l} 0.693,147,180,559,945,304 \\ + 2.302,585,092,994,045,668 \end{array} \right\} = 2.995,732,273,553,990,972;$
 or the logarithm of 20 is $2.995,732,273,553,990,972.$ Q. E. I.

$L. \frac{22}{1} \text{ is } = L. \frac{22}{11} + L. \frac{11}{1} = L. \frac{2}{1} + L. \frac{11}{1} = \left\{ \begin{array}{l} 0.693,147,180,559,945,304 \\ + 2.397,895,272,798,370,526 \end{array} \right\} = 3.091,042,453,358,315,830;$
 or the logarithm of 22 is $3.091,042,453,358,315,830.$ Q. E. I.

$L. \frac{168}{1} \text{ is } = L. \frac{168}{21} + L. \frac{21}{1} = L. \frac{8}{1} + L. \frac{21}{1} = \left\{ \begin{array}{l} 2.079,441,541,679,835,912 \\ + 3.044,522,437,723,422,973 \end{array} \right\} = 5.123,963,979,403,258,885.$

$L. \frac{169}{1} \text{ is } = L. \frac{169}{168} + L. \frac{168}{1} = \left\{ \begin{array}{l} 0.005,934,735,519,814,547 \\ + 5.123,963,979,403,258,885 \end{array} \right\} = 5.129,898,714,923,073,432.$

$L. \frac{13}{1} \text{ is } = \frac{1}{2} \times L. \frac{169}{1} = \frac{5.129,898,714,923,073,432}{2} = 2.564,949,357,461,536,716$; or the logarithm of 13 is 2.564,949,357,461,536,716. Q. E. I.

$L. \frac{288}{1} \text{ is } = L. \frac{288}{18} + L. \frac{18}{1} = L. \frac{16}{1} + L. \frac{18}{1} = \left\{ \begin{array}{l} 2.772,588,722,239,781,216 \\ + 2.890,371,757,896,164,670 \end{array} \right\} = 5.662,960,480,135,945,886.$

$L. \frac{289}{1} \text{ is } = L. \frac{289}{288} + L. \frac{288}{1} = \left\{ \begin{array}{l} 0.003,466,207,976,486,229 \\ + 5.662,960,480,135,945,886 \end{array} \right\} = 5.666,426,688,112,432,115.$

$L. \frac{17}{1} \text{ is } = \frac{1}{2} \times L. \frac{289}{1} = \frac{5.666,426,688,112,432,115}{2} = 2.833,213,344,056,216,057$; or the logarithm of 17 is 2.833,213,344,056,216,057. Q. E. I.

$L. \frac{360}{1} \text{ is } = L. \frac{360}{20} + L. \frac{20}{1} = L. \frac{18}{1} + L. \frac{20}{1} = \left\{ \begin{array}{l} 2.890,371,757,896,164,670 \\ + 2.995,732,273,553,990,972 \end{array} \right\} = 5.886,104,031,450,155,642.$

$L. \frac{361}{1} \text{ is } = L. \frac{361}{360} + L. \frac{360}{1} = \left\{ \begin{array}{l} 0.002,773,926,882,725,234 \\ + 5.886,104,031,450,155,642 \end{array} \right\} = 5.888,877,958,332,880,876.$

$L. \frac{19}{1} \text{ is } = \frac{1}{2} \times L. \frac{361}{1} = \frac{5.888,877,958,332,880,876}{2} = 2.944,438,979,166,440,438$; or the logarithm of 19 is 2.944,438,979,166,440,438. Q. E. I.

$L. \frac{528}{1} \text{ is } = L. \frac{528}{22} + L. \frac{22}{1} = L. \frac{24}{1} + L. \frac{22}{1} = \left\{ \begin{array}{l} 3.178,053,830,347,945,598 \\ + 3.091,042,453,358,315,830 \end{array} \right\} = 6.269,096,283,706,261,428.$

$L. \frac{529}{1} \text{ is } = L. \frac{529}{528} + L. \frac{528}{1} = \left\{ \begin{array}{l} 0.001,892,148,152,037,909 \\ + 6.269,096,283,706,261,428 \end{array} \right\} = 6.270,988,431,858,299,337.$

$L. \frac{23}{1} \text{ is } = \frac{1}{2} \times L. \frac{529}{1} = \frac{6.270,988,431,858,299,337}{2} = 3.135,494,215,929,149,668$; or the logarithm of the number 23 is 3.135,494,215,929,149,668. Q. E. I.

Concerning the logarithmick series invented by Dr. Wallis.

I shall now proceed to consider the series $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \mathcal{E}c$, which was invented by Dr. Wallis, and which, if A be supposed to be less than 1, is equal to the logarithm of the ratio of 1 to $1 - A$.

60. The proposition of which I here mean to establish the truth, is this.

If there be two different quantities, A and B , that are, both of them, less than 1, and of which, consequently, the several powers $A^2, A^3, A^4, A^5, A^6, \mathcal{E}c$, and $B^2, B^3, B^4, B^5, B^6, \mathcal{E}c$ are decreasing quantities, the two infinite serieses $A + \frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \mathcal{E}c$ and $B + \frac{B^2}{2} + \frac{B^3}{3} + \frac{B^4}{4} + \frac{B^5}{5} + \frac{B^6}{6} + \mathcal{E}c$ (which, it is evident, will be decreasing progressions,) will be measures, or logarithms, of the ratios of 1 to $1 - A$ and of 1 to $1 - B$; or the series $A +$

$\frac{A^2}{2} + \frac{A^3}{3} + \frac{A^4}{4} + \frac{A^5}{5} + \frac{A^6}{6} + \mathcal{E}c$, will be to the series $B + \frac{B^2}{2} + \frac{B^3}{3} + \frac{B^4}{4} + \frac{B^5}{5} + \frac{B^6}{6} + \mathcal{E}c$ in the same proportion as the ratio of 1 to $1 - A$ is to the ratio of 1 to $1 - B$.

Or, if we change the notation a little, and substitute the small letters k and q instead of the capital letters A and B respectively (which will be more agreeable to the notation now most in use in treating of these kinds of serieses), the proposition which we are to demonstrate will be as follows.

If there be two different quantities k and q , that are both of them less than 1, and of which, consequently, the several powers $k^2, k^3, k^4, k^5, k^6, \mathcal{E}c$, and $q^2, q^3, q^4, q^5, q^6, \mathcal{E}c$ will be decreasing quantities, the two infinite serieses $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c$ and $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \mathcal{E}c$, (which, it is evident, will be decreasing progressions) will be measures or logarithms, of the ratios of 1 to $1 - k$ and of 1 to $1 - q$, or the series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c$ will be to the series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \mathcal{E}c$ in the same proportion as the ratio of 1 to $1 - k$ to the ratio of 1 to $1 - q$.

In order to demonstrate this proposition it will be necessary to premise the following proposition, as a Lemma.

L E M M A III.

61. If x be any quantity less than 1, and n be any whole number whatsoever, the quantity $1 - x^{\frac{1}{n}}$, or $\sqrt[n]{1 - x}$, that is, the n^{th} root of the residual quantity $1 - x$, will be equal to the following infinite series of decreasing terms, to wit,

$$\begin{aligned} 1 - \frac{1}{n} \times x - \frac{1}{n} \times \frac{n-1}{2n} \times x^2 \\ - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times x^3 \\ - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times x^4 \\ - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n} \times x^5 \\ - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n} \times \frac{5n-1}{6n} \times x^6 - \mathcal{E}c \end{aligned}$$

ad infinitum, or (if we put $A = 1$, and $B = \frac{1}{n}$, and $C = \frac{1}{n} \times \frac{n-1}{2n}$, and $D = \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}$, and $E, F, G, H, I, K, \mathcal{E}c$ for the several following co-

efficients of the powers of x in this series, respectively), to the infinite series $1 - \frac{1}{n}$
 $\times A x - \frac{n-1}{2n} \times B x^2 - \frac{2n-1}{3n} \times C x^3 - \frac{3n-1}{4n} \times D x^4 - \frac{4n-1}{5n} \times E x^5 -$
 $\frac{5n-1}{6n} \times F x^6 - \frac{6n-1}{7n} \times G x^7 - \frac{7n-1}{8n} \times H x^8 - \frac{8n-1}{9n} \times I x^9 - \frac{9n-1}{10n}$
 $\times K x^{10} - \text{\textit{\textcircled{E}c ad infinitum}}$; in which series all the terms after the first term 1
 are marked with the sign $-$, or are to be subtracted from the said first term, and
 the several co-efficients of x , x^2 , x^3 , x^4 , x^5 , x^6 , x^7 , x^8 , x^9 , x^{10} , $\text{\textcircled{E}c}$, are derived,
 or generated, from the first term 1 by the continual, or successive, multiplication
 of the fractions $\frac{1}{n}$, $\frac{n-1}{2n}$, $\frac{2n-1}{3n}$, $\frac{3n-1}{4n}$, $\frac{4n-1}{5n}$, $\frac{5n-1}{6n}$, $\frac{6n-1}{7n}$, $\frac{7n-1}{8n}$, $\frac{8n-1}{9n}$,
 $\frac{9n-1}{10n}$, $\text{\textcircled{E}c}$, which are therefore called their *generating fractions*.

62. The law by which all these generating fractions after the first fraction $\frac{1}{n}$
 are formed, or derived, from the said first fraction $\frac{1}{n}$ and from each other, is as
 follows. The numerator $n-1$ of the second fraction $\frac{n-1}{2n}$ is formed from 1,
 the numerator of the first fraction $\frac{1}{n}$, by the addition of $n-2$, or by the addi-
 tion of n and the subtraction of 2; for $1 + n - 2 = n - 1$. And the nume-
 rators of all the following fractions after the second fraction $\frac{n-1}{2n}$, to wit, the nu-
 merators $2n-1$, $3n-1$, $4n-1$, $5n-1$, $6n-1$, $7n-1$, $8n-1$,
 $9n-1$, $\text{\textcircled{E}c}$, are formed from the numerator $n-1$ of the said second fraction,
 and from each other by the continual addition of n to every preceding numerator.
 And the denominators of the second fraction, $\frac{n-1}{2n}$, and the third fraction, $\frac{2n-1}{3n}$,
 and all the following fractions, to wit, the denominators $2n$, $3n$, $4n$, $5n$, $6n$,
 $7n$, $8n$, $9n$, $\text{\textcircled{E}c}$, are formed from n , the denominator of the first fraction, $\frac{1}{n}$
 and from each other by the continual addition of n to the preceding deno-
 minator.

63. This is another branch of the above-mentioned celebrated theorem of Sir
 Isaac Newton for finding the roots of a binomial quantity. For it extends to the
 roots of a residual quantity, as $1 - x$, or of the difference of 1 and x , as well as to
 the roots of a binomial quantity, properly so called, as $1 + x$, or the sum of 1 and x .
 And the foregoing manner of expressing this theorem in this case of the n^{th} root
 of $1 - x$ is, as I apprehend, the most intelligible and convenient way of expressing
 it that can be chosen. This series is derived from the series $1 - \frac{m}{1} A x +$

$$\frac{m-1}{2} B x^2 - \frac{m-2}{3} C x^3 + \frac{m-3}{4} D x^4 - \frac{m-4}{5} E x^5 + \frac{m-5}{6} F x^6 - \mathfrak{E}^2 c \text{ or } 1 -$$

$$\frac{m}{1} \times 1 \times x + \frac{m}{1} \times \frac{m-1}{2} \times x^2 - \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times x^3 + \frac{m}{1} \times \frac{m-1}{2}$$

$$\times \frac{m-2}{3} \times \frac{m-3}{4} \times x^4 - \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times x^5 + \frac{m}{1}$$

$$\times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times \frac{m-5}{6} \times x^6 - \mathfrak{E}^2 c, \text{ (which is } = \overline{1-x}^m \text{)}$$

by substituting $\frac{1}{x}$ in its terms instead of m .

64. Coroll. 1. It follows from the foregoing Lemma, that $1 - \sqrt[n]{1 - x}$, or the excess of 1 above the n^{th} root of the residual quantity $1 - x$, is equal to the infinite series $\frac{1}{n} A x + \frac{n-1}{2n} B x^2 + \frac{2n-1}{3n} C x^3 + \frac{3n-1}{4n} D x^4 + \frac{4n-1}{5n} E x^5 + \frac{5n-1}{6n} F x^6 + \frac{6n-1}{7n} G x^7 + \frac{7n-1}{8n} H x^8 + \frac{8n-1}{9n} I x^9 + \frac{9n-1}{10n} K x^{10} + \mathcal{E}c$, *ad infinitum*; in which A is, as before, = 1; and B is = $\frac{1}{n} A$, or the co-efficient of x ; and C is = $\frac{n-1}{2n} B$, or the co-efficient of x^2 ; and D is = $\frac{2n-1}{3n} C$, or the co-efficient of x^3 ; and the following capital letters, E, F, G, H, I, K, $\mathcal{E}c$, are equal to, or stand for, the co-efficients of x^4 , x^5 , x^6 , x^7 , x^8 , x^9 , and the other following powers of x , and in which all the terms are marked with the sign +, or added to each other.

65. Coroll. 2. If n be a very large number, as, for example, the ninth power of a million, or 1 with 54 cyphers annexed to it, or 1,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000, (which Mr. Locke, in his Essay on Human Understanding, calls a *nonillion*,) the quantity $\frac{1}{\sqrt[n]{1-x}}$, or the excess of 1 above the n^{th} root of the residual quantity $1 - x$, will be very nearly equal to the series $\frac{x}{n} + \frac{x^2}{2n} + \frac{x^3}{3n} + \frac{x^4}{4n} + \frac{x^5}{5n} + \frac{x^6}{6n} + \&c$
ad infinitum, or to $\frac{1}{n}$ $\times$ the series $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \&c$ ad
infinitum. And the number n may be taken of so great a magnitude that the ratio of
 $\frac{1}{n}$ $\times$ the series $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \&c$ ad infinitum to $\frac{1}{\sqrt[n]{1-x}}$
 $\frac{1}{\sqrt[n]{1-x}}$ shall approach as near to a ratio of equality as we please.

For, when n is equal to a nonillion, or any such very great number, it is evident that $n - 1$, and $2n - 1$ and $3n - 1$ and $4n - 1$ and $5n - 1$, &c, will be very nearly equal to n , $2n$, $3n$, $4n$, $5n$, &c, on account of the immense magnitudes of the numbers n , $2n$, $3n$, $4n$, $5n$, &c, in comparison of the unit which is subtracted from them. Therefore the series $\frac{1}{n} A x + \frac{n-1}{2n} B x^2 +$

$\frac{2n-1}{3^n} C x^3 + \frac{3n-1}{4^n} D x^4 + \frac{4n-1}{5^n} E x^5 + \frac{5n-1}{6^n} F x^6 + \mathcal{E}c$ (which is $= 1 - \overline{1-x}^{\frac{1}{n}}$) will in this case be very nearly $= \frac{1}{n} A x + \frac{n}{2n} B x^2 + \frac{2n}{3n} C x^3 + \frac{3n}{4n} D x^4 + \frac{4n}{5n} E x^5 + \frac{5n}{6n} F x^6 + \mathcal{E}c$. $= \frac{1}{n} A x + \frac{1}{2} B x^2 + \frac{2}{3} C x^3 + \frac{3}{4} D x^4 + \frac{4}{5} E x^5 + \frac{5}{6} F x^6 + \mathcal{E}c = \frac{1}{n} \times 1 \times x + \frac{1}{2} \times \frac{1}{n} \times 1 \times x^2 + \frac{2}{3} \times \frac{1}{2} \times \frac{1}{n} \times 1 \times x^3 + \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{n} \times 1 \times x^4 + \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{n} \times 1 \times x^5 + \frac{5}{6} \times \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{n} \times 1 \times x^6 + \mathcal{E}c = \frac{1}{n} x + \frac{1}{2n} x^2 + \frac{1}{3n} x^3 + \frac{1}{4n} x^4 + \frac{1}{5n} x^5 + \frac{1}{6n} x^6 + \mathcal{E}c = \frac{1}{n} \times x + \frac{1}{n} \times \frac{x^2}{2} + \frac{1}{n} \times \frac{x^3}{3} + \frac{1}{n} \times \frac{x^4}{4} + \frac{1}{n} \times \frac{x^5}{5} + \frac{1}{n} \times \frac{x^6}{6} + \mathcal{E}c = \frac{1}{n} \times \text{the series } x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \mathcal{E}c$. Therefore $1 - \overline{1-x}^{\frac{1}{n}}$ will in this case be very nearly equal to $\frac{1}{n} \times \text{the series } x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \mathcal{E}c$ *ad infinitum*. And it is evident that, if any ratio whatsoever be assigned that differs very little from a ratio of equality, n may be taken of so great a magnitude that the proportion of $1 - \overline{1-x}^{\frac{1}{n}}$ to $\frac{1}{n} \times \text{the series } x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \mathcal{E}c$ shall approach still nearer to a ratio of equality than such assigned ratio.

Q. E. D.

66. This lemma being premised, the main proposition may be stated and demonstrated in the manner following.

THEOREM II.

If k and q are any two small quantities less than 1, whereof we will suppose k to be the greater; the ratio of 1 to $1 - k$ will be to the ratio of 1 to $1 - q$ in the same proportion as the infinite series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c$ to the infinite series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \mathcal{E}c$.

DEMONSTRATION.

Let n be put, as before, for any very large number, as, for example, for a nonillion, or the ninth power of a million.

3

Then,

Then, by the 2d Corollary of the foregoing Lemma, we shall have $1 - \sqrt[n]{1 - k}$, very nearly, $= \frac{1}{n} \times$ the series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c \text{ ad infinitum}$, and $1 - \sqrt[n]{1 - q}$, very nearly, $= \frac{1}{n} \times$ the series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \mathcal{E}c \text{ ad infinitum}$. And, by increasing the magnitude of n , each of these ratios may be made to come as near to a ratio of equality as we please. Therefore the proportion of $1 - \sqrt[n]{1 - k}$ to $1 - \sqrt[n]{1 - q}$ is, in this case of the very great magnitude of n , very nearly the same as that of $\frac{1}{n} \times$ the series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c \text{ ad infinitum}$ to $\frac{1}{n} \times$ the series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \mathcal{E}c \text{ ad infinitum}$, and consequently, as that of n times $\frac{1}{n} \times$ the series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c \text{ ad infinitum}$ to n times $\frac{1}{n} \times$ the series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \mathcal{E}c \text{ ad infinitum}$, or as that of the series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c \text{ ad infinitum}$ to the series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \mathcal{E}c \text{ ad infinitum}$; that is, the proportion of the excess of 1 above the n^{th} root of the residual quantity $1 - k$ to the excess of 1 above the n^{th} root of the residual quantity $1 - q$, is, in this case of the very great magnitude of n , very nearly the same as that of the series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c \text{ ad infinitum}$ to the series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \mathcal{E}c \text{ ad infinitum}$.

Now the ratio of 1 to $1 - k$ is to the ratio of 1 to $1 - q$ in the same proportion as any given part of the former ratio is to the like part of the latter ratio, and consequently, as the n^{th} , or nonillionth, part of the former ratio is to the n^{th} , or nonillionth, part of the latter ratio. But

the ratio of 1 to the n^{th} root of $1 - k$ or to $\sqrt[n]{1 - k}$ is the n^{th} part of the ratio of 1 to $1 - k$; and the ratio of 1 to the n^{th} root of $1 - q$ is the n^{th} part of the ratio of 1 to $1 - q$. Therefore the ratio of 1 to $1 - k$ is to the ratio of 1 to $1 - q$ in the same proportion as the very small ratio of 1 to the n^{th} root of $1 - k$, or to $\sqrt[n]{1 - k}$, is to the very small ratio of 1 to the n^{th} root of $1 - q$, or to $\sqrt[n]{1 - q}$.

But it has been shewn above, in Lemma 1, article 11, that when three quantities are very nearly equal to each other, the ratio of the greatest of the three to the least will be to the ratio of the middle quantity to the least in very nearly

the same proportion as the excess of the greatest quantity above the least is to the excess of the middle quantity above the least; and hence it follows, *dividendo*, that the ratio of the greatest of the three to the least will be to the excess of the said ratio above the ratio of the middle quantity to the least, or to the ratio of the greatest quantity to the middle quantity, in the same proportion as the excess of the greatest quantity above the least is to the difference whereby the said excess exceeds the excess of the middle quantity above the least, or to the excess of the greatest quantity above the middle quantity: or, in other words, if L , M , and N are three quantities that are very nearly equal to each other, whereof L is the greatest and N is the least; it appears from Lemma 1, article 11, that the ratio of L to N will be to the ratio of M to N very nearly in the same proportion as the difference $L - N$ is to the difference $M - N$; and consequently, *dividendo*, the ratio of L to N will be to the excess of the ratio of L to N above the ratio of M to N , that is, to the ratio of L to M , very nearly in the same proportion as the difference $L - N$ is to the excess of the difference $L - N$ above the difference $M - N$, that is, to the difference $L - M$. Therefore in these three quantities, to wit, 1 and the n^{th} , or nonillionth, root of $1 - q$, and the n^{th} , or nonillionth, root of $1 - k$,

or 1, $\sqrt[n]{1 - q}$, and $\sqrt[n]{1 - k}$, (in which the excesses of the first quantity 1 above the two latter quantities are so extremely small, that they will not appear before the 54th place of decimal fractions), we may conclude, that the ratio of the greatest, to wit, 1, to the least, to wit, $\sqrt[n]{1 - k}$, will be to the ratio of 1, the greatest, to $\sqrt[n]{1 - q}$, the middle quantity, very nearly in the same proportion as $1 - \sqrt[n]{1 - k}$, the difference between the greatest and the least, is to $1 - \sqrt[n]{1 - q}$, the difference between the greatest and the middle quantity.

But it has been before shewn, that the ratio of 1 to $1 - k$ is to the ratio of 1 to $1 - q$ in the same proportion as the very small ratio of 1 to $\sqrt[n]{1 - k}$ is to the very small ratio of 1 to $\sqrt[n]{1 - q}$.

Therefore the ratio of 1 to $1 - k$ will be to the ratio of 1 to $1 - q$ very nearly in the same proportion as the excess of 1 above $\sqrt[n]{1 - k}$ to the excess of 1 above $\sqrt[n]{1 - q}$.

But it has been shewn that, in this case of the very great magnitude of n , the excess of 1 above $\sqrt[n]{1 - k}$ is to the excess of 1 above $\sqrt[n]{1 - q}$, very nearly in the same proportion as the series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c \text{ ad infinitum}$ to the series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \mathcal{E}c \text{ ad infinitum}$.

Therefore

Therefore the ratio of 1 to $1 - k$ will be to the ratio of 1 to $1 - q$, in the same proportion as the series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \&c$ *ad infinitum* to the series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \&c$ *ad infinitum*.

Q. E. D.

The foregoing demonstration expressed in fewer words.

67. The foregoing demonstration may be expressed in fewer words, as follows :

Let n be put as before, for any very large number ; as, for example, for a nonillion, or the ninth power of a million.

Then it will follow from Lemma 3, coroll. 2, that $1 - \sqrt[n]{1 - k}$ will be very nearly equal to $\frac{1}{n} \times$ the infinite series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \frac{k^7}{7} + \&c$, and that $1 - \sqrt[n]{1 - q}$ will, in like manner, be, very nearly, equal to $\frac{1}{n} \times$ the infinite series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \frac{q^7}{7} + \&c$. Therefore $1 - \sqrt[n]{1 - k}$ will be to $1 - \sqrt[n]{1 - q}$, very nearly, in the same proportion as $\frac{1}{n} \times$ the infinite series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \&c$ to $\frac{1}{n} \times$ the infinite series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \&c$, that is, in the same proportion as the n^{th} part of the infinite series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \&c$ to the n^{th} part of the infinite series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \&c$, and consequently in the same proportion as the whole series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \&c$ *ad infinitum*, to the whole series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \&c$ *ad infinitum*.

Now the ratio of 1 to $1 - k$ is to the ratio of 1 to $1 - q$ in the same proportion as the n^{th} part of the former ratio is to the n^{th} part of the latter ratio, and consequently, in the same proportion as the ratio of 1 to $\sqrt[n]{1 - k}$ is to the ratio of 1 to $\sqrt[n]{1 - q}$.

But, because $\sqrt[n]{1 - k}$ and $\sqrt[n]{1 - q}$ approach extremely near to an equality with 1, it follows from Lemma 1, that the ratio of 1 to $\sqrt[n]{1 - k}$ will be to the ratio of

of $\sqrt[n]{1-q}$ to $\sqrt[n]{1-k}$ in very nearly the same proportion as the excess of 1 above $\sqrt[n]{1-k}$ is to the excess of $\sqrt[n]{1-q}$ above $\sqrt[n]{1-k}$, or (putting $R. \frac{1}{\sqrt[n]{1-k}}$ for the

ratio of 1 to $\sqrt[n]{1-k}$, and $R. \frac{\sqrt[n]{1-q}}{\sqrt[n]{1-k}}$ for the ratio of $\sqrt[n]{1-q}$ to $\sqrt[n]{1-k}$) that $R.$

$\frac{1}{\sqrt[n]{1-k}}$ will be to $R. \frac{\sqrt[n]{1-q}}{\sqrt[n]{1-k}}$ very nearly in the same proportion as $1 - \sqrt[n]{1-k}$

is to $\sqrt[n]{1-q} - \sqrt[n]{1-k}$. And therefore it follows, *dividendo*, that the first of these four quantities will be to the excess of the first above the second in very nearly the same proportion as the third quantity to the excess of the third above the fourth,

that is, that $R. \frac{1}{\sqrt[n]{1-k}}$ will be to $R. \frac{1}{\sqrt[n]{1-k}} - R. \frac{\sqrt[n]{1-q}}{\sqrt[n]{1-k}}$ very nearly in the same

proportion as $1 - \sqrt[n]{1-k}$ is to $1 - \sqrt[n]{1-k} - \sqrt[n]{1-q} + \sqrt[n]{1-k}$, or (because

$R. \frac{1}{\sqrt[n]{1-k}} - R. \frac{\sqrt[n]{1-q}}{\sqrt[n]{1-k}} = R. \frac{1 - \sqrt[n]{1-q}}{\sqrt[n]{1-k}}$, and $1 - \sqrt[n]{1-k} - \sqrt[n]{1-q} + \sqrt[n]{1-k} = 1 - \sqrt[n]{1-q}$,

is $= 1 - \sqrt[n]{1-q}$), that $R. \frac{1}{\sqrt[n]{1-k}}$

will be to $R. \frac{1}{\sqrt[n]{1-q}}$ very nearly in the same proportion as $1 - \sqrt[n]{1-k}$ to $1 -$

$\sqrt[n]{1-q}$. Therefore the ratio of 1 to $1 - k$ will be to the ratio of 1 to $1 - q$ very nearly in the same proportion as $1 - \sqrt[n]{1-k}$ is to $1 - \sqrt[n]{1-q}$.

But it has been shewn that, in this case of the very great magnitude of n , the proportion of $1 - \sqrt[n]{1-k}$ to $1 - \sqrt[n]{1-q}$ is very nearly the same as that of the series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \&c$ *ad infinitum* to the series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \&c$ *ad infinitum*.

Therefore

Therefore the ratio of 1 to $1 - k$ will be the ratio of 1 to $1 - q$ in the same proportion as the series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \&c$ *ad infinitum* is to the series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \&c$ *ad infinitum*.

Q. E. D.

The same demonstration expressed with still greater brevity.

68. And, if still greater brevity of expression be desired, the foregoing demonstration may be expressed in a still conciser manner, as follows.

Let n be put, as before, for any very large number, as, for example, for a nonillion, or the ninth power of a million.

Then will $R. \frac{1}{1-k}$ be to $R. \frac{1}{1-q}$ in the same proportion as $\frac{1}{n} \times R. \frac{1}{1-k}$ to $\frac{1}{n} \times R. \frac{1}{1-q}$, that is, (by the nature of roots,) as $R. \frac{1}{(1-k)^n}$ to $R. \frac{1}{(1-q)^n}$ that is

by Lemma 1, (on account of the near approach of $\sqrt[n]{1-k}$ and $\sqrt[n]{1-q}$ to an equality with 1,) as $1 - \frac{k}{n}$ to $1 - \frac{q}{n}$, that is, by Lemma 3^d, coroll. 2, (on account of the very great magnitude of the number n ,) as $\frac{1}{n} \times$ the infinite series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \&c$ to $\frac{1}{n} \times$ the infinite series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \&c$, and consequently (multiplying both sides by n) as the infinite series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \&c$ to the infinite series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \&c$. Q. E. D.

SCHOLIUM.

69. This proposition is *accurately* true, though, when n is of any finite magnitude, how great soever, some of the intermediate propositions by means of which it is proved, are only *very nearly* true. For, as these intermediate propositions are not limited in the degree in which they approach to truth or accuracy, but may be made to come as near to being accurately true as we please, by increasing the number n , the conclusion derived from them must be accurately true. For, if it were supposed to be not accurately true, but only to approach within certain limits of the truth, we might increase the magnitude of n till the said conclusion was made to approach nearer to being accurately true than within the assigned limits; which would be contrary to the supposition of its having approached only as near

as

as the said assigned limits to the truth, and consequently would prove that the said supposition was false. Therefore the said conclusion does not only approach within certain limits of the truth, but is accurately true.

70. We have now, I hope, sufficiently established the truth of Dr. Wallis's proposition, to wit, "that, if q be made to represent successively several different fractions, or quantities less than 1, the several values of the infinite series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \&c$ will be proportional to the several ratios of 1 to the corresponding values of $1 - q$ respectively, or will be the logarithms of those ratios." It remains that we illustrate the use of this series in the business of computing logarithms by applying it to a few examples. I shall therefore now proceed to apply it to the computation of the logarithms of the nine following ratios, to wit, the ratio of 10 to 9, the ratio of 11 to 10, the ratio of 81 to 80, the ratio of 121 to 120, the ratio of 2401 to 2400, the ratio of 169 to 168, the ratio of 289 to 288, the ratio of 361 to 360, and the ratio of 529 to 528; which are the ratios of which we computed the logarithms above in the nine examples of the use of the former series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c$, that was invented by Mr. Mercator. See above, art. 24, 25, 26, 27, 28, 29, 30, 31, and 32. By the help of these nine logarithms we shall be able to find the logarithms of the ratios of 2 to 1, 3 to 1, 4 to 1, 5 to 1, 6 to 1, 7 to 1, 8 to 1, 9 to 1, 10 to 1, 11 to 1, 12 to 1, 13 to 1, 14 to 1, 15 to 1, 16 to 1, 17 to 1, 18 to 1, 19 to 1, 20 to 1, 21 to 1, 22 to 1, 23 to 1, and 24 to 1, and the logarithms of all greater ratios that are composed of these ratios, by mere addition and subtraction; which will make the utility of this series sufficiently manifest. And, as in computing the logarithms of the said nine ratios, the numerator of the fraction that is equal to q is always 1, and the denominator of it is a whole number, I think it will be most convenient to substitute $\frac{1}{m}$ instead of q in the residual quantity $1 - q$ and the infinite series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \&c$, whereby the said residual quantity will be changed into $1 - \frac{1}{m}$, and the said series will be changed into the series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \&c$, or the logarithm of the ratio of 1 to $1 - \frac{1}{m}$ will be the series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \&c$ *ad infinitum*.

71. If we should have occasion to express this series in such a manner as to point out the generating fractions by the successive multiplication of which its second and other following terms are derived from the first term $\frac{1}{m}$ and from each other,

it may be done by putting the capital letter A for the whole first term $\frac{1}{m}$, and B for the whole second term $\frac{1}{2m^2}$, and c for the whole third term $\frac{1}{3m^3}$, and d for the whole fourth term $\frac{1}{4m^4}$, and E, F, G, H, I, K, L, M, and the following capital letters of the alphabet, for the whole fifth, sixth, seventh, eighth, ninth, tenth, eleventh, twelfth, and other following terms of the series; by which substitution the said Series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \frac{1}{7m^7} + \frac{1}{8m^8} + \frac{1}{9m^9} + \frac{1}{10m^{10}} + \&c$ will be converted into the series $\frac{1}{m} + \frac{A}{2m} + \frac{2B}{3m} + \frac{3C}{4m} + \frac{4D}{5m} + \frac{5E}{6m} + \frac{6F}{7m} + \frac{7G}{8m} + \frac{8H}{9m} + \frac{9I}{10m} + \&c$. But the terms of the series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \frac{1}{7m^7} + \frac{1}{8m^8} + \frac{1}{9m^9} + \frac{1}{10m^{10}} + \&c$ are so extremely simple and concise that there seems to be no occasion to change its form.

E X A M P L E S
OF THE
COMPUTATION OF THE LOGARITHMS
OF THE
RATIOS OF 1 TO SEVERAL NUMBERS

Denoted by the residual quantity $1 - \frac{1}{m}$, by means of the infinite series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \frac{1}{7m^7} + \frac{1}{8m^8} + \frac{1}{9m^9} + \frac{1}{10m^{10}} + \frac{1}{11m^{11}} + \frac{1}{12m^{12}} + \frac{1}{13m^{13}} + \frac{1}{14m^{14}} + \frac{1}{15m^{15}} + \frac{1}{16m^{16}} + \frac{1}{17m^{17}} + \frac{1}{18m^{18}} + \&c,$

INVENTED BY DR. WALLIS.

E X A M P L E I.

72. **L**ET it be required to find by means of the aforesaid series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \&c$ *ad infinitum*, the logarithm of the ratio of 10 to 9 (or of 10 to $10 - 1$, or of $\frac{10}{10}$ to $\frac{10-1}{10}$), or of 1 to $1 - \frac{1}{10}$.

Here m is $= 10$, and $\frac{1}{m}$ is $= \frac{1}{10} = 0.100,000,000,000,000,000$.

We

We shall therefore have,

$$\frac{1}{m^2} (= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{10} = \frac{0.100,000,000,000,000,000}{10}) = 0.010,000,000,000,000,000 ;$$

$$\text{And } \frac{1}{m^3} (= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{10} = \frac{0.010,000,000,000,000,000}{10}) = 0.001,000,000,000,000,000 ;$$

$$\text{And } \frac{1}{m^4} (= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{10} = \frac{0.001,000,000,000,000,000}{10}) = 0.000,100,000,000,000,000 ;$$

$$\text{And } \frac{1}{m^5} (= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{10} = \frac{0.000,100,000,000,000,000}{10}) = 0.000,010,000,000,000,000 ;$$

$$\text{And } \frac{1}{m^6} (= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{10} = \frac{0.000,010,000,000,000,000}{10}) = 0.000,001,000,000,000,000 ;$$

$$\text{And } \frac{1}{m^7} (= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{10} = \frac{0.000,001,000,000,000,000}{10}) = 0.000,000,100,000,000,000 ;$$

$$\text{And } \frac{1}{m^8} (= \frac{1}{m^7} \times \frac{1}{m} = \frac{1}{m^7} \times \frac{1}{10} = \frac{0.000,000,100,000,000,000}{10}) = 0.000,000,010,000,000,000 ;$$

$$\text{And } \frac{1}{m^9} (= \frac{1}{m^8} \times \frac{1}{m} = \frac{1}{m^8} \times \frac{1}{10} = \frac{0.000,000,010,000,000,000}{10}) = 0.000,000,001,000,000,000 ;$$

$$\text{And } \frac{1}{m^{10}} (= \frac{1}{m^9} \times \frac{1}{m} = \frac{1}{m^9} \times \frac{1}{10} = \frac{0.000,000,001,000,000,000}{10}) = 0.000,000,000,100,000,000 ;$$

$$\text{And } \frac{1}{m^{11}} (= \frac{1}{m^{10}} \times \frac{1}{m} = \frac{1}{m^{10}} \times \frac{1}{10} = \frac{0.000,000,000,100,000,000}{10}) = 0.000,000,000,010,000,000 ;$$

$$\text{And } \frac{1}{m^{12}} (= \frac{1}{m^{11}} \times \frac{1}{m} = \frac{1}{m^{11}} \times \frac{1}{10} = \frac{0.000,000,000,010,000,000}{10}) = 0.000,000,000,001,000,000 ;$$

$$\text{And } \frac{1}{m^{13}} (= \frac{1}{m^{12}} \times \frac{1}{m} = \frac{1}{m^{12}} \times \frac{1}{10} = \frac{0.000,000,000,001,000,000}{10}) = 0.000,000,000,000,100,000 ;$$

$$\text{And } \frac{1}{m^{14}} (= \frac{1}{m^{13}} \times \frac{1}{m} = \frac{1}{m^{13}} \times \frac{1}{10} = \frac{0.000,000,000,000,100,000}{10}) = 0.000,000,000,000,010,000 ;$$

$$\text{And } \frac{1}{m^{15}} (= \frac{1}{m^{14}} \times \frac{1}{m} = \frac{1}{m^{14}} \times \frac{1}{10} = \frac{0.000,000,000,000,010,000}{10}) = 0.000,000,000,000,001,000 ;$$

$$\text{And } \frac{1}{m^{16}} (= \frac{1}{m^{15}} \times \frac{1}{m} = \frac{1}{m^{15}} \times \frac{1}{10} = \frac{0.000,000,000,000,001,000}{10}) = 0.000,000,000,000,000,100 ;$$

$$\text{And } \frac{1}{m^{17}} (= \frac{1}{m^{16}} \times \frac{1}{m} = \frac{1}{m^{16}} \times \frac{1}{10} = \frac{0.000,000,000,000,000,100}{10}) = 0.000,000,000,000,000,010 ;$$

And consequently

$$\frac{1}{2m^2} (= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.010,000,000,000,000,000}{2}) = 0.005,000,000,000,000,000 ;$$

$$\text{And } \frac{1}{3m^3} (= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.001,000,000,000,000,000}{3}) = 0.000,333,333,333,333,333 ;$$

$$\text{And } \frac{1}{4m^4} (= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,100,000,000,000,000}{4}) = 0.000,025,000,000,000,000;$$

$$\text{And } \frac{1}{5m^5} (= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,010,000,000,000,000}{5}) = 0.000,002,000,000,000,000;$$

$$\text{And } \frac{1}{6m^6} (= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,001,000,000,000,000}{6}) = 0.000,000,166,666,666,666;$$

$$\text{And } \frac{1}{7m^7} (= \frac{1}{m^7} \times \frac{1}{7} = \frac{0.000,000,100,000,000,000}{7}) = 0.000,000,014,285,714,285;$$

$$\text{And } \frac{1}{8m^8} (= \frac{1}{m^8} \times \frac{1}{8} = \frac{0.000,000,010,000,000,000}{8}) = 0.000,000,001,250,000,000;$$

$$\text{And } \frac{1}{9m^9} (= \frac{1}{m^9} \times \frac{1}{9} = \frac{0.000,000,001,000,000,000}{9}) = 0.000,000,000,111,111,111;$$

$$\text{And } \frac{1}{10m^{10}} (= \frac{1}{m^{10}} \times \frac{1}{10} = \frac{0.000,000,000,100,000,000}{10}) = 0.000,000,000,010,000,000;$$

$$\text{And } \frac{1}{11m^{11}} (= \frac{1}{m^{11}} \times \frac{1}{11} = \frac{0.000,000,000,010,000,000}{11}) = 0.000,000,000,000,909,090;$$

$$\text{And } \frac{1}{12m^{12}} (= \frac{1}{m^{12}} \times \frac{1}{12} = \frac{0.000,000,000,001,000,000}{12}) = 0.000,000,000,000,083,333;$$

$$\text{And } \frac{1}{13m^{13}} (= \frac{1}{m^{13}} \times \frac{1}{13} = \frac{0.000,000,000,000,100,000}{13}) = 0.000,000,000,000,007,692;$$

$$\text{And } \frac{1}{14m^{14}} (= \frac{1}{m^{14}} \times \frac{1}{14} = \frac{0.000,000,000,000,010,000}{14}) = 0.000,000,000,000,000,714;$$

$$\text{And } \frac{1}{15m^{15}} (= \frac{1}{m^{15}} \times \frac{1}{15} = \frac{0.000,000,000,000,001,000}{15}) = 0.000,000,000,000,000,066;$$

$$\text{And } \frac{1}{16m^{16}} (= \frac{1}{m^{16}} \times \frac{1}{16} = \frac{0.000,000,000,000,000,100}{16}) = 0.000,000,000,000,000,006;$$

$$\text{And } \frac{1}{17m^{17}} (= \frac{1}{m^{17}} \times \frac{1}{17} = \frac{0.000,000,000,000,000,010}{17}) = 0.000,000,000,000,000,000.$$

Therefore the series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \frac{1}{7m^7} + \frac{1}{8m^8}$
 $+ \frac{1}{9m^9} + \frac{1}{10m^{10}} + \frac{1}{11m^{11}} + \frac{1}{12m^{12}} + \frac{1}{13m^{13}} + \frac{1}{14m^{14}} + \frac{1}{15m^{15}} + \frac{1}{16m^{16}} +$
 $\&c$ is =

$$\begin{aligned}
 & 0;100,000,000,000,000,000, \\
 & + ;..5,000,000,000,000,000, \\
 & + ;...333,333,333,333,333, \\
 & + ;...25,000,000,000,000, \\
 & + ;...2,000,000,000,000, \\
 & + ;...166,666,666,666, \\
 & + ;...14,285,714,285, \\
 & + ;...1,250,000,000, \\
 & + ;...111,111,111, \\
 & + ;...10,000,000, \\
 & + ;...909,090, \\
 & + ;...83,333, \\
 & + ;...7,692, \\
 & + ;...714, \\
 & + ;...66, \\
 & + ;...6, \\
 & = 0.105,360,515,657,826,296.
 \end{aligned}$$

Therefore this number 0.105,360,515,657,826,296 is the logarithm of the ratio of 1 to $1 - \frac{1}{10}$ or of 10 to 9. Q. E. I.

N. B. This value of the logarithm of the ratio of 10 to 9 is less than its true value by about 5 in the 18th place of decimal fractions, or 0.000,000,000,000,000,005. For, if we add 0.000,000,000,000,000,005 to 0.105,360,515,657,826,296, their sum will be 0.105,360,515,657,826,301, which agrees with the true logarithm of this ratio in all its figures, the more accurate value of this logarithm being 0.105,360,515,657,826,301,227.

EXAMPLE II.

73. Let it be required to find by means of the same infinite series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \text{&c}$ the logarithm of the ratio of 11 to 10 (or of 11 to $11 - 1$, or of $\frac{11}{11}$ to $\frac{11-1}{11}$), or of 1 to $1 - \frac{1}{11}$.

Here m is = 11, and $\frac{1}{m}$ is = $\frac{1}{11} = 0.090,909,090,909,090,909$, &c.

We shall therefore have

$$\frac{1}{m^2} (= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{11} = \frac{0.090,909,090,909,090,909}{11}) = 0.008,264,462,809,917,355;$$

$$\text{And } \frac{1}{m^3} (= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{11} = \frac{0.008,264,462,809,917,355}{11}) = 0.000,751,314,800,901,577;$$

And

$$\text{And } \frac{1}{m^4} (= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{11} = \frac{0.000,751,314,800,901,577}{11}) = 0.000,068,301,345,536,507;$$

$$\text{And } \frac{1}{m^5} (= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{11} = \frac{0.000,068,301,345,536,507}{11}) = 0.000,006,209,213,230,591;$$

$$\text{And } \frac{1}{m^6} (= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{11} = \frac{0.000,006,209,213,230,591}{11}) = 0.000,000,564,473,930,053;$$

$$\text{And } \frac{1}{m^7} (= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{11} = \frac{0.000,000,564,473,930,053}{11}) = 0.000,000,051,315,811,823;$$

$$\text{And } \frac{1}{m^8} (= \frac{1}{m^7} \times \frac{1}{m} = \frac{1}{m^7} \times \frac{1}{11} = \frac{0.000,000,051,315,811,823}{11}) = 0.000,000,004,665,073,802;$$

$$\text{And } \frac{1}{m^9} (= \frac{1}{m^8} \times \frac{1}{m} = \frac{1}{m^8} \times \frac{1}{11} = \frac{0.000,000,004,665,073,802}{11}) = 0.000,000,000,424,097,618;$$

$$\text{And } \frac{1}{m^{10}} (= \frac{1}{m^9} \times \frac{1}{m} = \frac{1}{m^9} \times \frac{1}{11} = \frac{0.000,000,000,424,097,618}{11}) = 0.000,000,000,038,554,328;$$

$$\text{And } \frac{1}{m^{11}} (= \frac{1}{m^{10}} \times \frac{1}{m} = \frac{1}{m^{10}} \times \frac{1}{11} = \frac{0.000,000,000,038,554,328}{11}) = 0.000,000,000,003,504,938;$$

$$\text{And } \frac{1}{m^{12}} (= \frac{1}{m^{11}} \times \frac{1}{m} = \frac{1}{m^{11}} \times \frac{1}{11} = \frac{0.000,000,000,003,504,938}{11}) = 0.000,000,000,000,318,630;$$

$$\text{And } \frac{1}{m^{13}} (= \frac{1}{m^{12}} \times \frac{1}{m} = \frac{1}{m^{12}} \times \frac{1}{11} = \frac{0.000,000,000,000,318,630}{11}) = 0.000,000,000,000,028,966;$$

$$\text{And } \frac{1}{m^{14}} (= \frac{1}{m^{13}} \times \frac{1}{m} = \frac{1}{m^{13}} \times \frac{1}{11} = \frac{0.000,000,000,000,028,966}{11}) = 0.000,000,000,000,002,633;$$

$$\text{And } \frac{1}{m^{15}} (= \frac{1}{m^{14}} \times \frac{1}{m} = \frac{1}{m^{14}} \times \frac{1}{11} = \frac{0.000,000,000,000,002,633}{11}) = 0.000,000,000,000,000,239;$$

$$\text{And } \frac{1}{m^{16}} (= \frac{1}{m^{15}} \times \frac{1}{m} = \frac{1}{m^{15}} \times \frac{1}{11} = \frac{0.000,000,000,000,000,239}{11}) = 0.000,000,000,000,000,021;$$

$$\text{And } \frac{1}{m^{17}} (= \frac{1}{m^{16}} \times \frac{1}{m} = \frac{1}{m^{16}} \times \frac{1}{11} = \frac{0.000,000,000,000,000,021}{11}) = 0.000,000,000,000,000,001;$$

And consequently

$$\frac{1}{2m^2} (= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.008,264,462,809,917,355}{2}) = 0.004,132,231,404,958,677;$$

$$\text{And } \frac{1}{3m^3} (= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,751,314,800,901,577}{3}) = 0.000,250,438,266,967,192;$$

$$\text{And } \frac{1}{4m^4} (= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,068,301,345,536,507}{4}) = 0.000,017,075,336,384,126;$$

$$\text{And } \frac{1}{5m^5} (= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,006,209,213,230,591}{5}) = 0.000,001,241,842,646,118;$$

And

$$\text{And } \frac{1}{6m^6} (= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,000,564,473,930,053}{6}) = 0.000,000,094,078,988,342;$$

$$\text{And } \frac{1}{7m^7} (= \frac{1}{m^7} \times \frac{1}{7} = \frac{0.000,000,051,315,811,823}{7}) = 0.000,000,007,330,830,260;$$

$$\text{And } \frac{1}{8m^8} (= \frac{1}{m^8} \times \frac{1}{8} = \frac{0.000,000,004,665,073,802}{8}) = 0.000,000,000,583,134,225;$$

$$\text{And } \frac{1}{9m^9} (= \frac{1}{m^9} \times \frac{1}{9} = \frac{0.000,000,000,424,097,618}{9}) = 0.000,000,000,047,121,957;$$

$$\text{And } \frac{1}{10m^{10}} (= \frac{1}{m^{10}} \times \frac{1}{10} = \frac{0.000,000,000,038,554,328}{10}) = 0.000,000,000,003,855,432;$$

$$\text{And } \frac{1}{11m^{11}} (= \frac{1}{m^{11}} \times \frac{1}{11} = \frac{0.000,000,000,003,504,938}{11}) = 0.000,000,000,000,318,630;$$

$$\text{And } \frac{1}{12m^{12}} (= \frac{1}{m^{12}} \times \frac{1}{12} = \frac{0.000,000,000,000,318,630}{12}) = 0.000,000,000,000,026,552;$$

$$\text{And } \frac{1}{13m^{13}} (= \frac{1}{m^{13}} \times \frac{1}{13} = \frac{0.000,000,000,000,028,966}{13}) = 0.000,000,000,000,002,228;$$

$$\text{And } \frac{1}{14m^{14}} (= \frac{1}{m^{14}} \times \frac{1}{14} = \frac{0.000,000,000,000,002,633}{14}) = 0.000,000,000,000,000,188;$$

$$\text{And } \frac{1}{15m^{15}} (= \frac{1}{m^{15}} \times \frac{1}{15} = \frac{0.000,000,000,000,000,239}{15}) = 0.000,000,000,000,000,015;$$

$$\text{And } \frac{1}{16m^{16}} (= \frac{1}{m^{16}} \times \frac{1}{16} = \frac{0.000,000,000,000,000,021}{16}) = 0.000,000,000,000,000,001;$$

Therefore the series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \frac{1}{7m^7} + \frac{1}{8m^8} +$
 $\frac{1}{9m^9} + \frac{1}{10m^{10}} + \frac{1}{11m^{11}} + \frac{1}{12m^{12}} + \frac{1}{13m^{13}} + \frac{1}{14m^{14}} + \frac{1}{15m^{15}} + \frac{1}{16m^{16}} + \&c$
 is =

$$\begin{aligned} & 0;090,909,090,909,090,909, \\ & + ;..4,132,231,404,958,677, \\ & + ;...250,438,266,967,192, \\ & + ;...17,075,336,384,126, \\ & + ;...1,241,842,646,118, \\ & + ;...94,078,988,342, \\ & + ;...7,330,830,260, \\ & + ;...583,134,225, \\ & + ;...47,121,957, \\ & + ;...3,855,432, \\ & + ;...318,630, \\ & + ;...26,552, \\ & + ;...2,228, \\ & + ;...188, \\ & + ;...15, \\ & + ;...1, \\ & = 0.095,310,179,804,324,852. \end{aligned}$$

Therefore this number 0.095,310,179,804,324,852, is the logarithm of the ratio of 1 to $1 - \frac{1}{11}$, or of 11 to 10. Q. E. I.

This number agrees with the value of the same logarithm found above in article 25 by Mercator's series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \text{Ec.}$ in all the figures but the last, that value being 0.095,310,179,804,324,858.

EXAMPLE III.

74. Let it be required to find by means of the same series the logarithm of the ratio of 81 to 80, or of 81 to $81 - 1$ (or of $\frac{81}{81}$ to $\frac{81-1}{81}$), or of 1 to $1 - \frac{1}{81}$.

Here m is = 81, and $\frac{1}{m}$ is = $\frac{1}{81} = 0.012,345,679,012,345,679$.

We shall therefore have

$$\frac{1}{m^2} (= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{81} = \frac{0.012,345,679,012,345,679}{81}) = 0.000,152,415,790,275,872;$$

$$\text{And } \frac{1}{m^3} (= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{81} = \frac{0.000,152,415,790,275,872}{81}) = 0.000,001,881,676,423,158;$$

$$\text{And } \frac{1}{m^4} (= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{81} = \frac{0.000,001,881,676,423,158}{81}) = 0.000,000,023,230,573,125;$$

$$\text{And } \frac{1}{m^5} (= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{81} = \frac{0.000,000,023,230,573,125}{81}) = 0.000,000,000,286,797,199;$$

$$\text{And } \frac{1}{m^6} (= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{81} = \frac{0.000,000,000,286,797,199}{81}) = 0.000,000,000,003,540,706;$$

$$\text{And } \frac{1}{m^7} (= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{81} = \frac{0.000,000,000,003,540,706}{81}) = 0.000,000,000,000,043,712;$$

$$\text{And } \frac{1}{m^8} (= \frac{1}{m^7} \times \frac{1}{m} = \frac{1}{m^7} \times \frac{1}{81} = \frac{0.000,000,000,000,043,712}{81}) = 0.000,000,000,000,000,539;$$

$$\text{And } \frac{1}{m^9} (= \frac{1}{m^8} \times \frac{1}{m} = \frac{1}{m^8} \times \frac{1}{81} = \frac{0.000,000,000,000,000,539}{81}) = 0.000,000,000,000,000,006;$$

And consequently

$$\frac{1}{2m^2} (= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,152,415,790,275,872}{2}) = 0.000,076,207,895,137,936;$$

$$\text{And } \frac{1}{3m^3} (= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,001,881,676,423,158}{3}) = 0.000,000,627,225,474,386;$$

And

$$\text{And } \frac{1}{4m^4} (= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,000,023,230,573,125}{4}) = 0.000,000,005,807,643,281;$$

$$\text{And } \frac{1}{5m^5} (= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,286,797,199}{5}) = 0.000,000,000,057,359,439;$$

$$\text{And } \frac{1}{6m^6} (= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,003,540,706}{6}) = 0.000,000,000,000,590,117;$$

$$\text{And } \frac{1}{7m^7} (= \frac{1}{m^7} \times \frac{1}{7} = \frac{0.000,000,000,000,043,712}{7}) = 0.000,000,000,000,006,244;$$

$$\text{And } \frac{1}{8m^8} (= \frac{1}{m^8} \times \frac{1}{8} = \frac{0.000,000,000,000,000,539}{8}) = 0.000,000,000,000,000,067;$$

$$\text{And } \frac{1}{9m^9} (= \frac{1}{m^9} \times \frac{1}{9} = \frac{0.000,000,000,000,000,006}{9}) = 0.000,000,000,000,000,000.$$

Therefore the series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \frac{1}{7m^7} + \frac{1}{8m^8} + \mathcal{E}c$ is =

$$\begin{array}{r} 0,012,345,679,012,345,679, \\ + ;...076,207,895,137,936, \\ + ;...627,225,474,386, \\ + ;...5,807,643,281, \\ + ;...57,359,439, \\ + ;...590,117, \\ + ;...6,244, \\ + ;...67, \\ + ;...0, \\ \hline = 0.012,422,519,998,557,149. \end{array}$$

Therefore this number 0.012,422,519,998,557,149 is the logarithm of the ratio of 1 to $1 - \frac{1}{81}$, or of 81 to 80. Q. E. I.

This number agrees with the value of the same logarithm found above in article 26, by Mercator's series, in all but the two last figures, that value being 0.012,422,519,998,557,152.

EXAMPLE IV.

75. Let it be required to find by means of the same series the logarithm of the ratio of 121 to 120, (or of 121 to $121 - 1$, or of $\frac{121}{121}$ to $\frac{121-1}{121}$) or of 1 to $1 - \frac{1}{121}$.

Here m is = 121, and $\frac{1}{m}$ is = $\frac{1}{121} = 0.008,264,462,809,917,355.$
Q q

We

We shall therefore have,

$$\frac{1}{m^2} (= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{121} = \frac{0.008,264,462,809,917,355}{121}) = 0.000,068,301,345,536,507;$$

$$\text{And } \frac{1}{m^3} (= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{121} = \frac{0.000,068,301,345,536,507}{121}) = 0.000,000,564,473,930,053;$$

$$\text{And } \frac{1}{m^4} (= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{121} = \frac{0.000,000,564,473,930,053}{121}) = 0.000,000,004,665,073,802;$$

$$\text{And } \frac{1}{m^5} (= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{121} = \frac{0.000,000,004,665,073,802}{121}) = 0.000,000,000,038,554,328;$$

$$\text{And } \frac{1}{m^6} (= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{121} = \frac{0.000,000,000,038,554,328}{121}) = 0.000,000,000,000,318,630;$$

$$\text{And } \frac{1}{m^7} (= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{121} = \frac{0.000,000,000,000,318,630}{121}) = 0.000,000,000,000,002,633;$$

$$\text{And } \frac{1}{m^8} (= \frac{1}{m^7} \times \frac{1}{m} = \frac{1}{m^7} \times \frac{1}{121} = \frac{0.000,000,000,000,002,633}{121}) = 0.000,000,000,000,000,021;$$

$$\text{And } \frac{1}{m^9} (= \frac{1}{m^8} \times \frac{1}{m} = \frac{1}{m^8} \times \frac{1}{121} = \frac{0.000,000,000,000,000,021}{121}) = 0.000,000,000,000,000,000;$$

And consequently

$$\frac{1}{2m^2} (= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,068,301,345,536,507}{2}) = 0.000,034,150,672,768,253;$$

$$\text{And } \frac{1}{3m^3} (= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,000,564,473,930,053}{3}) = 0.000,000,188,157,976,684;$$

$$\text{And } \frac{1}{4m^4} (= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,000,004,665,073,802}{4}) = 0.000,000,001,166,268,450;$$

$$\text{And } \frac{1}{5m^5} (= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,038,554,328}{5}) = 0.000,000,000,007,710,865;$$

$$\text{And } \frac{1}{6m^6} (= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,000,318,630}{6}) = 0.000,000,000,000,053,105;$$

$$\text{And } \frac{1}{7m^7} (= \frac{1}{m^7} \times \frac{1}{7} = \frac{0.000,000,000,000,002,633}{7}) = 0.000,000,000,000,000,376;$$

$$\text{And } \frac{1}{8m^8} (= \frac{1}{m^8} \times \frac{1}{8} = \frac{0.000,000,000,000,000,021}{8}) = 0.000,000,000,000,000,002.$$

Therefore the series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \frac{1}{7m^7} + \frac{1}{8m^8} + \text{Ec}$ is =

$$\begin{array}{r}
 0;008,264,462,809,917,355, \\
 + \quad ;\dots, 34,150,672,768,253, \\
 + \quad ;\dots, 188,157,976,684, \\
 + \quad ;\dots, 1,166,268,450, \\
 + \quad ;\dots, 7,710,865, \\
 + \quad ;\dots, 53,105, \\
 + \quad ;\dots, 376, \\
 + \quad ;\dots, 2, \\
 \hline
 = 0.008,298,802,814,695,090.
 \end{array}$$

Therefore this number 0.008,298,802,814,695,090, is the logarithm of the ratio of 1 to $1 - \frac{1}{121}$, or of 121 to 120. Q. E. I.

This number agrees with the value of the same logarithm found above in article 27, by Mercator's series, in all but the last figure, that value being 0.008,298,802,814,695,093.

EXAMPLE V.

76. Let it be required to find by means of the same series the logarithm of the ratio of 2401 to 2400, (or of 2401 to $2401 - 1$, or of $\frac{2401}{2401}$ to $\frac{2401-1}{2401}$), or of 1 to $1 - \frac{1}{2401}$.

Here m is $= 2401$, and $\frac{1}{m}$ is $= \frac{1}{2401} = 0.000,416,493,127,863,390, \&c.$

We shall therefore have

$$\frac{1}{m^2} (= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{2401} = \frac{0.000,416,493,127,863,390}{2401}) = 0.000,000,173,466,525,557;$$

$$\text{And } \frac{1}{m^3} (= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{2401} = \frac{0.000,000,173,466,525,557}{2401}) = 0.000,000,000,072,247,615;$$

$$\text{And } \frac{1}{m^4} (= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{2401} = \frac{0.000,000,000,072,247,615}{2401}) = 0.000,000,000,000,030,090;$$

$$\text{And } \frac{1}{m^5} (= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{2401} = \frac{0.000,000,000,000,030,090}{2401}) = 0.000,000,000,000,000,012;$$

$$\text{And } \frac{1}{m^6} (= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{2401} = \frac{0.000,000,000,000,000,012}{2401}) = 0.000,000,000,000,000,000;$$

And consequently

$$\frac{1}{2m^2} (= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,000,173,466,525,557}{2}) = 0.000,000,086,733,262,778;$$

Qq 2

And

$$\text{And } \frac{1}{3m^3} (= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,000,000,072,247,615}{3}) = 0.000,000,000,024,082,538;$$

$$\text{And } \frac{1}{4m^4} (= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,000,000,000,030,090}{4}) = 0.000,000,000,000,007,522;$$

$$\text{And } \frac{1}{5m^5} (= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,000,000,012}{5}) = 0.000,000,000,000,000,002.$$

Therefore the series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \text{\&c}$ is in this case =

$$\begin{array}{r} 0.000,416,493,127,863,390, \\ + ; \dots, \dots, 86,733,262,778, \\ + ; \dots, \dots, \dots, 24,082,538, \\ + ; \dots, \dots, \dots, \dots, 7,522, \\ + ; \dots, \dots, \dots, \dots, \dots, 2, \\ \hline = 0.000,416,579,885,216,230. \end{array}$$

Therefore this number 0.000,416,579,885,216,230 is the logarithm of the ratio of 1 to $1 - \frac{1}{2401}$, or of 2401 to 2400. Q. E. I.

This number agrees with the value of the same logarithm found above in article 28, by Mercator's series, in all the figures except the last, that value of it being 0.000,416,579,885,216,232.

EXAMPLE VI.

77. Let it be required to find by the same series the logarithm of the ratio of 169 (which is the square of 13) to 168, (or of the ratio of 169 to $169 - 1$, or of $\frac{169}{169}$ to $\frac{169-1}{169}$), or of 1 to $1 - \frac{1}{169}$.

Here m is = 169, and $\frac{1}{m}$ is = $\frac{1}{169} = 0.005,917,159,763,313,609, \text{\&c}.$

We shall therefore have

$$\frac{1}{m^2} (= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{169} = \frac{0.005,917,159,763,313,609}{169}) = 0.000,035,012,779,664,577;$$

$$\text{And } \frac{1}{m^3} (= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{169} = \frac{0.000,035,012,779,664,577}{169}) = 0.000,000,207,176,211,033;$$

$$\text{And } \frac{1}{m^4} (= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{169} = \frac{0.000,000,207,176,211,033}{169}) = 0.000,000,001,225,894,739;$$

$$\text{And } \frac{1}{m^5} (= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{169} = \frac{0.000,000,001,225,894,739}{169}) = 0.000,000,000,007,253,815;$$

And

$$\text{And } \frac{1}{m^6} (= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{169} = \frac{0.000,000,000,007,253,815}{169}) = 0.000,000,000,000,042,921;$$

$$\text{And } \frac{1}{m^7} (= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{169} = \frac{0.000,000,000,000,042,921}{169}) = 0.000,000,000,000,000,253;$$

$$\text{And } \frac{1}{m^8} (= \frac{1}{m^7} \times \frac{1}{m} = \frac{1}{m^7} \times \frac{1}{169} = \frac{0.000,000,000,000,000,253}{169}) = 0.000,000,000,000,000,001;$$

And consequently

$$\frac{1}{2m^2} (= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,035,012,779,664,577}{2}) = 0.000,017,506,389,832,288;$$

$$\text{And } \frac{1}{3m^3} (= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,000,207,176,211,033}{3}) = 0.000,000,069,058,737,011;$$

$$\text{And } \frac{1}{4m^4} (= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,000,001,225,894,739}{4}) = 0.000,000,000,306,473,684;$$

$$\text{And } \frac{1}{5m^5} (= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,007,253,815}{5}) = 0.000,000,000,001,450,763;$$

$$\text{And } \frac{1}{6m^6} (= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,000,042,921}{6}) = 0.000,000,000,000,007,153;$$

$$\text{And } \frac{1}{7m^7} (= \frac{1}{m^7} \times \frac{1}{7} = \frac{0.000,000,000,000,000,253}{7}) = 0.000,000,000,000,000,036;$$

$$\text{And } \frac{1}{8m^8} (= \frac{1}{m^8} \times \frac{1}{8} = \frac{0.000,000,000,000,000,001}{8}) = 0.000,000,000,000,000,000.$$

Therefore the series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \frac{1}{7m^7} + \frac{1}{8m^8} + \mathcal{E}$ is =

$$\begin{array}{r} 0;005,917,159,763,313,609, \\ + ;\dots, 17,506,389,832,288, \\ + ;\dots, \dots, 69,058,737,011, \\ + ;\dots, \dots, \dots, 306,473,684, \\ + ;\dots, \dots, \dots, \dots, 1,450,763, \\ + ;\dots, \dots, \dots, \dots, \dots, 7,153, \\ + ;\dots, \dots, \dots, \dots, \dots, 36, \\ + ;\dots, \dots, \dots, \dots, \dots, \dots, \\ \hline = 0.005,934,735,519,814,544. \end{array}$$

Therefore this number 0.005,934,735,519,814,544, is the logarithm of the ratio of 1 to $1 - \frac{1}{169}$, or of 169 to 168. Q. E. I.

This number agrees with the value of the same logarithm found above in article 29, by Mercator's series, in all the figures except the last, that value of it being 0.005,934,735,519,814,547.

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EXAMPLE VII.

78. Let it be required to find by means of the same series the logarithm of the ratio of 289 (which is the square of 17) to 288, (or of the ratio of 289 to $289 - 1$, or of $\frac{289}{289}$ to $\frac{289-1}{289}$) or of 1 to $1 - \frac{1}{289}$.

Here m is $= 289$, and $\frac{1}{m}$ is $= \frac{1}{289} = 0.003,460,207,612,456,747, \&c.$

We shall therefore have

$$\frac{1}{m^2} (= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{289} = \frac{0.003,460,207,612,456,747}{289}) = 0.000,011,973,036,721,303;$$

$$\text{And } \frac{1}{m^3} (= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{289} = \frac{0.000,011,973,036,721,303}{289}) = 0.000,000,041,429,192,807;$$

$$\text{And } \frac{1}{m^4} (= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{289} = \frac{0.000,000,041,429,192,807}{289}) = 0.000,000,000,143,353,608;$$

$$\text{And } \frac{1}{m^5} (= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{289} = \frac{0.000,000,000,143,353,608}{289}) = 0.000,000,000,000,496,033;$$

$$\text{And } \frac{1}{m^6} (= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{289} = \frac{0.000,000,000,000,496,033}{289}) = 0.000,000,000,000,001,716;$$

$$\text{And } \frac{1}{m^7} (= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{289} = \frac{0.000,000,000,000,001,716}{289}) = 0.000,000,000,000,000,005;$$

And consequently

$$\frac{1}{2m^2} (= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,011,973,036,721,303}{2}) = 0.000,005,986,518,360,651;$$

$$\text{And } \frac{1}{3m^3} (= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,000,041,429,192,807}{3}) = 0.000,000,013,809,730,935;$$

$$\text{And } \frac{1}{4m^4} (= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,000,000,143,353,608}{4}) = 0.000,000,000,035,838,402;$$

$$\text{And } \frac{1}{5m^5} (= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,000,496,033}{5}) = 0.000,000,000,000,099,206;$$

$$\text{And } \frac{1}{6m^6} (= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,000,001,716}{6}) = 0.000,000,000,000,000,286;$$

$$\text{And } \frac{1}{7m^7} (= \frac{1}{m^7} \times \frac{1}{7} = \frac{0.000,000,000,000,000,005}{7}) = 0.000,000,000,000,000,000.$$

Therefore the series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \frac{1}{7m^7} + \&c$ is in this case =

$$\begin{array}{r}
 0;003,460,207,612,456,747, \\
 + \quad ; \dots, \dots 5,986,518,360,651, \\
 + \quad ; \dots, \dots, 13,809,730,935, \\
 + \quad ; \dots, \dots, \dots 35,838,402, \\
 + \quad ; \dots, \dots, \dots, 99,206, \\
 + \quad ; \dots, \dots, \dots, \dots, 286, \\
 + \quad ; \dots, \dots, \dots, \dots, \dots, \\
 \hline
 = 0;003,466,207,976,486,227.
 \end{array}$$

Therefore this number 0.003,466,207,976,486,227, is the logarithm of the ratio of 1 to $1 - \frac{1}{289}$, or of 289 to 288. Q. E. I.

This number agrees with the value of the same logarithm found above in article 30, by Mercator's series, in all the figures except the last, that value being 0.003,466,207,976,486,229.

EXAMPLE VIII.

79. Let it be required to find by means of the same series the logarithm of the ratio of 361 (which is the square of 19) to 360, (or of 361 to $361 - 1$, or of $\frac{361}{361}$ to $\frac{361-1}{361}$) or of 1 to $1 - \frac{1}{361}$.

Here m is $= 361$, and $\frac{1}{m}$ is $= \frac{1}{361} = 0.002,770,083,102,493,074$.

We shall therefore have

$$\begin{aligned}
 \frac{1}{m^2} & (= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{361} = \frac{0.002,770,083,102,493,074}{361}) = 0.000,007,673,360,394,717; \\
 \text{And } \frac{1}{m^3} & (= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{361} = \frac{0.000,007,673,360,394,717}{361}) = 0.000,000,021,255,845,968; \\
 \text{And } \frac{1}{m^4} & (= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{361} = \frac{0.000,000,021,255,845,968}{361}) = 0.000,000,000,058,880,459; \\
 \text{And } \frac{1}{m^5} & (= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{361} = \frac{0.000,000,000,058,880,459}{361}) = 0.000,000,000,000,163,103; \\
 \text{And } \frac{1}{m^6} & (= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{361} = \frac{0.000,000,000,000,163,103}{361}) = 0.000,000,000,000,000,451; \\
 \text{And } \frac{1}{m^7} & (= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{361} = \frac{0.000,000,000,000,000,451}{361}) = 0.000,000,000,000,000,001;
 \end{aligned}$$

And

And consequently

$$\frac{1}{2m^2} (= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,007,673,360,394,717}{2}) = 0.000,003,836,680,197,358;$$

$$\text{And } \frac{1}{3m^3} (= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,000,021,255,845,968}{3}) = 0.000,000,007,085,281,989;$$

$$\text{And } \frac{1}{4m^4} (= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,000,000,058,880,459}{4}) = 0.000,000,000,014,720,114;$$

$$\text{And } \frac{1}{5m^5} (= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,000,163,103}{5}) = 0.000,000,000,000,032,620;$$

$$\text{And } \frac{1}{6m^6} (= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,000,000,451}{6}) = 0.000,000,000,000,000,075;$$

$$\text{And } \frac{1}{7m^7} (= \frac{1}{m^7} \times \frac{1}{7} = \frac{0.000,000,000,000,000,001}{7}) = 0.000,000,000,000,000,000.$$

Therefore the series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \text{etc}$ is =

$$\begin{aligned} & 0;002,770,083,102,493,074, \\ & + ;\dots\dots 3,836,680,197,358, \\ & + ;\dots\dots\dots 7,085,281,989, \\ & + ;\dots\dots\dots\dots 14,720,114, \\ & + ;\dots\dots\dots\dots\dots 32,620, \\ & + ;\dots\dots\dots\dots\dots\dots 75, \\ & = 0.002,773,926,882,725,230. \end{aligned}$$

Therefore this number 0.002,773,926,882,725,230, is the logarithm of the ratio of 1 to $1 - \frac{1}{361}$, or of 361 to 360. Q. E. I.

This number, 0.002,773,926,882,725,230, agrees with the value of the logarithm of the same ratio found above in article 31, by Mercator's series, in all the figures except the last, that value being 0.002,773,926,882,725,234.

EXAMPLE IX.

80. Lastly, let it be required to find by means of the same series the logarithm of the ratio of 529 (which is the square of 23) to 528, (or of 529 to $529 - 1$, or of $\frac{529}{529}$ to $\frac{529-1}{529}$), or of 1 to $1 - \frac{1}{529}$.

Here m is = 529, and $\frac{1}{m}$ is = $\frac{1}{529} = 0.001,890,359,168,241,965$.

We shall therefore have

$$\frac{1}{m^2} (= \frac{1}{m} \times \frac{1}{m} = \frac{1}{m} \times \frac{1}{529} = \frac{0.001,890,359,168,241,965}{529}) = 0.000,003,573,457,784,956;$$

$$\text{And } \frac{1}{m^3} (= \frac{1}{m^2} \times \frac{1}{m} = \frac{1}{m^2} \times \frac{1}{529} = \frac{0.000,003,573,457,784,956}{529}) = 0.000,000,006,755,118,686;$$

$$\text{And } \frac{1}{m^4} (= \frac{1}{m^3} \times \frac{1}{m} = \frac{1}{m^3} \times \frac{1}{529} = \frac{0.000,000,006,755,118,686}{529}) = 0.000,000,000,012,769,600;$$

$$\text{And } \frac{1}{m^5} (= \frac{1}{m^4} \times \frac{1}{m} = \frac{1}{m^4} \times \frac{1}{529} = \frac{0.000,000,000,012,769,600}{529}) = 0.000,000,000,000,024,139;$$

$$\text{And } \frac{1}{m^6} (= \frac{1}{m^5} \times \frac{1}{m} = \frac{1}{m^5} \times \frac{1}{529} = \frac{0.000,000,000,000,024,139}{529}) = 0.000,000,000,000,000,045;$$

$$\text{And } \frac{1}{m^7} (= \frac{1}{m^6} \times \frac{1}{m} = \frac{1}{m^6} \times \frac{1}{529} = \frac{0.000,000,000,000,000,045}{529}) = 0.000,000,000,000,000,000;$$

And consequently

$$\frac{1}{2m^2} (= \frac{1}{m^2} \times \frac{1}{2} = \frac{0.000,003,573,457,784,956}{2}) = 0.000,001,786,728,892,478;$$

$$\text{And } \frac{1}{3m^3} (= \frac{1}{m^3} \times \frac{1}{3} = \frac{0.000,000,006,755,118,686}{3}) = 0.000,000,002,251,706,228;$$

$$\text{And } \frac{1}{4m^4} (= \frac{1}{m^4} \times \frac{1}{4} = \frac{0.000,000,000,012,769,600}{4}) = 0.000,000,000,003,192,400;$$

$$\text{And } \frac{1}{5m^5} (= \frac{1}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,000,024,139}{5}) = 0.000,000,000,000,004,827;$$

$$\text{And } \frac{1}{6m^6} (= \frac{1}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,000,000,045}{6}) = 0.000,000,000,000,000,007.$$

Therefore the series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \&c$ is, in this case, =

$$\begin{aligned} & 0;001,890,359,168,241,965, \\ & + ; \dots, \dots 1,786,728,892,478, \\ & + ; \dots, \dots, \dots 2,251,706,228, \\ & + ; \dots, \dots, \dots, \dots 3,192,400, \\ & + ; \dots, \dots, \dots, \dots, \dots 4,827, \\ & + ; \dots, \dots, \dots, \dots, \dots, \dots 7, \\ & = 0.001,892,148,152,037,905. \end{aligned}$$

Therefore this number 0.001,892,148,152,037,905, is the logarithm of the ratio of 1 to $1 - \frac{1}{529}$ or of 529 to 528. Q. E. I.

This value of the logarithm of the ratio of 529 to 528, agrees with the value of it found above in article 32, by Mercator's series, in all the figures except the last, that value being 0.001,892,148,152,037,909.

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81. As

81. As the value of these nine logarithms obtained in the preceding examples, by means of the series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \mathcal{E}c$, invented by Dr. Wallis, agree so nearly with the values of them obtained above, by means of the series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \mathcal{E}c$, invented by Mercator, there is no reason to apprehend that any mistakes have been made in computing them; but we may justly be confident that they are exact in all the places of figures to which they agree with each other. And thus these two serieses serve as checks upon each other, or as means of discovering any mistakes that may have been made in computing logarithms by either of them, and of confirming, or proving the truth of, the calculations by which they have been obtained, when those calculations have been made without an error. And this is, indeed, the principal benefit that can be derived from the possession of both these serieses, rather than of only one of them, since each of them seems just as fit as the other for the purpose of computing logarithms.

Of the Identity of the Logarithms of any given Ratios obtained by the Series
 $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$ ad infinitum, which was invented by Mr. Mercator, with the Logarithms of the same Ratios obtained by the Series
 $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \&c$ ad infinitum, which was invented by Dr. Wallis.

82. We have seen in the foregoing examples of the computation of logarithms, by the two foregoing serieses, that the logarithm of the same ratio is always the same number, whether it be computed by Mr. Mercator's series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c$, or by Dr. Wallis's series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c$. Thus, for example, the logarithm of the ratio of 10 to 9, obtained in article 24, by means of the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c$, or $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \mathcal{E}c$, is = 0.105,360,515,657,826,302; and the logarithm of the same ratio of 10 to 9, obtained in article 72, by means of the series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c$, or $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \mathcal{E}c$, is = 0.105,360,515,657,826,296; which agrees with the former logarithm of it, obtained in article 24, in all but the three last figures. The logarithms, therefore, of two different ratios obtained by means of Dr. Wallis's series, are not only proportional to the logarithms of the same two ratios obtained by means of Mr. Mercator's series, (as they necessarily must be, because both pairs of logarithms are proportional to, or measures of, the same two ratios,) but they are equal

to them; so that the two serieses $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c$, and $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c$, exhibit the logarithms of the ratios of 1 to $1 - k$, and of $1 + k$ to 1, in one and the same system, or, in other words, the series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c$, *ad infinitum*, is to the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c$, *ad infinitum*, in the same proportion as the ratio of 1 to $1 - k$ is to the ratio of $1 + k$ to 1. Now "that this should be so," is by no means self-evident; but it may be demonstrated in the manner following.

THEOREM III.

83. If k be any quantity less than 1, the series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c$ *ad infinitum*, will be the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c$ *ad infinitum*, in the same proportion as the ratio of 1 to $1 - k$ to the ratio of $1 + k$ to 1.

DEMONSTRATION.

Let q be any quantity less than k , and consequently, *a fortiori*, less than 1.

Then will the series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c$ *ad infinitum* be to the series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \mathcal{E}c$ *ad infinitum*, in the same proportion as the ratio of 1 to $1 - k$ is to the ratio of 1 to $1 - q$. This is demonstrated in theorem 2, article 66.

Let q be extremely small, as, for example, equal to $\frac{1}{1000,000^9}$, or the quotient of the division of 1 by a nonillion, or by the ninth power of a million, that is, to 0.000000,000000,000000,000000,000000,000000,000000,000000,000000,000000, or an unit in the 54th place of decimal fractions, then will the whole series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \mathcal{E}c$ *ad infinitum* be very nearly equal to its first term q . Therefore in this case the series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c$ will be to the single term q very nearly in the same proportion as the ratio of 1 to $1 - k$ to the ratio of 1 to $1 - q$.

Further, by theorem 1, article 18, the series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \mathcal{E}c$ *ad infinitum* is to the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c$ *ad infinitum*,
R r 2

use of; and, therefore, are often called *Napier's logarithms*. They are also often called *natural logarithms*; because in most of the methods of computing logarithms they occur more readily, as measures of ratios, than any other numbers. They are also often called *hyperbolic logarithms*; because they express the magnitudes of the asymptotick areas of an hyperbola, when the square, or parallelogram, of the hyperbola is called 1; as it usually is when mathematicians apply numbers to the mensuration of its areas. But if any other area, either greater or less than the square, or parallelogram of the hyperbola, should be denoted by 1, (which is entirely a matter of choice) these logarithms would no longer express the magnitudes of the said asymptotick areas: and therefore the name of *hyperbolic* seems to have been given them without sufficient reason. If the asymptotick area of an hyperbola that corresponds to the ratio of 10 to 1, or that is bounded by two ordinates, of which the greater is equal to ten times the lesser, should be called 1, the numeral expressions of the magnitudes of the asymptotick areas of such hyperbola would be the logarithms of Briggs's system, or those which are now in common use. But it is evident, that the logarithms of the same ratios in all systems must be proportional to each other, because they are measures of, or proportional to, the same quantities, to wit, the same ratios. Thus, for example, the logarithm of the ratio of 10 to 1 in any system of logarithms whatsoever, must be to the logarithm of the ratio of 2 to 1 in the same system in the same proportion as the logarithm of the former ratio found above in article 41 to the logarithm of the latter ratio found above in article 35, that is, as 2.302,585,092,994,045,668 to 0.693,147,180,559,945,304; because in both systems the logarithms of the said ratios are proportional to the ratios themselves. And, *permutando*, the logarithm of any one ratio in one system of logarithms, must be to the logarithm of the same ratio in any other system in the same proportion as the logarithm of any other ratio in the former system to the logarithm of the said other ratio in the latter system; and consequently the logarithms of any one system may be derived from those of any other system by increasing or diminishing them all in the same proportion. In Briggs's system of logarithms the logarithm of the ratio of 10 to 1 is an unit, or 1; and consequently that of the ratio of 100 to 1 (which is double of the ratio of 10 to 1) is 2, and that of the ratio of 1000 to 1 (which is triple of the ratio of 10 to 1) is 3, and so on of the other powers of 10. The logarithms of this system will therefore be less than the logarithms of the same ratios in Napier's system in the proportion of 1 (the logarithm of the ratio of 10 to 1, in Briggs's system,) to 2.302,585,092,994,045,668, which is the logarithm of the ratio of 10 to 1 in Napier's system. Therefore, in order to derive Briggs's logarithms from Napier's, we must diminish the latter in the proportion of 1 to 2.302,585,092,994,045,668, or divide them by 2.302,585,092,994,045,668, or (which comes to the same thing) multiply them by the fraction $\frac{1}{2.302,585,092,994,045,668}$, or its equal, the decimal fraction 0.434,294,481,903,251,830.

85. The division of 1 by the long number 2.302,585,092,994,045,668, in order to obtain the decimal fraction 0.434,294,481,903,251,830, is a troublesome and tedious operation, but a very important one in the business of computing logarithms: I shall therefore, for the reader's more complete satisfaction, set it down at full length; and, in order to avoid mistakes in the calculation in the multiplications of the long divisor 2.302,585,092,994,045,668 by the several figures in the quotient, I shall raise the several products of the multiplication of this divisor into the numbers 2, 3, 4, 5, 6, 7, 8, and 9, (which are all the figures that can occur in the quotient) by addition instead of multiplication, before I begin to perform the operation of division; and then, when I come to the operation of division, I shall make use of these products that have been thus previously found by the simple operations of adding one number to another, in which it is not likely we should make any mistake. These additions are as follows:

$$\begin{array}{r}
 2.302,585,092,994,045,668, \\
 2.302,585,092,994,045,668, \\
 \hline
 4.605,170,185,988,091,336, \\
 2.302,585,092,994,045,668, \\
 \hline
 6.907,755,278,982,137,004, \\
 2.302,585,092,994,045,668, \\
 \hline
 9.210,340,371,976,182,672, \\
 2.302,585,092,994,045,668, \\
 \hline
 11.512,925,464,970,228,340, \\
 2.302,585,092,994,045,668, \\
 \hline
 13.815,510,557,964,274,008, \\
 2.302,585,092,994,045,668, \\
 \hline
 16.118,095,650,958,319,676, \\
 2.302,585,092,994,045,668, \\
 \hline
 18.420,680,743,952,365,344, \\
 2.302,585,092,994,045,668, \\
 \hline
 20.723,265,836,946,411,012.
 \end{array}$$

It appears therefore that

$$\begin{array}{l}
 2 \times 2.302,585,092,994,045,668 \text{ is } = 4.605,170,185,988,091,336; \\
 \text{And that } 3 \times 2.302,585,092,994,045,668 \text{ is } = 6.907,755,278,982,137,004; \\
 \text{And that } 4 \times 2.302,585,092,994,045,668 \text{ is } = 9.210,340,371,976,182,672; \\
 \text{And that } 5 \times 2.302,585,092,994,045,668 \text{ is } = 11.512,925,464,970,228,340; \\
 \text{And that } 6 \times 2.302,585,092,994,045,668 \text{ is } = 13.815,510,557,964,274,008; \\
 \text{And that } 7 \times 2.302,585,092,994,045,668 \text{ is } = 16.118,095,650,958,319,676; \\
 \text{And that } 8 \times 2.302,585,092,994,045,668 \text{ is } = 18.420,680,743,952,365,344; \\
 \text{And that } 9 \times 2.302,585,092,994,045,668 \text{ is } = 20.723,265,836,946,411,012.
 \end{array}$$

Having thus found the products of the multiplication of the long number 2.302,585,092,994,045,668 into the several numbers 2, 3, 4, 5, 6, 7, 8, and

and 9, in a manner that is hardly liable to error, we may venture to proceed to the grand operation of dividing 1 by the said long number; which is as follows:

$$\begin{array}{r}
 2.302,585,092,994,045,668.) \quad (0.434,294,481,903,251,830. \\
 1.000,000,000,000,000,000,000,000,000,000,000,000,000,000, \\
 \underline{9210340371976182672} \\
 .7896596280238173280 \\
 \underline{6907755278982137004} \\
 .9888410012560362760 \\
 \underline{9210340371976182672} \\
 .6780696405841800880 \\
 \underline{4605170185988091336} \\
 21755262198537095440 \\
 \underline{20723265836946411012} \\
 .10319963615906844280 \\
 \underline{9210340371976182672} \\
 .11096232439306616080 \\
 \underline{9210340371976182672} \\
 .18858920673304334080 \\
 \underline{18420680743952365344} \\
 ..4382399293519687360 \\
 \underline{2302585092994045668} \\
 20798142005256416920 \\
 \underline{20723265836946411012} \\
 ...7487616831000590800 \\
 \underline{6907755278982137004} \\
 .5798615520184537960 \\
 \underline{4605170185988091336} \\
 11934453341964466240 \\
 \underline{11512925464970228340} \\
 ..4215278769942379000 \\
 \underline{2302585092994045668} \\
 1912693676948333320 \\
 \underline{18420680743952365344} \\
 ..7062560255309679760 \\
 \underline{6907755278982137004} \\
 .1548049763275427560
 \end{array}$$

We may conclude therefore that 0.434,294,481,903,251,830 is the decimal fraction by which Napier's logarithm of any ratio must be multiplied, in order to produce Briggs's logarithm of the same ratio.

N. B.

N. B. This number is somewhat greater than the truth; the more accurate value of the decimal fraction that is equal to $\frac{1}{2.302,585,092,994,045,668}$, $\&c$, being 0.434,294,481,903,251,827,651,128.

Of the Computation of Briggs's Logarithms, by Means of the Two foregoing Infinite Serieses of Mr. Mercator and Dr. Wallis.

86. Whenever it is required to derive the logarithm of any given ratio in Briggs's system from the logarithm of the same ratio in Napier's system, the most direct way of obtaining it will be, either to divide the latter logarithm (which is supposed to be already known) by the long number 2.302,585,092,994,045,668, or to multiply it into the decimal fraction 0.434,294,481,903,251,830; and of these operations I conceive the latter, to wit, the multiplication by the decimal fraction 0.434,294,481,903,251,830, to be rather the less laborious. But if we are desirous of computing a whole table of Briggs's logarithms, or a very great number of them, the labour of performing either of these operations for every different logarithm we wanted to compute would be found almost intolerable. In such a case, therefore, it would be adviseable to perform one of these operations only with respect to the few original logarithms of small ratios, which are computed by means of either Mercator's series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \&c$ ad infinitum, or Dr. Wallis's series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \&c$ ad infinitum, and from which the logarithms of greater ratios are afterwards derived by the easy operations of addition and subtraction, or by multiplication or division by the small numbers 2, 3, or 4. Thus, for example, if we wanted to compute the logarithms of the ratios of 2 to 1, 3 to 1, 4 to 1, 5 to 1, 6 to 1, 7 to 1, 8 to 1, 9 to 1, 11 to 1, 12 to 1, 13 to 1, 14 to 1, 15 to 1, 16 to 1, 17 to 1, 18 to 1, 19 to 1, 20 to 1, 21 to 1, 22 to 1, 23 to 1, and 24 to 1, according to Briggs's system, it would be most convenient to perform this operation of multiplication by the decimal fraction 0.434,294,481,903,251,830, only with respect to the logarithms of the small original ratios of 10 to 9, of 11 to 10, of 81 to 80, of 121 to 120, of 2401 to 2400, of 169 to 168, of 289 to 288, of 361 to 360, and of 529 to 528; after which we might derive the logarithms of the greater ratios of 2 to 1, 3 to 1, 4 to 1, 5 to 1, 6 to 1, 7 to 1, 8 to 1, 9 to 1, 11 to 1, 12 to 1, 13 to 1, 14 to 1, 15 to 1, 16 to 1, 17 to 1, 18 to 1, 19 to 1, 20 to 1, 21 to 1, 22 to 1, 23 to 1, and 24 to 1, from the said lesser logarithms by addition or subtraction, or by multiplication and division of them by the small numbers 2, 3, and 4, in the manner above set forth. And even in multiplying those small original logarithms into the said long decimal fraction, it will be of importance to seek out the easiest way of proceeding, in order to lessen the labour of so tedious an operation. Now there are two ways of multiplying

multiplying one of these original logarithms represented by Mercator's series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \mathcal{E}c$, or Dr. Wallis's series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \mathcal{E}c$, into the decimal fraction 0.434,294,481,903,251,830. For we may either compute the value of the said series to the intended degree of exactness, and then multiply the value so found into the said decimal fraction, or we may multiply each of the terms of the said series into the said decimal fraction at the time we are computing them. By proceeding in the former method, the logarithm sought will be = 0.434,294,481,903,251,830 $\times$ the series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \mathcal{E}c$, or 0.434,294,481,903,251,830 $\times$ the series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \mathcal{E}c$; or if, for the sake of brevity, we denote the said long multiplier 0.434,294,481,903,251,830 by the capital letter M, the logarithm sought will be = M $\times$ the series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \mathcal{E}c$, or M $\times$ the series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \mathcal{E}c$, according as Mercator's or Doctor Wallis's series is made use of; and by proceeding in the latter method, the logarithm sought will be = to the series $\frac{M}{m} - \frac{M}{2m^2} + \frac{M}{3m^3} - \frac{M}{4m^4} + \frac{M}{5m^5} - \frac{M}{6m^6} + \mathcal{E}c$, or to the series $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} + \mathcal{E}c$; of which serieses it is evident that the former, to wit, $\frac{M}{m} - \frac{M}{2m^2} + \frac{M}{3m^3} - \frac{M}{4m^4} + \frac{M}{5m^5} - \frac{M}{6m^6} + \mathcal{E}c$, is = M $\times$ the series $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \mathcal{E}c$, and the latter, to wit, $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} + \mathcal{E}c$, is = M $\times$ the series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \mathcal{E}c$. But the latter method of proceeding (by which the logarithm sought is derived from the series $\frac{M}{m} - \frac{M}{2m^2} + \frac{M}{3m^3} - \frac{M}{4m^4} + \frac{M}{5m^5} - \frac{M}{6m^6} + \mathcal{E}c$, or the series $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} + \mathcal{E}c$) is much the more convenient of the two for the purposes of calculation. In order, therefore, to find Briggs's logarithms of the ratios of the several numbers 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, and 24, to 1, it would be adviseable to compute the logarithms of the several lesser ratios of 10 to 9, 11 to 10, 81 to 80, 121 to 120, 2401 to 2400, 169 to 168, 289 to 288, 361 to 360, and 529 to 528, by means of one of the two serieses $\frac{M}{m} - \frac{M}{2m^2} + \frac{M}{3m^3} - \frac{M}{4m^4} + \frac{M}{5m^5} - \frac{M}{6m^6} + \mathcal{E}c$, and $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} + \mathcal{E}c$. This I shall now proceed to do by means of the latter of these two serieses, to wit, the series $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} + \mathcal{E}c$ *ad infinitum*.

E X A M P L E S

OF THE

COMPUTATION OF BRIGGS'S LOGARITHMS

OF THE

RATIOS OF 1 TO SEVERAL FRACTIONAL NUMBERS,
OR NUMBERS LESS THAN 1,

Denoted by the residual quantity $1 - \frac{1}{m}$,

by means of the infinite series $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} + \mathcal{E}c$, which is derived from the series $\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \mathcal{E}c$, (invented by Dr. Wallis,)

by multiplying the several terms into the decimal fraction M , or 0.434,294,
481,903,251,830.

E X A M P L E I.

87. **L**ET it be required to find by means of the said series the logarithm of the ratio of 10 to 9, (or of 10 to $10 - 1$, or of $\frac{10}{10}$ to $\frac{10-1}{10}$,) or of 1 to $1 - \frac{1}{10}$.

Here m is = 10, and consequently $\frac{M}{m}$ is = $\frac{M}{10} = \frac{0.434,294,481,903,251,830}{10}$
= 0.043,429,448,190,325,183.

We

We shall therefore have

$$\frac{M}{m^2} (= \frac{M}{m} \times \frac{I}{m} = \frac{M}{m} \times \frac{I}{10} = \frac{0.043,429,448,190,325,183}{10}) = 0.004,342,944,819,032,518;$$

$$\text{And } \frac{M}{m^3} (= \frac{M}{m^2} \times \frac{I}{m} = \frac{M}{m^2} \times \frac{I}{10} = \frac{0.004,342,944,819,032,518}{10}) = 0.000,434,294,481,903,251;$$

$$\text{And } \frac{M}{m^4} (= \frac{M}{m^3} \times \frac{I}{m} = \frac{M}{m^3} \times \frac{I}{10} = \frac{0.000,434,294,481,903,251}{10}) = 0.000,043,429,448,190,325;$$

$$\text{And } \frac{M}{m^5} (= \frac{M}{m^4} \times \frac{I}{m} = \frac{M}{m^4} \times \frac{I}{10} = \frac{0.000,043,429,448,190,325}{10}) = 0.000,004,342,944,819,032;$$

$$\text{And } \frac{M}{m^6} (= \frac{M}{m^5} \times \frac{I}{m} = \frac{M}{m^5} \times \frac{I}{10} = \frac{0.000,004,342,944,819,032}{10}) = 0.000,000,434,294,481,903;$$

$$\text{And } \frac{M}{m^7} (= \frac{M}{m^6} \times \frac{I}{m} = \frac{M}{m^6} \times \frac{I}{10} = \frac{0.000,000,434,294,481,903}{10}) = 0.000,000,043,429,448,190;$$

$$\text{And } \frac{M}{m^8} (= \frac{M}{m^7} \times \frac{I}{m} = \frac{M}{m^7} \times \frac{I}{10} = \frac{0.000,000,043,429,448,190}{10}) = 0.000,000,004,342,944,819;$$

$$\text{And } \frac{M}{m^9} (= \frac{M}{m^8} \times \frac{I}{m} = \frac{M}{m^8} \times \frac{I}{10} = \frac{0.000,000,004,342,944,819}{10}) = 0.000,000,000,434,294,481;$$

$$\text{And } \frac{M}{m^{10}} (= \frac{M}{m^9} \times \frac{I}{m} = \frac{M}{m^9} \times \frac{I}{10} = \frac{0.000,000,000,434,294,481}{10}) = 0.000,000,000,043,429,448;$$

$$\text{And } \frac{M}{m^{11}} (= \frac{M}{m^{10}} \times \frac{I}{m} = \frac{M}{m^{10}} \times \frac{I}{10} = \frac{0.000,000,000,043,429,448}{10}) = 0.000,000,000,004,342,944;$$

$$\text{And } \frac{M}{m^{12}} (= \frac{M}{m^{11}} \times \frac{I}{m} = \frac{M}{m^{11}} \times \frac{I}{10} = \frac{0.000,000,000,004,342,944}{10}) = 0.000,000,000,000,434,294;$$

$$\text{And } \frac{M}{m^{13}} (= \frac{M}{m^{12}} \times \frac{I}{m} = \frac{M}{m^{12}} \times \frac{I}{10} = \frac{0.000,000,000,000,434,294}{10}) = 0.000,000,000,000,043,429;$$

$$\text{And } \frac{M}{m^{14}} (= \frac{M}{m^{13}} \times \frac{I}{m} = \frac{M}{m^{13}} \times \frac{I}{10} = \frac{0.000,000,000,000,043,429}{10}) = 0.000,000,000,000,004,342;$$

$$\text{And } \frac{M}{m^{15}} (= \frac{M}{m^{14}} \times \frac{I}{m} = \frac{M}{m^{14}} \times \frac{I}{10} = \frac{0.000,000,000,000,004,342}{10}) = 0.000,000,000,000,000,434;$$

$$\text{And } \frac{M}{m^{16}} (= \frac{M}{m^{15}} \times \frac{I}{m} = \frac{M}{m^{15}} \times \frac{I}{10} = \frac{0.000,000,000,000,000,434}{10}) = 0.000,000,000,000,000,043;$$

$$\text{And } \frac{M}{m^{17}} (= \frac{M}{m^{16}} \times \frac{I}{m} = \frac{M}{m^{16}} \times \frac{I}{10} = \frac{0.000,000,000,000,000,043}{10}) = 0.000,000,000,000,000,004;$$

And consequently

$$\frac{M}{2m^2} (= \frac{M}{m^2} \times \frac{I}{2} = \frac{0.004,342,944,819,032,518}{2}) = 0.002,171,472,409,516,259;$$

$$\text{And } \frac{M}{3m^3} (= \frac{M}{m^3} \times \frac{I}{3} = \frac{0.000,434,294,481,903,251}{3}) = 0.000,144,764,827,301,083;$$

$$\text{And } \frac{M}{4m^4} (= \frac{M}{m^4} \times \frac{I}{4} = \frac{0.000,043,429,448,190,325}{4}) = 0.000,010,857,362,047,581;$$

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And

$$\text{And } \frac{M}{5m^5} (= \frac{M}{m^5} \times \frac{1}{5} = \frac{0.000,004,342,944,819,032}{5}) = 0.000,000,868,588,963,806;$$

$$\text{And } \frac{M}{6m^6} (= \frac{M}{m^6} \times \frac{1}{6} = \frac{0.000,000,434,294,481,903}{6}) = 0.000,000,072,382,413,650;$$

$$\text{And } \frac{M}{7m^7} (= \frac{M}{m^7} \times \frac{1}{7} = \frac{0.000,000,043,429,448,190}{7}) = 0.000,000,006,204,206,884;$$

$$\text{And } \frac{M}{8m^8} (= \frac{M}{m^8} \times \frac{1}{8} = \frac{0.000,000,004,342,944,819}{8}) = 0.000,000,000,542,868,102;$$

$$\text{And } \frac{M}{9m^9} (= \frac{M}{m^9} \times \frac{1}{9} = \frac{0.000,000,000,434,294,481}{9}) = 0.000,000,000,048,254,942;$$

$$\text{And } \frac{M}{10m^{10}} (= \frac{M}{m^{10}} \times \frac{1}{10} = \frac{0.000,000,000,043,429,448}{10}) = 0.000,000,000,004,342,944;$$

$$\text{And } \frac{M}{11m^{11}} (= \frac{M}{m^{11}} \times \frac{1}{11} = \frac{0.000,000,000,004,342,944}{11}) = 0.000,000,000,000,394,813;$$

$$\text{And } \frac{M}{12m^{12}} (= \frac{M}{m^{12}} \times \frac{1}{12} = \frac{0.000,000,000,000,434,294}{12}) = 0.000,000,000,000,036,191;$$

$$\text{And } \frac{M}{13m^{13}} (= \frac{M}{m^{13}} \times \frac{1}{13} = \frac{0.000,000,000,000,043,429}{13}) = 0.000,000,000,000,003,340;$$

$$\text{And } \frac{M}{14m^{14}} (= \frac{M}{m^{14}} \times \frac{1}{14} = \frac{0.000,000,000,000,004,342}{14}) = 0.000,000,000,000,000,310;$$

$$\text{And } \frac{M}{15m^{15}} (= \frac{M}{m^{15}} \times \frac{1}{15} = \frac{0.000,000,000,000,000,434}{15}) = 0.000,000,000,000,000,028;$$

$$\text{And } \frac{M}{16m^{16}} (= \frac{M}{m^{16}} \times \frac{1}{16} = \frac{0.000,000,000,000,000,043}{16}) = 0.000,000,000,000,000,002;$$

$$\text{And } \frac{M}{17m^{17}} (= \frac{M}{m^{17}} \times \frac{1}{17} = \frac{0.000,000,000,000,000,004}{17}) = 0.000,000,000,000,000,000.$$

$$\text{Therefore the series } \frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} + \frac{M}{7m^7} + \frac{M}{8m^8} + \frac{M}{9m^9} + \frac{M}{10m^{10}} + \frac{M}{11m^{11}} + \frac{M}{12m^{12}} + \frac{M}{13m^{13}} + \frac{M}{14m^{14}} + \frac{M}{15m^{15}} + \frac{M}{16m^{16}} + \frac{M}{17m^{17}} + \text{Ec is } =$$

$$\begin{aligned} & 0;043,429,448,190,325,183, \\ & + ;..2,171,472,409,516,259, \\ & + ;...144,764,827,301,083, \\ & + ;...10,857,362,047,581, \\ & + ;...868,588,963,806, \\ & + ;...72,382,413,650, \\ & + ;...6,204,206,884, \\ & + ;...542,868,102, \\ & + ;...48,254,942, \\ & + ;...4,342,944, \\ & + ;...394,813, \\ & + ;...36,191, \\ & + ;...3,340, \\ & + ;...310, \\ & + ;...28, \\ & + ;...2, \\ & = 0.045,757,490,560,675,118. \end{aligned}$$

Therefore

Therefore this number 0.045,757,490,560,675,118 is Briggs's logarithm of the ratio of 1 to $1 - \frac{1}{10}$, or of 10 to 9. Q. E. I.

This value of the logarithm of the ratio of 10 to 9 is exact in all the figures, except the two last, which ought to be 25 instead of 18; the more exact value of this logarithm being 0.045,757,490,560,675,125,410, as will appear by subtracting Briggs's logarithm of 9, as computed by Mr. Abraham Sharp, to wit, 0.954,242,509,439,324,874,590, &c, from Briggs's logarithm of 10, to wit, 1.000,000,000,000,000,000,000.

EXAMPLE II.

88. Let it be required to find by means of the same series Briggs's logarithm of the ratio of 11 to 10, (or of 11 to $11 - 1$, or of $\frac{11}{11}$ to $\frac{11-1}{11}$), or of 1 to $1 - \frac{1}{11}$.

Here m is = 11, and $\frac{1}{m}$ is = $\frac{1}{11}$, and consequently $\frac{M}{m}$ is ($= M \times \frac{1}{11} = \frac{M}{11}$) = $\frac{0.434,294,481,903,251,830}{11}$ = 0.039,481,316,536,659,257.

We shall therefore have

$$\frac{M}{m^2} (= \frac{M}{m} \times \frac{1}{m} = \frac{M}{m} \times \frac{1}{11} = \frac{0.039,481,316,536,659,257}{11}) = 0.003,589,210,594,241,750;$$

$$\text{And } \frac{M}{m^3} (= \frac{M}{m^2} \times \frac{1}{m} = \frac{M}{m^2} \times \frac{1}{11} = \frac{0.003,589,210,594,241,750}{11}) = 0.000,326,291,872,203,795;$$

$$\text{And } \frac{M}{m^4} (= \frac{M}{m^3} \times \frac{1}{m} = \frac{M}{m^3} \times \frac{1}{11} = \frac{0.000,326,291,872,203,795}{11}) = 0.000,029,662,897,473,072;$$

$$\text{And } \frac{M}{m^5} (= \frac{M}{m^4} \times \frac{1}{m} = \frac{M}{m^4} \times \frac{1}{11} = \frac{0.000,029,662,897,473,072}{11}) = 0.000,002,696,627,043,006;$$

$$\text{And } \frac{M}{m^6} (= \frac{M}{m^5} \times \frac{1}{m} = \frac{M}{m^5} \times \frac{1}{11} = \frac{0.000,002,696,627,043,006}{11}) = 0.000,000,245,147,913,000;$$

$$\text{And } \frac{M}{m^7} (= \frac{M}{m^6} \times \frac{1}{m} = \frac{M}{m^6} \times \frac{1}{11} = \frac{0.000,000,245,147,913,000}{11}) = 0.000,000,022,286,173,909;$$

$$\text{And } \frac{M}{m^8} (= \frac{M}{m^7} \times \frac{1}{m} = \frac{M}{m^7} \times \frac{1}{11} = \frac{0.000,000,022,286,173,909}{11}) = 0.000,000,002,026,015,809;$$

$$\text{And } \frac{M}{m^9} (= \frac{M}{m^8} \times \frac{1}{m} = \frac{M}{m^8} \times \frac{1}{11} = \frac{0.000,000,002,026,015,809}{11}) = 0.000,000,000,184,183,255;$$

$$\text{And } \frac{M}{m^{10}} (= \frac{M}{m^9} \times \frac{1}{m} = \frac{M}{m^9} \times \frac{1}{11} = \frac{0.000,000,000,184,183,255}{11}) = 0.000,000,000,016,743,932;$$

$$\text{And } \frac{M}{m^{11}} (= \frac{M}{m^{10}} \times \frac{1}{m} = \frac{M}{m^{10}} \times \frac{1}{11} = \frac{0.000,000,000,016,743,932}{11}) = 0.000,000,000,001,522,175;$$

$$\text{And } \frac{M}{m^{12}} (= \frac{M}{m^{11}} \times \frac{1}{m} = \frac{M}{m^{11}} \times \frac{1}{11} = \frac{0.000,000,000,001,522,175}{11}) = 0.000,000,000,000,138,379;$$

And

$$\text{And } \frac{M}{m^{13}} (= \frac{M}{m^{12}} \times \frac{1}{m} = \frac{M}{m^{12}} \times \frac{1}{11} = \frac{0.000,000,000,000,138,379}{11}) = 0.000,000,000,000,012,579;$$

$$\text{And } \frac{M}{m^{14}} (= \frac{M}{m^{13}} \times \frac{1}{m} = \frac{M}{m^{13}} \times \frac{1}{11} = \frac{0.000,000,000,000,012,579}{11}) = 0.000,000,000,000,001,143;$$

$$\text{And } \frac{M}{m^{15}} (= \frac{M}{m^{14}} \times \frac{1}{m} = \frac{M}{m^{14}} \times \frac{1}{11} = \frac{0.000,000,000,000,001,143}{11}) = 0.000,000,000,000,000,103;$$

$$\text{And } \frac{M}{m^{16}} (= \frac{M}{m^{15}} \times \frac{1}{m} = \frac{M}{m^{15}} \times \frac{1}{11} = \frac{0.000,000,000,000,000,103}{11}) = 0.000,000,000,000,000,009;$$

And consequently

$$\frac{M}{2m^2} (= \frac{M}{m^2} \times \frac{1}{2} = \frac{0.003,589,210,594,241,750}{2}) = 0.001,794,605,297,120,875;$$

$$\text{And } \frac{M}{3m^3} (= \frac{M}{m^3} \times \frac{1}{3} = \frac{0.000,326,291,872,203,795}{3}) = 0.000,108,763,957,401,265;$$

$$\text{And } \frac{M}{4m^4} (= \frac{M}{m^4} \times \frac{1}{4} = \frac{0.000,029,662,897,473,072}{4}) = 0.000,007,415,724,368,268;$$

$$\text{And } \frac{M}{5m^5} (= \frac{M}{m^5} \times \frac{1}{5} = \frac{0.000,002,696,627,043,006}{5}) = 0.000,000,539,325,408,601;$$

$$\text{And } \frac{M}{6m^6} (= \frac{M}{m^6} \times \frac{1}{6} = \frac{0.000,000,245,147,913,000}{6}) = 0.000,000,040,857,985,500;$$

$$\text{And } \frac{M}{7m^7} (= \frac{M}{m^7} \times \frac{1}{7} = \frac{0.000,000,022,286,173,909}{7}) = 0.000,000,003,183,739,129;$$

$$\text{And } \frac{M}{8m^8} (= \frac{M}{m^8} \times \frac{1}{8} = \frac{0.000,000,002,026,015,809}{8}) = 0.000,000,000,253,251,976;$$

$$\text{And } \frac{M}{9m^9} (= \frac{M}{m^9} \times \frac{1}{9} = \frac{0.000,000,000,184,183,255}{9}) = 0.000,000,000,020,464,806;$$

$$\text{And } \frac{M}{10m^{10}} (= \frac{M}{m^{10}} \times \frac{1}{10} = \frac{0.000,000,000,016,743,932}{10}) = 0.000,000,000,001,674,393;$$

$$\text{And } \frac{M}{11m^{11}} (= \frac{M}{m^{11}} \times \frac{1}{11} = \frac{0.000,000,000,001,522,175}{11}) = 0.000,000,000,000,138,379;$$

$$\text{And } \frac{M}{12m^{12}} (= \frac{M}{m^{12}} \times \frac{1}{12} = \frac{0.000,000,000,000,138,379}{12}) = 0.000,000,000,000,011,531;$$

$$\text{And } \frac{M}{13m^{13}} (= \frac{M}{m^{13}} \times \frac{1}{13} = \frac{0.000,000,000,000,012,579}{13}) = 0.000,000,000,000,000,967;$$

$$\text{And } \frac{M}{14m^{14}} (= \frac{M}{m^{14}} \times \frac{1}{14} = \frac{0.000,000,000,000,001,143}{14}) = 0.000,000,000,000,000,081;$$

$$\text{And } \frac{M}{15m^{15}} (= \frac{M}{m^{15}} \times \frac{1}{15} = \frac{0.000,000,000,000,000,103}{15}) = 0.000,000,000,000,000,006.$$

Therefore the series $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} + \frac{M}{7m^7} + \frac{M}{8m^8} + \frac{M}{9m^9} +$
 $\frac{M}{10m^{10}} + \frac{M}{11m^{11}} + \frac{M}{12m^{12}} + \frac{M}{13m^{13}} + \frac{M}{14m^{14}} + \frac{M}{15m^{15}} + \mathcal{E}c$ is =

$$\begin{aligned}
 & 0;039,481,316,536,659,257, \\
 & + ;..1,794,605,297,120,875, \\
 & + ;...108,763,957,401,265, \\
 & + ;....7,415,724,368,268, \\
 & + ;.....539,325,408,601, \\
 & + ;.....40,857,985,500, \\
 & + ;.....3,183,739,129, \\
 & + ;.....253,251,976, \\
 & + ;.....20,464,806, \\
 & + ;.....1,674,393, \\
 & + ;.....138,379, \\
 & + ;.....11,531, \\
 & + ;.....967, \\
 & + ;.....81, \\
 & + ;.....6, \\
 & = 0.041,392,685,158,225,034.
 \end{aligned}$$

Therefore this number 0.041,392,685,158,225,034, is Briggs's logarithm of the ratio of 1 to $1 - \frac{1}{11}$, or of 11 to 10. Q. E. I.

This value of the logarithm of the ratio of 11 to 10 is exact in all the figures except the two last, its more exact value being 0.041,392,685,158,225,040,750.

EXAMPLE III.

89. Let it be required to find by means of the same series Briggs's logarithm of the ratio of 81 to 80, (or of 81 to $81 - 1$, or of $\frac{81}{81}$ to $\frac{81-1}{81}$), or of 1 to $1 - \frac{1}{81}$.

Here m is $= 81$, and $\frac{1}{m}$ is $= \frac{1}{81}$, and consequently $\frac{M}{m}$ is $(= M \times \frac{1}{m} = M \times \frac{1}{81} = \frac{M}{81} = \frac{0.434,294,481,903,251,830}{81}) = 0.005,361,660,270,410,516$.

We shall therefore have,

$$\begin{aligned}
 \frac{M}{m^2} & (= \frac{M}{m} \times \frac{1}{m} = \frac{M}{m} \times \frac{1}{81} = \frac{0.005,361,660,270,410,516}{81}) = 0.000,066,193,336,671,734; \\
 \text{And } \frac{M}{m^3} & (= \frac{M}{m^2} \times \frac{1}{m} = \frac{M}{m^2} \times \frac{1}{81} = \frac{0.000,066,193,336,671,734}{81}) = 0.000,000,817,201,687,305; \\
 \text{And } \frac{M}{m^4} & (= \frac{M}{m^3} \times \frac{1}{m} = \frac{M}{m^3} \times \frac{1}{81} = \frac{0.000,000,817,201,687,305}{81}) = 0.000,000,010,088,909,719; \\
 \text{And } \frac{M}{m^5} & (= \frac{M}{m^4} \times \frac{1}{m} = \frac{M}{m^4} \times \frac{1}{81} = \frac{0.000,000,010,088,909,719}{81}) = 0.000,000,000,124,554,440; \\
 \text{And } \frac{M}{m^6} & (= \frac{M}{m^5} \times \frac{1}{m} = \frac{M}{m^5} \times \frac{1}{81} = \frac{0.000,000,000,124,554,440}{81}) = 0.000,000,000,001,537,709;
 \end{aligned}$$

$$\text{And } \frac{M}{m^7} (= \frac{M}{m^6} \times \frac{1}{m} = \frac{M}{m^6} \times \frac{1}{81} = \frac{0.000,000,000,001,537,709}{81}) = 0.000,000,000,000,018,984;$$

$$\text{And } \frac{M}{m^8} (= \frac{M}{m^7} \times \frac{1}{m} = \frac{M}{m^7} \times \frac{1}{81} = \frac{0.000,000,000,000,018,984}{81}) = 0.000,000,000,000,000,234;$$

$$\text{And } \frac{M}{m^9} (= \frac{M}{m^8} \times \frac{1}{m} = \frac{M}{m^8} \times \frac{1}{81} = \frac{0.000,000,000,000,000,234}{81}) = 0.000,000,000,000,000,002;$$

And consequently

$$\frac{M}{2m^2} (= \frac{M}{m^2} \times \frac{1}{2} = \frac{0.000,066,193,336,671,734}{2}) = 0.000,033,096,668,335,867;$$

$$\text{And } \frac{M}{3m^3} (= \frac{M}{m^3} \times \frac{1}{3} = \frac{0.000,000,817,201,687,305}{3}) = 0.000,000,272,400,562,435;$$

$$\text{And } \frac{M}{4m^4} (= \frac{M}{m^4} \times \frac{1}{4} = \frac{0.000,000,010,088,909,719}{4}) = 0.000,000,002,522,227,429;$$

$$\text{And } \frac{M}{5m^5} (= \frac{M}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,124,554,440}{5}) = 0.000,000,000,024,910,888;$$

$$\text{And } \frac{M}{6m^6} (= \frac{M}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,001,537,709}{6}) = 0.000,000,000,000,256,284;$$

$$\text{And } \frac{M}{7m^7} (= \frac{M}{m^7} \times \frac{1}{7} = \frac{0.000,000,000,000,018,984}{7}) = 0.000,000,000,000,002,712;$$

$$\text{And } \frac{M}{8m^8} (= \frac{M}{m^8} \times \frac{1}{8} = \frac{0.000,000,000,000,000,234}{8}) = 0.000,000,000,000,000,029;$$

$$\text{And } \frac{M}{9m^9} (= \frac{M}{m^9} \times \frac{1}{9} = \frac{0.000,000,000,000,000,002}{9}) = 0.000,000,000,000,000,000.$$

Therefore the series $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} + \frac{M}{7m^7} + \frac{M}{8m^8} + \text{Et}$
is =

$$\begin{aligned} & 0;005,361,660,270,410,516, \\ & + ;\dots, 33,096,668,335,867, \\ & + ;\dots, 272,400,562,435, \\ & + ;\dots, 2,522,227,429, \\ & + ;\dots, 24,910,888, \\ & + ;\dots, 256,284, \\ & + ;\dots, 2,712, \\ & + ;\dots, 29, \\ & + ;\dots, 0, \\ & = 0.005,395,031,886,706,160. \end{aligned}$$

Therefore this number 0.005,395,031,886,706,160 is Briggs's logarithm of the ratio of 1 to $1 - \frac{1}{81}$, or of 81 to 80. Q. E. I.

This value of the logarithm of the ratio of 81 to 80 is exact in all but the last place of figures, in which there ought to be a 3 instead of a cypher; the more exact value of this logarithm being the excess of Briggs's logarithm of the ratio of 80 to 1, above

above Briggs's logarithm of the ratio of 80 to 1, that is, (according to Mr. Abraham Sharp's computation of those logarithms,) to the excess of 1.908, 485,018,878,649,749,180, above 1.903,089,986,991,943,585,641, or to 0.005, 395,031,886,706,163,539.

EXAMPLE IV.

90. Let it be required to find by means of the same series Briggs's logarithm of the ratio of 121 to 120, (or of 121 to $121 - 1$, or of $\frac{121}{121}$ to $\frac{121-1}{121}$) or of 1 to $1 - \frac{1}{121}$.

Here m is = 121, and $\frac{1}{m}$ is = $\frac{1}{121}$, and consequently $\frac{M}{m}$ is (= $M \times \frac{1}{m} = M \times \frac{1}{121} = \frac{M}{121} = \frac{0.434,294,481,903,251,830}{121}$) = 0.003,589,210,594,241,750.

We shall therefore have

$$\frac{M}{m^2} (= \frac{M}{m} \times \frac{1}{m} = \frac{M}{m} \times \frac{1}{121} = \frac{0.003,589,210,594,241,750}{121}) = 0.000,029,662,897,473,072;$$

$$\text{And } \frac{M}{m^3} (= \frac{M}{m^2} \times \frac{1}{m} = \frac{M}{m^2} \times \frac{1}{121} = \frac{0.000,029,662,897,473,072}{121}) = 0.000,000,245,147,913,000;$$

$$\text{And } \frac{M}{m^4} (= \frac{M}{m^3} \times \frac{1}{m} = \frac{M}{m^3} \times \frac{1}{121} = \frac{0.000,000,245,147,913,000}{121}) = 0.000,000,002,026,015,809;$$

$$\text{And } \frac{M}{m^5} (= \frac{M}{m^4} \times \frac{1}{m} = \frac{M}{m^4} \times \frac{1}{121} = \frac{0.000,000,002,026,015,809}{121}) = 0.000,000,000,016,743,932;$$

$$\text{And } \frac{M}{m^6} (= \frac{M}{m^5} \times \frac{1}{m} = \frac{M}{m^5} \times \frac{1}{121} = \frac{0.000,000,000,016,743,932}{121}) = 0.000,000,000,000,138,379;$$

$$\text{And } \frac{M}{m^7} (= \frac{M}{m^6} \times \frac{1}{m} = \frac{M}{m^6} \times \frac{1}{121} = \frac{0.000,000,000,000,138,379}{121}) = 0.000,000,000,000,001,143;$$

$$\text{And } \frac{M}{m^8} (= \frac{M}{m^7} \times \frac{1}{m} = \frac{M}{m^7} \times \frac{1}{121} = \frac{0.000,000,000,000,001,143}{121}) = 0.000,000,000,000,000,009;$$

And consequently

$$\frac{M}{2m^2} (= \frac{M}{m^2} \times \frac{1}{2} = \frac{0.000,029,662,897,473,072}{2}) = 0.000,014,831,448,736,536;$$

$$\text{And } \frac{M}{3m^3} (= \frac{M}{m^3} \times \frac{1}{3} = \frac{0.000,000,245,147,913,000}{3}) = 0.000,000,081,715,971,000;$$

$$\text{And } \frac{M}{4m^4} (= \frac{M}{m^4} \times \frac{1}{4} = \frac{0.000,000,002,026,015,809}{4}) = 0.000,000,000,506,503,952;$$

$$\text{And } \frac{M}{5m^5} (= \frac{M}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,016,743,932}{5}) = 0.000,000,000,003,348,786;$$

T t

And

$$\text{And } \frac{M}{6m^6} (= \frac{M}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,000,138,379}{6}) = 0.000,000,000,000,023,063;$$

$$\text{And } \frac{M}{7m^7} (= \frac{M}{m^7} \times \frac{1}{7} = \frac{0.000,000,000,000,001,143}{7}) = 0.000,000,000,000,000,163;$$

$$\text{And } \frac{M}{8m^8} (= \frac{M}{m^8} \times \frac{1}{8} = \frac{0.000,000,000,000,000,009}{8}) = 0.000,000,000,000,000,001.$$

Therefore the series $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} + \frac{M}{7m^7} + \frac{M}{8m^8} + \dots$ will be =

$$\begin{array}{r} 0;003,589,210,594,241,750, \\ + ;\dots, 14,831,448,736,536, \\ + ;\dots, \dots, 81,715,971,000, \\ + ;\dots, \dots, \dots, 506,503,952, \\ + ;\dots, \dots, \dots, 3,348,786, \\ + ;\dots, \dots, \dots, 23,063, \\ + ;\dots, \dots, \dots, 163, \\ + ;\dots, \dots, \dots, 1, \\ \hline = 0.003,604,124,268,825,251. \end{array}$$

Therefore this number 0.003,604,124,268,825,251, is Briggs's logarithm of the ratio of 1 to $1 - \frac{1}{121}$, or of 121 to 120. Q. E. I.

This value of the logarithm of the ratio of 121 to 120 is exact in all the figures except the last, which ought to be a 3 instead of an unit; for the more accurate value of this logarithm is 0.003,604,124,268,825,253,778, as will appear from subtracting the logarithm of the ratio of 120 to 1 from the logarithm of the ratio of 121 to 1; which subtraction may be performed as follows. The logarithm of 11 is (according to Mr. Sharp's computation) 1.041,392,685,158,225,040,750; the double of which, to wit, 2.082,785,370,316,450,081,500, is the logarithm of 121, which is the square of 11. The logarithm of 12 is (according to Mr. Sharp's computation) = 1.079,181,246,047,624,827,722; and consequently the logarithm of 10 times 12, or 120, will be = 1 + 1.079,181,246,047,624,827,722, or 2.079,181,246,047,624,827,722; which being subtracted from 2.082,785,370,316,450,081,500, leaves 0.003,604,124,268,825,253,778 for the logarithm of the ratio of 121 to 120.

EXAMPLE V.

91. Let it be required to find by means of the same series the logarithm of the ratio of 2401 to 2400, (or of 2401 to $2401 - 1$, or of $\frac{2401}{2401}$ to $\frac{2401-1}{2401}$), or of 1 to $1 - \frac{1}{2401}$.

Here

Here m is $= 2401$, and $\frac{1}{m}$ is $= \frac{1}{2401}$, and $\frac{M}{m}$ is $(= \frac{M}{2401} = \frac{0.434,294,481,903,251,830}{2401})$
 $= 0.000,180,880,667,181,695$.

We shall therefore have

$$\frac{M}{m^2} (= \frac{M}{m} \times \frac{1}{m} = \frac{M}{m} \times \frac{1}{2401} = \frac{0.000,180,880,667,181,695}{2401}) = 0.000,000,075,335,554,844;$$

$$\text{And } \frac{M}{m^3} (= \frac{M}{m^2} \times \frac{1}{m} = \frac{M}{m^2} \times \frac{1}{2401} = \frac{0.000,000,075,335,554,844}{2401}) = 0.000,000,000,031,376,740;$$

$$\text{And } \frac{M}{m^4} (= \frac{M}{m^3} \times \frac{1}{m} = \frac{M}{m^3} \times \frac{1}{2401} = \frac{0.000,000,000,031,376,740}{2401}) = 0.000,000,000,000,013,068;$$

$$\text{And } \frac{M}{m^5} (= \frac{M}{m^4} \times \frac{1}{m} = \frac{M}{m^4} \times \frac{1}{2401} = \frac{0.000,000,000,000,013,068}{2401}) = 0.000,000,000,000,000,005;$$

And consequently

$$\frac{M}{2m^2} (= \frac{M}{m^2} \times \frac{1}{2} = \frac{0.000,000,075,335,554,844}{2}) = 0.000,000,037,667,777,422;$$

$$\text{And } \frac{M}{3m^3} (= \frac{M}{m^3} \times \frac{1}{3} = \frac{0.000,000,000,031,376,740}{3}) = 0.000,000,000,010,458,913;$$

$$\text{And } \frac{M}{4m^4} (= \frac{M}{m^4} \times \frac{1}{4} = \frac{0.000,000,000,000,013,068}{4}) = 0.000,000,000,000,003,267;$$

$$\text{And } \frac{M}{5m^5} (= \frac{M}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,000,000,005}{5}) = 0.000,000,000,000,000,001.$$

Therefore the series $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \&c$ will be =

$$\begin{array}{r} 0.000,180,880,667,181,695, \\ + \text{ ; } \dots, \dots, 37,667,777,422, \\ + \text{ ; } \dots, \dots, \dots, 10,458,913, \\ + \text{ ; } \dots, \dots, \dots, \dots, 3,267, \\ + \text{ ; } \dots, \dots, \dots, \dots, \dots, 1, \\ \hline = 0.000,180,918,345,421,298. \end{array}$$

Therefore this number 0.000,180,918,345,421,298 is the logarithm of the ratio of 1 to $1 - \frac{1}{2401}$, or of 2401 to 2400. Q. E. I.

This value of the logarithm of 2401 to 2400 is exact in all the figures except the last; its more exact value being 0.000,180,918,345,421,299,912, as will appear by subtracting the logarithm of 2400 from the logarithm of 2401, which may be done as follows. According to Mr. Sharp's computation, the logarithm of 7 is $= 0.845,098,040,014,256,830,712$; consequently the logarithm of 2401 (which is the fourth power of 7) will be $= 4 \times 0.845,098,040,014,256,830,712 = 3.380,392,160,057,027,322,848$. And the logarithm of 24 is $= 1.380,211,241,711,606,022,936$; and consequently the logarithm of 24×100 , or 2400, is $= 2 + 1.380,211,241,$

$711,606,022,936 = 3.380,211,241,711,606,022,936$. And, if this last logarithm be subtracted from $3.380,392,160,057,027,322,848$, (which is the logarithm of 2401,) the remainder will be $0.000,180,918,345,421,299,912$.

EXAMPLE VI.

92. Let it be required to find by means of the same series the logarithm of the ratio of 169 to 168, (or of 169 to $169 - 1$, or of $\frac{169}{169}$ to $\frac{169-1}{169}$) or of 1 to $1 - \frac{1}{169}$.

Here m is $= 169$, and $\frac{1}{m}$ is $= \frac{1}{169}$, and $\frac{M}{m}$ is $(= \frac{M}{169} = \frac{0.434,294,481,903,251,830}{169} = 0.002,569,789,833,747,052$.

We shall therefore have

$$\frac{M}{m^2} (= \frac{M}{m} \times \frac{1}{m} = \frac{M}{m} \times \frac{1}{169} = \frac{0.002,569,789,833,747,052}{169}) = 0.000,015,205,857,004,420;$$

$$\text{And } \frac{M}{m^3} (= \frac{M}{m^2} \times \frac{1}{m} = \frac{M}{m^2} \times \frac{1}{169} = \frac{0.000,015,205,857,004,420}{169}) = 0.000,000,089,975,485,233;$$

$$\text{And } \frac{M}{m^4} (= \frac{M}{m^3} \times \frac{1}{m} = \frac{M}{m^3} \times \frac{1}{169} = \frac{0.000,000,089,975,485,233}{169}) = 0.000,000,000,532,399,320;$$

$$\text{And } \frac{M}{m^5} (= \frac{M}{m^4} \times \frac{1}{m} = \frac{M}{m^4} \times \frac{1}{169} = \frac{0.000,000,000,532,399,320}{169}) = 0.000,000,000,003,150,291;$$

$$\text{And } \frac{M}{m^6} (= \frac{M}{m^5} \times \frac{1}{m} = \frac{M}{m^5} \times \frac{1}{169} = \frac{0.000,000,000,003,150,291}{169}) = 0.000,000,000,000,018,640;$$

$$\text{And } \frac{M}{m^7} (= \frac{M}{m^6} \times \frac{1}{m} = \frac{M}{m^6} \times \frac{1}{169} = \frac{0.000,000,000,000,018,640}{169}) = 0.000,000,000,000,000,110;$$

$$\text{And } \frac{M}{m^8} (= \frac{M}{m^7} \times \frac{1}{m} = \frac{M}{m^7} \times \frac{1}{169} = \frac{0.000,000,000,000,000,110}{169}) = 0.000,000,000,000,000,000;$$

And consequently

$$\frac{M}{2m^2} (= \frac{M}{m^2} \times \frac{1}{2} = \frac{0.000,015,205,857,004,420}{2}) = 0.000,007,602,928,502,210;$$

$$\text{And } \frac{M}{3m^3} (= \frac{M}{m^3} \times \frac{1}{3} = \frac{0.000,000,089,975,485,233}{3}) = 0.000,000,029,991,828,411;$$

$$\text{And } \frac{M}{4m^4} (= \frac{M}{m^4} \times \frac{1}{4} = \frac{0.000,000,000,532,399,320}{4}) = 0.000,000,000,133,099,830;$$

$$\text{And } \frac{M}{5m^5} (= \frac{M}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,003,150,291}{5}) = 0.000,000,000,000,630,058;$$

And

$$\text{And } \frac{M}{6m^6} (= \frac{M}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,000,018,640}{6}) = 0.000,000,000,000,003,106;$$

$$\text{And } \frac{M}{7m^7} (= \frac{M}{m^7} \times \frac{1}{7} \times \frac{0.000,000,000,000,000,110}{7}) = 0.000,000,000,000,000,015.$$

Therefore the series $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} + \frac{M}{7m^7} + \&c$ will be =

$$\begin{aligned} & 0;002,569,789,833,747,052, \\ & + ; \dots, 7,602,928,502,210, \\ & + ; \dots, 29,991,828,411, \\ & + ; \dots, 133,099,830, \\ & + ; \dots, 630,058, \\ & + ; \dots, 3,106, \\ & + ; \dots, 15, \\ & = 0.002,577,422,887,810,682. \end{aligned}$$

Therefore this number 0.002,577,422,887,810,682 is Briggs's logarithm of the ratio of 1 to $1 - \frac{1}{169}$, or of 169 to 168. Q. E. I.

This value of the logarithm of the ratio of 169 to 168 is exact in all the figures except the last, which should be a 4 instead of a 2; the more accurate value of this logarithm being 0.002,577,422,887,810,684,765. This will appear from Mr. Sharp's computations, in the following manner. The logarithm of 13, according to Mr. Sharp, is = 1.113,943,352,306,836,769,206; therefore the logarithm of 169 (which is the square of 13) is = $2 \times 1.113,943,352,306,836,769,206$, or 2.227,886,704,613,673,538,412. The logarithm of 12 is = 1.079,181,246,047,624,827,722; and the logarithm of 14 is 1.146,128,035,678,238,025,925. Therefore the logarithm of 168 (which is = 12×14) is =

$$\left. \begin{aligned} & 1.079,181,246,047,624,827,722 \\ & + 1.146,128,035,678,238,025,925 \end{aligned} \right\} = 2.225,309,281,725,862,853,647;$$

which being subtracted from 2.227,886,704,613,673,538,412 (which is the logarithm of 169) leaves 0.002,577,422,887,810,684,765 for the logarithm of the ratio of 169 to 168. Q. E. D.

EXAMPLE VII.

93. Let it be required to find by means of the same series the logarithm of the ratio of 289 to 288, (or of 289 to $289 - 1$, or of $\frac{289}{289}$ to $\frac{289-1}{289}$), or of 1 to $1 - \frac{1}{289}$.

$$- \frac{1}{289}.$$

Here m is = 289, and $\frac{1}{m}$ is = $\frac{1}{289}$, and consequently $\frac{M}{m}$ is (= $\frac{M}{289} = \frac{0.434,294,481,903,251,830}{289}$) = 0.001,502,749,072,329,591.

We

We shall therefore have

$$\frac{M}{m^2} (= \frac{M}{m} \times \frac{I}{m} = \frac{M}{m} \times \frac{I}{289} = \frac{0.001,502,749,072,329,591}{289}) = 0.000,005,199,823,779,687;$$

$$\text{And } \frac{M}{m^3} (= \frac{M}{m^2} \times \frac{I}{m} = \frac{M}{m^2} \times \frac{I}{289} = \frac{0.000,005,199,823,779,687}{289}) = 0.000,000,017,992,469,825;$$

$$\text{And } \frac{M}{m^4} (= \frac{M}{m^3} \times \frac{I}{m} = \frac{M}{m^3} \times \frac{I}{289} = \frac{0.000,000,017,992,469,825}{289}) = 0.000,000,000,062,257,681;$$

$$\text{And } \frac{M}{m^5} (= \frac{M}{m^4} \times \frac{I}{m} = \frac{M}{m^4} \times \frac{I}{289} = \frac{0.000,000,000,062,257,681}{289}) = 0.000,000,000,000,215,424;$$

$$\text{And } \frac{M}{m^6} (= \frac{M}{m^5} \times \frac{I}{m} = \frac{M}{m^5} \times \frac{I}{289} = \frac{0.000,000,000,000,215,424}{289}) = 0.000,000,000,000,000,745;$$

$$\text{And } \frac{M}{m^7} (= \frac{M}{m^6} \times \frac{I}{m} = \frac{M}{m^6} \times \frac{I}{289} = \frac{0.000,000,000,000,000,745}{289}) = 0.000,000,000,000,000,002;$$

And consequently

$$\frac{M}{2m^2} (= \frac{M}{m^2} \times \frac{I}{2} = \frac{0.000,005,199,823,779,687}{2}) = 0.000,002,599,911,889,843;$$

$$\text{And } \frac{M}{3m^3} (= \frac{M}{m^3} \times \frac{I}{3} = \frac{0.000,000,017,992,469,825}{3}) = 0.000,000,005,997,489,941;$$

$$\text{And } \frac{M}{4m^4} (= \frac{M}{m^4} \times \frac{I}{4} = \frac{0.000,000,000,062,257,681}{4}) = 0.000,000,000,015,564,420;$$

$$\text{And } \frac{M}{5m^5} (= \frac{M}{m^5} \times \frac{I}{5} = \frac{0.000,000,000,000,215,424}{5}) = 0.000,000,000,000,043,084;$$

$$\text{And } \frac{M}{6m^6} (= \frac{M}{m^6} \times \frac{I}{6} = \frac{0.000,000,000,000,000,745}{6}) = 0.000,000,000,000,000,124;$$

$$\text{And } \frac{M}{7m^7} (= \frac{M}{m^7} \times \frac{I}{7} = \frac{0.000,000,000,000,000,002}{7}) = 0.000,000,000,000,000,000.$$

Therefore the series $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} + \frac{M}{7m^7} + \mathcal{E}c$
is =

$$\begin{aligned} & 0;001,502,749,072,329,591, \\ & + ;\dots\dots\dots 2,599,911,889,843, \\ & + ;\dots\dots\dots 5,997,489,941, \\ & + ;\dots\dots\dots 15,564,420, \\ & + ;\dots\dots\dots 43,084, \\ & + ;\dots\dots\dots 124, \\ & + ;\dots\dots\dots, \\ & = 0.001,505,354,997,317,003. \end{aligned}$$

Therefore this number 0.001,505,354,997,317,003 is Briggs's logarithm of the ratio of 1 to $1 - \frac{I}{289}$, or of 289 to 288.

Q. E. I.

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This value of Briggs's logarithm of the ratio of 289 to 288 is exact in all the figures except the last, which should be a 6 instead of a 3; the more accurate value of this logarithm being 0.001,505,354,997,317,006,423. This will appear from Mr. Sharp's computations, in the following manner. The logarithm of 17, according to Mr. Sharp's computation of it, is 1.230,448,921,378,273,928,540; therefore the logarithm of 289 (which is the square of 17) will be $= 2 \times 1.230,448,921,378,273,928,540$, or to 2.460,897,842,756,547,857,080. The logarithm of 16 is $= 1.204,119,982,655,924,780,854$; and the logarithm of 18 is $= 1.255,272,505,103,306,069,803$. Therefore the logarithm of 288 (which is $= 16 \times 18$) is $= 1.204,119,982,655,924,780,854 + 1.255,272,505,103,306,069,803 = 2.459,392,487,759,230,850,657$; which being subtracted from the logarithm of 289, which has been shewn to be 2.460,897,842,756,547,857,080, will leave 0.001,505,354,997,317,006,423 for the logarithm of the ratio of 289 to 288.

EXAMPLE VIII.

94. Let it be required to find by means of the same series the logarithm of the ratio of 361 to 360, (or of 361 to $361 - 1$, or of $\frac{361}{361}$ to $\frac{361-1}{361}$), or of 1 to $1 - \frac{1}{361}$

Here m is $= 361$, and $\frac{1}{m}$ is $= \frac{1}{361}$, and $\frac{M}{m}$ is $(= \frac{M}{361} = \frac{0.434,294,481,903,251,830}{361}) = 0.001,203,031,805,826,182$.

We shall therefore have

$$\frac{M}{m^2} (= \frac{M}{m} \times \frac{1}{m} = \frac{M}{m} \times \frac{1}{361} = \frac{0.001,203,031,805,826,182}{361}) = 0.000,003,332,498,077,080;$$

$$\text{And } \frac{M}{m^3} (= \frac{M}{m^2} \times \frac{1}{m} = \frac{M}{m^2} \times \frac{1}{361} = \frac{0.000,003,332,498,077,080}{361}) = 0.000,000,009,231,296,612;$$

$$\text{And } \frac{M}{m^4} (= \frac{M}{m^3} \times \frac{1}{m} = \frac{M}{m^3} \times \frac{1}{361} = \frac{0.000,000,009,231,296,612}{361}) = 0.000,000,000,025,571,458;$$

$$\text{And } \frac{M}{m^5} (= \frac{M}{m^4} \times \frac{1}{m} = \frac{M}{m^4} \times \frac{1}{361} = \frac{0.000,000,000,025,571,458}{361}) = 0.000,000,000,000,070,835;$$

$$\text{And } \frac{M}{m^6} (= \frac{M}{m^5} \times \frac{1}{m} = \frac{M}{m^5} \times \frac{1}{361} = \frac{0.000,000,000,000,070,835}{361}) = 0.000,000,000,000,000,196;$$

$$\text{And } \frac{M}{m^7} (= \frac{M}{m^6} \times \frac{1}{m} = \frac{M}{m^6} \times \frac{1}{361} = \frac{0.000,000,000,000,000,196}{361}) = 0.000,000,000,000,000,000;$$

And consequently

$$\frac{M}{2m^2} (= \frac{M}{m^2} \times \frac{1}{2} = \frac{0.000,003,332,498,077,080}{2}) = 0.000,001,666,249,038,540;$$

$$\text{And } \frac{M}{3m^3} (= \frac{M}{m^3} \times \frac{1}{3} = \frac{0.000,000,009,231,296,612}{3}) = 0.000,000,003,077,098,870;$$

And

$$\text{And } \frac{M}{4m^4} (= \frac{M}{m^4} \times \frac{1}{4} = \frac{0.000,000,000,025,571,458}{4}) = 0.000,000,000,006,392,864;$$

$$\text{And } \frac{M}{5m^5} (= \frac{M}{m^5} \times \frac{1}{5} = \frac{0.000,000,000,000,070,835}{5}) = 0.000,000,000,000,014,167;$$

$$\text{And } \frac{M}{6m^6} (= \frac{M}{m^6} \times \frac{1}{6} = \frac{0.000,000,000,000,000,196}{6}) = 0.000,000,000,000,000,032.$$

Therefore the series $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} + \text{Ec}$ is =

$$\begin{aligned} & 0;001,203,031,805,826,182, \\ & + ; \dots, \dots, 1,666,249,038,540, \\ & + ; \dots, \dots, \dots, 3,077,098,870, \\ & + ; \dots, \dots, \dots, \dots, 6,392,864, \\ & + ; \dots, \dots, \dots, \dots, \dots, 14,167, \\ & + ; \dots, \dots, \dots, \dots, \dots, 32, \\ & = 0.001,204,701,138,370,655. \end{aligned}$$

Therefore this number 0.001,204,701,138,370,655 is Briggs's logarithm of the ratio of 1 to $1 - \frac{1}{361}$, or of 361 to 360. Q. E. I.

This value of the logarithm of the ratio of 361 to 360 is exact in all the figures except the last, which ought to be an 8 instead of a 5; the more accurate value of this logarithm being 0.001,204,701,138,370,658,056. For, by Mr. Sharp's table of logarithms to 61 places of figures, it appears that the logarithm of 19 is 1.278,753,600,952,828,961,536; therefore the logarithm of 361 (which is the square of 19) will be $= 2 \times 1.278,753,600,952,828,961,536 = 2.557,507,201,905,657,923,072$. And the logarithm of 18 is 1.255,272,505,103,306,069,803, and that of 20 is 1.301,029,995,663,981,195,213; and consequently the logarithm of 360 (which is $= 18 \times 20$) will be $= 1.255,272,505,103,306,069,803 + 1.301,029,995,663,981,195,213 = 2.556,302,500,767,287,265,016$; which if we subtract from 2.557,507,201,905,657,923,072, the remainder, to wit, 0.001,204,701,138,370,658,056, will be the logarithm of the ratio of 361 to 360.

EXAMPLE IX.

95. Let it be required to find by means of the same series the logarithm of the ratio of 529 to 528, (or of 529 to $529 - 1$, or of $\frac{529}{529}$ to $\frac{529-1}{529}$), or of 1 to $1 - \frac{1}{529}$.

Here m is $= 529$, and $\frac{1}{m}$ is $= \frac{1}{529}$, and $\frac{M}{m}$ is $(= \frac{M}{529} = \frac{0.434,294,481,903,251,830}{529}) = 0.000,820,972,555,582,706$.

We shall therefore have

$$\frac{M}{m^2} (= \frac{M}{m} \times \frac{I}{m} = \frac{M}{m} \times \frac{I}{529} = \frac{0.000,820,972,555,582,706}{529}) = 0.000,001,551,932,997,320;$$

$$\text{And } \frac{M}{m^3} (= \frac{M}{m^2} \times \frac{I}{m} = \frac{M}{m^2} \times \frac{I}{529} = \frac{0.000,001,551,932,997,320}{529}) = 0.000,000,002,933,710,769;$$

$$\text{And } \frac{M}{m^4} (= \frac{M}{m^3} \times \frac{I}{m} = \frac{M}{m^3} \times \frac{I}{529} = \frac{0.000,000,002,933,710,769}{529}) = 0.000,000,000,005,545,767;$$

$$\text{And } \frac{M}{m^5} (= \frac{M}{m^4} \times \frac{I}{m} = \frac{M}{m^4} \times \frac{I}{529} = \frac{0.000,000,000,005,545,767}{529}) = 0.000,000,000,000,010,483;$$

$$\text{And } \frac{M}{m^6} (= \frac{M}{m^5} \times \frac{I}{m} = \frac{M}{m^5} \times \frac{I}{529} = \frac{0.000,000,000,000,010,483}{529}) = 0.000,000,000,000,000,019;$$

$$\text{And } \frac{M}{m^7} (= \frac{M}{m^6} \times \frac{I}{m} = \frac{M}{m^6} \times \frac{I}{529} = \frac{0.000,000,000,000,000,019}{529}) = 0.000,000,000,000,000,000;$$

And consequently

$$\frac{M}{2m^2} (= \frac{M}{m^2} \times \frac{I}{2} = \frac{0.000,001,551,932,997,320}{2}) = 0.000,000,775,966,498,660;$$

$$\text{And } \frac{M}{3m^3} (= \frac{M}{m^3} \times \frac{I}{3} = \frac{0.000,000,002,933,710,769}{3}) = 0.000,000,000,977,903,589;$$

$$\text{And } \frac{M}{4m^4} (= \frac{M}{m^4} \times \frac{I}{4} = \frac{0.000,000,000,005,545,767}{4}) = 0.000,000,000,001,386,441;$$

$$\text{And } \frac{M}{5m^5} (= \frac{M}{m^5} \times \frac{I}{5} = \frac{0.000,000,000,000,010,483}{5}) = 0.000,000,000,000,002,096;$$

$$\text{And } \frac{M}{6m^6} (= \frac{M}{m^6} \times \frac{I}{6} = \frac{0.000,000,000,000,000,019}{6}) = 0.000,000,000,000,000,003.$$

Therefore the series $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} + \&c$ will be =

$$\begin{aligned} &0.000,820,972,555,582,706, \\ &+ ; \dots, \dots, 775,966,498,660, \\ &+ ; \dots, \dots, \dots, 977,903,589, \\ &+ ; \dots, \dots, \dots, \dots, 1,386,441, \\ &+ ; \dots, \dots, \dots, \dots, \dots, 2,096, \\ &+ ; \dots, \dots, \dots, \dots, \dots, \dots, 3, \\ &= 0.000,821,749,501,373,495. \end{aligned}$$

Therefore this number 0.000,821,749,501,373,495 is Briggs's logarithm of the ratio of 1 to $1 - \frac{I}{529}$, or of 529 to 528. Q. E. I.

This value of the logarithm of the ratio of 529 to 528 is exact in all the figures except the last, which ought to be an 8 instead of a five; the more accurate value of this logarithm being 0.000,821,749,501,373,498,835. For, by Mr. Sharp's table, the logarithm of 23 is = 1.361,727,836,017,592,878,867; therefore the logarithm of 529 (which is the square of 23) will be = $2 \times 1.361,727,836,017,592,878,867$, or to 2.723,455,672,035,185,757,734. And the logarithm of 22 is 1.342,422,680,822,206,235,963, and that of 24 is 1.380,211,241,711,606,022,936. Therefore

fore the logarithm of 528 (which is $= 22 \times 24$) will be $= 1.342,422,680,822,206,235,963 + 1.380,211,241,711,606,022,936 = 2.722,633,922,533,812,258,899$; which if we subtract from 2.723,455,672,035,185,757,734, (which is the logarithm of 529) the remainder, to wit, 0.000,821,749,501,373,498,835, will be the logarithm of the ratio of 529 to 528.

96. It appears therefore from these examples that Briggs's logarithms of the ratios of 10 to 9, 11 to 10, 81 to 80, 121 to 120, 2401 to 2400, 169 to 168, 289 to 288, 361 to 360, and 529 to 528, are as follows, to wit:

$$L. \frac{10}{9} = 0.045,757,490,560,675,118;$$

$$L. \frac{11}{10} = 0.041,392,685,158,225,034;$$

$$L. \frac{81}{80} = 0.005,395,031,886,706,160;$$

$$L. \frac{121}{120} = 0.003,604,124,268,825,251;$$

$$L. \frac{2401}{2400} = 0.000,180,918,345,421,298;$$

$$L. \frac{169}{168} = 0.002,577,422,887,810,682;$$

$$L. \frac{289}{288} = 0.001,505,354,997,317,003;$$

$$L. \frac{361}{360} = 0.001,204,701,138,370,655;$$

$$\text{And } L. \frac{529}{528} = 0.000,821,749,501,373,495.$$

An Application of the foregoing Nine Logarithms of small Ratios in Briggs's System of Logarithms to the Investigation of the Logarithms of the Ratios of the first 23 natural Numbers (2, 3, 4, 5, &c, to 24 inclusively) to 1, in the same System.

97. From these original logarithms, which we have computed by means of the series $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} + \&c \text{ ad infinitum}$, we may deduce the logarithms of the ratios of 2 to 1, 3 to 1, 4 to 1, 5 to 1, 6 to 1, 7 to 1, and of every following number to 1, as far as 24, in the following manner.

$$L. \frac{11}{9} \text{ is } = L. \frac{11}{10} + L. \frac{10}{9} = \left\{ + \begin{matrix} 0.041,392,685,158,225,034 \\ + 0.045,757,490,560,675,118 \end{matrix} \right\} = 0.087,150,175,718,900,152.$$

$$L. \frac{121}{81} \text{ is } = 2 \times L. \frac{11}{9} = 2 \times 0.087,150,175,718,900,152 = 0.174,300,351,437,800,304.$$

$$L. \frac{121}{80} \text{ is } = L. \frac{121}{81} + L. \frac{81}{80} = \left\{ + \begin{matrix} 0.174,300,351,437,800,304 \\ + 0.005,395,031,886,706,160 \end{matrix} \right\} = 0.179,695,383,324,506,464.$$

$$L. \frac{120}{80} \text{ is } = L. \frac{121}{80} - L. \frac{121}{120} = \left\{ - \begin{matrix} 0.179,695,383,324,506,464 \\ - 0.003,604,124,268,825,251 \end{matrix} \right\} = 0.176,091,259,055,681,213.$$

$$L. \frac{3}{2} = L. \frac{120}{80} = 0.176,091,259,055,681,213.$$

$$L. \frac{9}{4} = 2 \times L. \frac{3}{2} = 2 \times 0.176,091,259,055,681,213 = 0.352,182,518,111,362,426.$$

$$L. \frac{81}{16} = 2 \times L. \frac{9}{4} = 2 \times 0.352,182,518,111,362,426 = 0.704,365,036,222,724,852.$$

$$L. \frac{80}{16} \text{ is } = L. \frac{81}{16} - L. \frac{81}{80} = \left\{ - \frac{0.704,365,036,222,724,852}{0.005,395,031,886,706,160} \right\} = 0.698,970,004,336,018,692.$$

$$L. \frac{5}{1} = L. \frac{80}{16} = 0.698,970,004,336,018,692.$$

Q. E. I.

This number 0.698,970,004,336,018,692 is exact in all the figures except the three last, which ought to be 804 instead of 692; the more accurate value of the logarithm of 5, in Briggs's system, being, according to Mr. Sharp's computation, 0.698,970,004,336,018,804,786.

$$L. \frac{10}{5} \text{ is } = L. \frac{10}{1} - L. \frac{5}{1} = \left\{ - \frac{1.000,000,000,000,000,000}{0.698,970,004,336,018,692} \right\} = 0.301,029,995,663,981,308.$$

$$L. \frac{2}{1} \text{ is } = L. \frac{10}{5} = 0.301,029,995,663,981,308.$$

Q. E. I.

This number 0.301,029,995,663,981,308 is exact in all but the three last figures; the more accurate value of the logarithm of 2, according to Mr. Sharp's computation, being 0.301,029,995,663,981,195,213.

$$L. \frac{4}{1} \text{ is } = 2 \times L. \frac{2}{1} = 2 \times 0.301,029,995,663,981,308 = 0.602,059,991,327,962,616.$$

Q. E. I.

$$L. \frac{8}{1} \text{ is } = 3 \times L. \frac{2}{1} = 3 \times 0.301,029,995,663,981,308 = 0.903,089,986,991,943,924.$$

Q. E. I.

$$L. \frac{16}{1} \text{ is } = 2 \times L. \frac{4}{1} = 2 \times 0.602,059,991,327,962,616 = 1.204,119,982,655,925,232.$$

Q. E. I.

$$L. \frac{20}{1} \text{ is } = L. \frac{20}{10} = L. \frac{10}{1} = L. \frac{2}{1} + L. \frac{10}{1} = \left\{ + \frac{0.301,029,995,663,981,308}{1.000,000,000,000,000,000} \right\} = 1.301,029,995,663,981,308.$$

Q. E. I.

$$L. \frac{9}{1} \text{ is } = L. \frac{10}{1} - L. \frac{10}{9} = \left\{ - \frac{1.000,000,000,000,000,000}{0.045,757,490,560,675,118} \right\} = 0.954,242,509,439,324,882.$$

Q. E. I.

$$L. \frac{3}{1} \text{ is } = \frac{1}{2} \times L. \frac{9}{1} = \frac{0.954,242,509,439,324,882}{2} = 0.477,121,254,719,662,441.$$

Q. E. I.

$$L. \frac{6}{1} \text{ is } = L. \frac{6}{3} + L. \frac{3}{1} = L. \frac{2}{1} + L. \frac{3}{1} = \left\{ + \frac{0.301,029,995,663,981,308}{0.477,121,254,719,662,441} \right\} = 0.778,151,250,383,643,749.$$

Q. E. I.

$$L. \frac{18}{1} \text{ is } = L. \frac{18}{9} + L. \frac{9}{1} = L. \frac{2}{1} + L. \frac{9}{1} = \left\{ + \frac{0.301,029,995,663,981,308}{0.954,242,509,439,324,882} \right\} = 1.255,272,505,103,306,190.$$

Q. E. I.

$$L. \frac{15}{1} \text{ is } = L. \frac{15}{5} + L. \frac{5}{1} = L. \frac{3}{1} + L. \frac{5}{1} = \left\{ + \frac{0.477,121,254,719,662,441}{0.698,970,004,336,018,692} \right\} = 1.176,091,259,055,681,133.$$

Q. E. I.

$$L. \frac{12}{1} \text{ is } = L. \frac{12}{6} + L. \frac{6}{1} = L. \frac{2}{1} + L. \frac{6}{1} = \left\{ + \begin{matrix} 0.301,029,995,663,981,308 \\ 0.778,151,250,383,643,749 \end{matrix} \right\} = 1.079,181,246,047,625,057.$$

Q. E. I.

$$L. \frac{24}{1} \text{ is } = L. \frac{24}{12} + L. \frac{12}{1} = L. \frac{2}{1} + L. \frac{12}{1} = \left\{ + \begin{matrix} 0.301,029,995,663,981,308 \\ 1.079,181,246,047,625,057 \end{matrix} \right\} = 1.380,211,241,711,606,365.$$

Q. E. I.

$$L. \frac{2400}{1} \text{ is } = L. \frac{2400}{100} + L. \frac{100}{1} = L. \frac{24}{1} + 2 \times L. \frac{10}{1} = L. \frac{24}{1} + 2 = \left\{ + \begin{matrix} 1.380,211,241,711,606,365 \\ 2.000,000,000,000,000,000 \end{matrix} \right\} = 3.380,211,241,711,606,365.$$

$$L. \frac{2401}{1} \text{ is } = L. \frac{2401}{2400} + L. \frac{2400}{1} = \left\{ + \begin{matrix} 0.000,180,918,345,421,298 \\ 3.380,211,241,711,606,365 \end{matrix} \right\} = 3.380,392,160,057,027,663.$$

$$L. \frac{7}{1} \text{ is } = \frac{1}{4} \times L. \frac{2401}{1} = \frac{3.380,392,160,057,027,663}{4} = 0.845,098,040,014,256,915. \quad \text{Q. E. I.}$$

$$L. \frac{14}{1} \text{ is } = L. \frac{14}{7} + L. \frac{7}{1} = L. \frac{2}{1} + L. \frac{7}{1} = \left\{ + \begin{matrix} 0.301,029,995,663,981,308 \\ 0.845,098,040,014,256,915 \end{matrix} \right\} = 1.146,128,035,678,238,223.$$

Q. E. I.

$$L. \frac{21}{1} \text{ is } = L. \frac{21}{7} + L. \frac{7}{1} = L. \frac{3}{1} + L. \frac{7}{1} = \left\{ + \begin{matrix} 0.477,121,254,719,662,441 \\ 0.845,098,040,014,256,915 \end{matrix} \right\} = 1.322,219,294,733,919,356.$$

Q. E. I.

$$L. \frac{11}{1} \text{ is } = L. \frac{11}{10} + L. \frac{10}{1} = \left\{ + \begin{matrix} 0.041,392,685,158,225,034 \\ 1.000,000,000,000,000,000 \end{matrix} \right\} = 1.041,392,685,158,225,034. \quad \text{Q. E. I.}$$

$$L. \frac{22}{1} \text{ is } = L. \frac{22}{11} + L. \frac{11}{1} = L. \frac{2}{1} + L. \frac{11}{1} = \left\{ + \begin{matrix} 0.301,029,995,663,981,308 \\ 1.041,392,685,158,225,034 \end{matrix} \right\} = 1.342,422,680,822,206,342.$$

Q. E. I.

$$L. \frac{168}{1} \text{ is } = L. \frac{168}{14} + L. \frac{14}{1} = L. \frac{12}{1} + L. \frac{14}{1} = \left\{ + \begin{matrix} 1.079,181,246,047,625,057 \\ 1.146,128,035,678,238,223 \end{matrix} \right\} = 2.225,309,281,725,863,280.$$

$$L. \frac{169}{1} \text{ is } = L. \frac{169}{168} + L. \frac{168}{1} = \left\{ + \begin{matrix} 0.002,577,422,887,810,682 \\ 2.225,309,281,725,863,280 \end{matrix} \right\} = 2.227,886,704,613,673,962.$$

$$L. \frac{13}{1} \text{ is } = \frac{1}{2} \times L. \frac{169}{1} = \frac{1}{2} \times 2.227,886,704,613,673,962 = 1.113,943,352,306,836,981.$$

Q. E. I.

$$L. \frac{288}{1} \text{ is } = L. \frac{288}{18} + L. \frac{18}{1} = L. \frac{16}{1} + L. \frac{18}{1} = \left\{ + \begin{matrix} 1.204,119,982,655,925,232 \\ 1.255,272,505,103,306,190 \end{matrix} \right\} = 2.459,392,487,759,231,422.$$

$$L. \frac{289}{1} \text{ is } = L. \frac{289}{288} + L. \frac{288}{1} = \left\{ + \begin{matrix} 0.001,505,354,997,317,003 \\ 2.459,392,487,759,231,422 \end{matrix} \right\} = 2.460,897,842,756,548,425.$$

$$L. \frac{17}{1} \text{ is } = \frac{1}{2} \times L. \frac{289}{1} = \frac{2.460,897,842,756,548,425}{2} = 1.230,448,921,378,274,212. \quad \text{Q. E. I.}$$

$$L. \frac{360}{1} \text{ is } = L. \frac{360}{20} + L. \frac{20}{1} = L. \frac{18}{1} + L. \frac{20}{1} = \left\{ + \begin{matrix} 1.255,272,505,103,306,190 \\ 1.301,029,995,663,981,308 \end{matrix} \right\} = 2.556,302,500,767,287,498$$

$$L. \frac{361}{1} \text{ is } = L. \frac{361}{360} + L. \frac{360}{1} = \left\{ + \begin{matrix} 0.001,204,701,138,370,655 \\ 2.556,302,500,767,287,498 \end{matrix} \right\} = 2.557,507,201,905,658,153.$$

$$L. \frac{19}{1} \text{ is } = \frac{1}{2} \times L. \frac{361}{1} = \frac{2.557,507,201,905,658,153}{2} = 1.278,753,600,952,829,076. \quad \text{Q. E. I.}$$

$$L. \frac{528}{1} \text{ is } = L. \frac{528}{24} + L. \frac{24}{1} = L. \frac{22}{1} + L. \frac{24}{1} = \left\{ + \frac{1.342,422,680,822,206,342}{1.380,211,241,711,606,365} \right\} = 2.722,633,922,533,812,707.$$

$$L. \frac{529}{1} \text{ is } = L. \frac{529}{528} + L. \frac{528}{1} = \left\{ + \frac{0.000,821,749,501,373,495}{2.722,633,922,533,812,707} \right\} = 2.723,455,672,035,186,202.$$

$$L. \frac{23}{1} \text{ is } = \frac{1}{2} \times L. \frac{529}{1} = \frac{2.723,455,672,035,186,202}{2} = 1.361,727,836,017,593,101. \quad Q. E. I.$$

98. These logarithms, if ranged in their proper order, will be as follows :

$$L. \frac{2}{1} = 0.301,029,995,663,981,308 ;$$

$$L. \frac{3}{1} = 0.477,121,254,719,662,441 ;$$

$$L. \frac{4}{1} = 0.602,059,991,327,962,616 ;$$

$$L. \frac{5}{1} = 0.698,970,004,336,018,692 ;$$

$$L. \frac{6}{1} = 0.778,151,250,383,643,749 ;$$

$$L. \frac{7}{1} = 0.845,098,040,014,256,915 ;$$

$$L. \frac{8}{1} = 0.903,089,986,991,943,924 ;$$

$$L. \frac{9}{1} = 0.954,242,509,439,324,882 ;$$

$$L. \frac{10}{1} = 1.000,000,000,000,000,000 ;$$

$$L. \frac{11}{1} = 1.041,392,685,158,225,034 ;$$

$$L. \frac{12}{1} = 1.079,181,246,047,625,057 ;$$

$$L. \frac{13}{1} = 1.113,943,352,306,836,981 ;$$

$$L. \frac{14}{1} = 1.146,128,035,678,238,223 ;$$

$$L. \frac{15}{1} = 1.176,091,259,055,681,133 ;$$

$$L. \frac{16}{1} = 1.204,119,982,655,925,232 ;$$

$$L. \frac{17}{1} = 1.230,448,921,378,274,212 ;$$

$$L. \frac{18}{1} = 1.255,272,505,103,306,190 ;$$

$$L. \frac{19}{1} = 1.278,753,600,952,829,076 ;$$

$$L. \frac{20}{1} = 1.301,029,995,663,981,308 ;$$

$$L. \frac{21}{1} = 1.322,219,294,733,919,356 ;$$

$$L. \frac{22}{1} = 1.342,422,680,822,206,342 ;$$

$$L. \frac{23}{1} = 1.361,727,836,017,593,101 ;$$

$$\text{And } L. \frac{24}{1} = 1.380,211,241,711,606,365.$$

These values of the foregoing logarithms are exact in, at least, the first fourteen places of decimal fractions, and much the greater part of them are exact also in the fifteenth place ; their more accurate values, according to Mr. Sharp's computation, being as follows :

$$L. \frac{2}{1} = 0.301,029,995,663,981,195,213 ;$$

$$L. \frac{3}{1} = 0.477,121,254,719,662,437,295 ;$$

$$L. \frac{4}{1} = 0.602,059,991,327,962,390,427 ;$$

$$L. \frac{5}{1} = 0.698,970,004,336,018,804,786 ;$$

$$L. \frac{6}{1} = 0.778,151,250,383,643,632,508 ;$$

$$L. \frac{7}{1} = 0.845,098,040,014,256,830,712 ;$$

$$L. \frac{8}{1} = 0.903,089,986,991,943,585,641 ;$$

$$L. \frac{9}{1} = 0.954,242,509,439,324,874,590 ;$$

$$L. \frac{10}{1} = 1.000,000,000,000,000,000,000 ;$$

$$L. \frac{11}{1} = 1.041,392,685,158,225,040,750 ;$$

$$L. \frac{12}{1} = 1.079,181,246,047,624,827,722 ;$$

$$L. \frac{13}{1} = 1.113,943,352,306,836,769,206 ;$$

$$L. \frac{14}{1} = 1.146,128,035,678,238,025,925 ;$$

$$L. \frac{15}{1} = 1.176,091,259,055,681,242,081 ;$$

$$L. \frac{16}{1} = 1.204,119,982,655,924,780,854 ;$$

$$L. \frac{17}{1} = 1.230,448,921,378,273,928,540;$$

$$L. \frac{18}{1} = 1.255,272,505,103,306,069,803;$$

$$L. \frac{19}{1} = 1.278,753,600,952,828,961,536;$$

$$L. \frac{20}{1} = 1.301,029,995,663,981,195,213;$$

$$L. \frac{21}{1} = 1.322,219,294,733,919,268,007;$$

$$L. \frac{22}{1} = 1.342,422,680,822,206,235,963;$$

$$L. \frac{23}{1} = 1.361,727,836,017,592,878,867;$$

$$L. \frac{24}{1} = 1.380,211,241,711,606,022,936.$$

99. If we were to pursue this matter further, and to compute the logarithms of the ratios of numbers greater than 24 to 1, by means of the aforefaid serieses of Mercator and Wallis, the labour of the computation would continually grow less and less as the numbers increased. For, as to the logarithms of the ratios of all composite numbers (or numbers formed by the multiplication of other numbers) to 1, they, it is evident, might be found by merely adding together the logarithms of the ratios of the factors of which they are composed to 1. Thus, for example, we should have $L. \frac{25}{1} (= L. \frac{25}{5} + L. \frac{5}{1} = L. \frac{5}{1} + L. \frac{5}{1}) = 2 L. \frac{5}{1}$; and $L. \frac{26}{1} (= L. \frac{26}{13} + L. \frac{13}{1}) = L. \frac{2}{1} + L. \frac{13}{1}$; and $L. \frac{27}{1} (= L. \frac{27}{9} + L. \frac{9}{3} + L. \frac{3}{1} = L. \frac{3}{1} + L. \frac{3}{1} + L. \frac{3}{1}) = 3 L. \frac{3}{1}$; and $L. \frac{28}{1} (= L. \frac{28}{14} + L. \frac{14}{1}) = L. \frac{2}{1} + L. \frac{14}{1}$. Therefore the only logarithms for the computation of which it would be necessary to have recourse to one of the foregoing infinite serieses, would be those of the ratios of the following prime numbers, such as 29, 31, 37, &c, to 1. And, in computing these logarithms by means of the said serieses, it is evident that the labour of the computation must be continually less and less as the number becomes greater and greater, because the terms of the series will converge with a greater degree of swiftness as the number to which it relates increases. This will appear more clearly by reviewing the manner in which we found the logarithms of the ratios of the prime numbers 13, 17, 19, and 23, to 1, which was as follows.

A Review of the foregoing Manner of computing Logarithms.

100. We did not (as we might have done) compute by the serieses the logarithms of the ratios of 13 to 12, of 17 to 16, of 19 to 18, and of 23 to 22, but, in

in order to obtain serieses that would converge more swiftly, we squared these four prime numbers 13, 17, 19, and 23, whereby we obtained the numbers 169, 289, 361, and 529, and then we computed, by means of the serieses, the logarithms of the ratios of 169 to 168, of 289 to 288, of 361 to 360, and of 529 to 528, and added the logarithms so found to the logarithms of the ratios of 168 to 1, of 288 to 1, of 360 to 1, and of 528 to 1, which were derived from the logarithms already known by adding together the logarithms of the ratios of the factors of those numbers, to wit, the factors 12 and 14, or $13 - 1$ and $13 + 1$, 16 and 18, or $17 - 1$ and $17 + 1$, 18 and 20, or $19 - 1$ and $19 + 1$, and 22 and 24, or $23 - 1$, and $23 + 1$, to 1; and when we had thus found the logarithms of the ratios of 169, or 13^2 , and 289, or 17^2 , and 361, or 19^2 , and 529, or 23^2 , to 1, we took the halves of those logarithms, and thereby obtained the logarithms of the ratios of those prime numbers 13, 17, 19, and 23 themselves, to 1.

Now, if this method be pursued with respect to any greater prime number, after the logarithms of the ratios of all the lesser prime numbers to 1 have been computed, we must proceed as follows.

Let the prime number, of the ratio of which to 1 we want to find the logarithm, be called p .

Then, in the first place, it is evident that, since p is a prime number, and consequently an odd number, $p - 1$ and $p + 1$ must, both of them, be even numbers, or divisible by 2.

Secondly, since the logarithms of the ratios of all the prime numbers that are less than p to 1, are supposed to be already known, the logarithms of the ratios of the prime numbers that are factors of the composite numbers $p - 1$ and $p + 1$ to 1, must be already known; and consequently the logarithms of the ratios of the said composite numbers $p - 1$ and $p + 1$ to 1, may be derived from the logarithms already known, by mere addition.

Thirdly, the number $pp - 1$ is equal to $(p - 1) \times (p + 1)$, and consequently the logarithm of the ratio of $pp - 1$ to 1 will be equal to the sum of the logarithms of the ratios of $p - 1$ to 1, and of $p + 1$ to 1.

We must therefore find, by addition of the logarithms already known, the logarithms of the ratios of $p - 1$ to 1 and of $p + 1$ to 1, and add them together into one sum; whereby we shall obtain the logarithm of the ratio of $pp - 1$ to 1. We must then find, by one of the foregoing serieses, the logarithm of the ratio of pp to $pp - 1$, and add it to the logarithm of the ratio of $pp - 1$ to 1. The sum will be the logarithm of the ratio of pp to 1; and half this logarithm will be the logarithm of the ratio of the prime number p itself to 1. Q. E. I.

Now, in computing by one of the foregoing serieses the logarithm of the ratio of pp to $pp - 1$, it is evident that the greater the number pp is, and consequently the greater the number p is, the greater will be the swiftness with which the terms of the said logarithmick series will converge. There will therefore be less trouble in computing the logarithm of the ratios of 29, 31, 37, 41, 43, and every following prime number, to 1, than there was in computing the logarithm of the ratio of 23 to 1; which yet, we have seen, was not very great, since only six terms of the

infinite

Infinite series $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} + \mathcal{E}c$ were sufficient to enable us to find it exact to fifteen places of figures. In computing the logarithms of the ratios of 29, 31, and 37, to 1, by the method just described, or by first raising the squares of these prime numbers, to wit, 841, 961, and 1369, and finding by the said infinite series the logarithms of the ratios of those squares to the next lesser numbers 840, 960, and 1368, only five terms of the said series will be sufficient to give the logarithms sought to the same degree of exactness; and for the logarithms of the ratios of prime numbers greater than 100 to 1, only four terms of the said series will be sufficient for the same purpose.

Of the great Expedition of the foregoing Series in computing the Logarithms of the Ratios of prime Numbers greater than 1100 to 1.

101. Mr. Abraham Sharp has already computed the logarithms of the ratios of all prime numbers under 1100 to 1 to no less than 61 places of figures; and they are published with Sherwin's Tables of Logarithms, in the third edition of that useful book, published in the year 1741, in pages 36, 37, 38, and 39. These may therefore be made use of in finding the logarithm of the ratio of any prime number greater than 1100 to 1; as, for example, in finding the logarithm of the ratio of 1103 to 1. Now the square of 1103 is 1,216,609; by which, if we divide M, or 0.434,294,481,903,251,830, the quotient will be = 0.000,000,356,971, $\mathcal{E}c$; that is, according to the notation used above, m will be = 1,216,609, and $\frac{M}{m} = 0.000,000,356,971, \mathcal{E}c$. Therefore $\frac{M}{m^2}$ will be = $\frac{M}{m} \times \frac{1}{m} = 0.000,000,356,971, \mathcal{E}c \times \frac{1}{1,216,609} = \frac{0.000,000,356,971, \mathcal{E}c}{1,216,609}$, the quotient of which division will appear only in the 13th place of decimal fractions; and $\frac{M}{m^3}$ will be = $\frac{M}{m^2} \times \frac{1}{m} = \frac{M}{m^2} \times \frac{1}{1,216,609}$, the quotient of which division will appear only in the 19th place of decimal fractions. Therefore, if the logarithm of the ratio of 1,216,609 to 1, be sought only to 18 places of decimal fractions, (which is the number of places to which the logarithms found in the foregoing nine examples are carried,) we need only compute the two first terms $\frac{M}{m}$ and $\frac{M}{2m^2}$ of the infinite series $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} + \mathcal{E}c$. And therefore to find the logarithm of the ratio of the prime number 1103 to 1, and, *à fortiori*, to find the logarithm of the ratio of any prime number greater than 1103 to 1, to 18 places of figures, it will be sufficient to make use of only the two first terms of the series $\frac{M}{m} + \frac{M}{2m^2} + \frac{M}{3m^3} + \frac{M}{4m^4} + \frac{M}{5m^5} + \frac{M}{6m^6} + \mathcal{E}c$. An easier method of computing such a logarithm seems hardly to be expected, and needs not much to be desired.

End of the Review of the foregoing Method of computing Logarithms.

102. I have now, I hope, sufficiently explained the methods of applying these two infinite serieses $\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \frac{1}{4m^4} + \frac{1}{5m^5} - \frac{1}{6m^6} + \mathcal{E}c$, and

X x

$\frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + \frac{1}{4m^4} + \frac{1}{5m^5} + \frac{1}{6m^6} + \mathcal{E}c$ (which were invented by Mr. Mercator and Dr. Wallis) to the computation both of Napier's and Briggs's logarithms; by which their use and excellency have appeared most manifestly. I should therefore here put an end to these remarks, were it not that the abstruseness and subtlety of the reasonings contained in the demonstrations of the two principal theorems in article 18 and article 66 are so great, (notwithstanding the endeavours I have there used to facilitate them,) that I imagine a further attempt to state them clearly will not be unacceptable to the reader. Such an attempt I shall therefore now proceed to make.

A further Attempt to illustrate and facilitate the Demonstration of Theorem 1 in Article 18.

103. If there be two given ratios, that of E to F, and that of G to H, and a third ratio, to wit, that of x to y, that is continually varying while its terms x and y vary in their magnitude; and the said varying ratio of x to y continually approaches to each of the given ratios of E to F and of G to H, as to its limit, so that it may be made to come as near as we please to each of the said given ratios, though it can never be absolutely equal to either of them: I say, that, upon this supposition, the ratio of G to H must be equal to the ratio of E to F. This proposition I take to be so evident as not to stand in need of a demonstration. But, if it were thought to need a demonstration, it might, without much difficulty, be demonstrated *ex absurdo* after the manner of the ancients, by supposing the ratios of E to F and of G to H not to be equal to each other, and then shewing the contradictory consequences that would follow from such a supposition. But this, as it appears to me unnecessary, I shall omit.

104. To this proposition we may bring the demonstration contained in article 18, by proceeding in the manner following.

Let k and q be any two quantities less than 1, whereof k is the greater; such, for example, as $\frac{9}{10}$ and $\frac{5}{10}$. We are to shew, that the ratio of $1 + k$ to 1 will be to the ratio of $1 + q$ to 1 in the same proportion as the infinite series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c$ to the infinite series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \mathcal{E}c$, or that the ratio, or proportion, which the ratio of $1 + k$ to 1 bears to the ratio of $1 + q$ to 1, is equal to the ratio of the former of these two infinite serieses to the latter. Now this may be done by shewing that both these ratios (which are evidently fixt or given ratios) are the limits of a certain varying ratio, which we shall now proceed to describe and consider.

Let us suppose the ratio of $1 + k$ to 1 to be first bisected, or divided into two equal ratios, by extracting the square root of $1 + k$, or inserting a geometrical mean proportional between 1 and $1 + k$; and let the ratio of $1 + q$ to 1 be bisected in like manner, by extracting the square-root of $1 + q$, or inserting a geometrical mean proportional between 1 and $1 + q$. Let us then suppose the

the same two ratios of $1 + k$ to 1 and $1 + q$ to 1 to be quadrifected, or divided into four equal ratios, by the insertion of three geometrical mean proportionals between 1 and $1 + k$, and three like mean proportionals between 1 and $1 + q$. And then let us suppose the same ratios to be divided into eight equal ratios by the insertion of seven geometrical mean proportionals between their terms, and into sixteen equal ratios, and into thirty-two equal ratios, and into a greater and greater number of equal ratios continually; every new number of ratios into which they are divided being double of the number in the division next preceding.

Then it is evident, that, when these two ratios are divided into only two equal ratios by the insertion of only one geometrical mean proportional between 1 and $1 + k$, and one geometrical mean proportional between 1 and $1 + q$, these geometrical mean proportionals will be $\sqrt{1 + k}$ and $\sqrt{1 + q}$, or $1 + k^{\frac{1}{2}}$ and $1 + q^{\frac{1}{2}}$; and when they are divided into four equal ratios by the insertion of three geometrical mean proportionals between 1 and $1 + k$ and three geometrical mean proportionals between 1 and $1 + q$, the first, or least, of the first set of mean proportionals will be $\sqrt[4]{1 + k}$, or $1 + k^{\frac{1}{4}}$, and the first, or least, of the second set of mean proportionals will be $\sqrt[4]{1 + q}$, or $1 + q^{\frac{1}{4}}$; and when they are divided into eight equal ratios by the insertion of seven geometrical mean proportionals between 1 and $1 + k$ and seven such mean proportionals between 1 and $1 + q$, the first, or least, of the first set of mean proportionals will be $\sqrt[8]{1 + k}$, or $1 + k^{\frac{1}{8}}$, and the first, or least, of the second set of mean proportionals will be $\sqrt[8]{1 + q}$, or $1 + q^{\frac{1}{8}}$; and, in general, if n be the number of lesser ratios into which each of the said two ratios of $1 + k$ to 1 and $1 + q$ to 1 is divided by the insertion of $n - 1$ geometrical mean proportionals, the first, or least, of the first set of such mean proportionals will be $\sqrt[n]{1 + k}$, or $1 + k^{\frac{1}{n}}$, and the first, or least, of the second set of such mean proportionals will be $\sqrt[n]{1 + q}$, or $1 + q^{\frac{1}{n}}$.

Now during all these successive divisions of the ratios of $1 + k$ to 1 and of $1 + q$ to 1 into two, four, eight, sixteen, and the following greater numbers of equal parts, the proportion of the ratio of the first, or least, of the first set of mean proportionals (which are inserted between 1 and $1 + k$) to 1 to the ratio of the first, or least, of the second set of mean proportionals (which are inserted between 1 and $1 + q$) to 1 , will always be the same, whatever be the number of parts, or lesser ratios, into which the said ratios of $1 + k$ to 1 and $1 + q$ to 1 be supposed to be divided, and will be the same with the proportion of the ratio of $1 + k$ to 1 to the ratio of $1 + q$ to 1 . Thus, if these ratios are divided into only two parts, or lesser ratios, by the insertion of the mean proportionals $1 + k^{\frac{1}{2}}$ and $1 + q^{\frac{1}{2}}$, the proportion of the ratio of $1 + k^{\frac{1}{2}}$ to 1 to the ratio of $1 + q^{\frac{1}{2}}$ to 1 , will be the same with the proportion of the ratio of $1 + k$ itself to 1 to the ratio of $1 + q$ itself to 1 . For the two former ratios, to wit, the ratios of $1 + k^{\frac{1}{2}}$ to 1 and of $1 + q^{\frac{1}{2}}$ to 1 , are the halves of the two latter ratios, or the ratios of $1 + k$ to 1 and $1 + q$ to 1 ; and therefore the proportion of the two former ratios must be the same as that of

the two latter. And, in like manner, if the ratios of $1 + k$ to 1 and $1 + q$ to 1 are divided into four equal parts, or lesser ratios, by the insertion of three geometrical mean proportionals between 1 and $1 + k$ and of three like mean proportionals between 1 and $1 + q$, the ratio of $\sqrt[4]{1 + k}$ (the first, or least, mean proportional of the first set) to 1 will be to the ratio of $\sqrt[4]{1 + q}$ (the first, or least, mean proportional of the second set) to 1 in the same proportion as the ratio of $1 + k$ to 1 is to the ratio of $1 + q$ to 1 ; because the ratio of $\sqrt[4]{1 + k}$ to 1 is a fourth part of the ratio of $1 + k$ to 1 , and the ratio of $\sqrt[4]{1 + q}$ to 1 is a fourth part of the ratio of $1 + q$ to 1 , and the like parts of ratios (as well as of all other quantities) must be to each other in the same proportion as the whole quantities themselves. And, for the same reason, if each of the two ratios of $1 + k$ to 1 and of $1 + q$ to 1 be divided into any other and greater number of parts, or lesser ratios, denoted by n , by the insertion of $n - 1$ mean proportionals between 1 and $1 + k$ and the same number of mean proportionals between 1 and $1 + q$, the ratio of $\sqrt[n]{1 + k}$ (the first, or least, of the first set of mean proportionals) to 1 will be to the ratio of $\sqrt[n]{1 + q}$ (the first, or least, of the second set of mean proportionals) to 1 in the same proportion as the ratio of $1 + k$ to 1 is to the ratio of $1 + q$ to 1 . So that, while the number n increases successively from 2 to 4, 8, 16, 32, 64, &c, *ad infinitum*, the proportion of the ratio of $\sqrt[n]{1 + k}$ to 1 to the ratio of $\sqrt[n]{1 + q}$ to 1 remains always the same, and is equal to the proportion of the ratio of $1 + k$ to 1 to the ratio of $1 + q$ to 1 .

But, though the proportion of the ratio of $\sqrt[n]{1 + k}$ to 1 to the ratio of $\sqrt[n]{1 + q}$ to 1 continues always the same during all the increase of the number n from 2 to 4, 8, 16, 32, 64, &c, *ad infinitum*, the proportion of $\sqrt[n]{1 + k} - 1$, or the excess of the first, or least, of the first set of mean proportionals above 1 , to $\sqrt[n]{1 + q} - 1$, or the excess of the first, or least, of the second set of mean proportionals above 1 , does not continue always the same, but is continually varying. Thus, for example, if k be $= \frac{9}{10}$ and q be $= \frac{5}{10}$, and consequently $1 + k$ be $= 1 + \frac{9}{10}$, or 1.9 , and $1 + q$ be $= 1 + \frac{5}{10}$, or 1.5 , and n be supposed to be successively equal to 2, 4, and 8, we shall have $\sqrt[n]{1 + k}$ successively equal to $\sqrt[2]{1.9}$, $\sqrt[4]{1.9}$, and $\sqrt[8]{1.9}$, that is, to $1.378,404$, $1.174,054$, and $1.083,538$, and $\sqrt[n]{1 + q}$ successively equal to $\sqrt[2]{1.5}$, $\sqrt[4]{1.5}$, and $\sqrt[8]{1.5}$, that is, to $1.224,744$, $1.106,681$, and $1.051,989$. Therefore $\sqrt[n]{1 + k} - 1$ will be successively equal to $0.378,404$, $0.174,054$, and $0.083,538$, and $\sqrt[n]{1 + q} - 1$ will be successively equal to $0.224,744$, $0.106,681$, and $0.051,989$. Now, if we compare together the ratios of $\frac{9}{10}$ to $\frac{5}{10}$, or $0.900,000$ to $0.500,000$, or k and q , or $\sqrt[2]{1 + k} - 1$ and $\sqrt[2]{1 + q} - 1$, and of $0.378,404$ to $0.224,744$, or $\sqrt[4]{1 + k} - 1$ to $\sqrt[4]{1 + q} - 1$, and of $0.174,054$ to $0.106,681$, or $\sqrt[8]{1 + k} - 1$ to $\sqrt[8]{1 + q} - 1$, and of $0.083,538$ to $0.051,989$, or $\sqrt[n]{1 + k} - 1$ to $\sqrt[n]{1 + q} - 1$, we shall find that the first of these ratios is greater than the second, and the second than the

the third, and the third than the fourth; or that the ratio of $\sqrt[n]{1+k} - 1$ to $\sqrt[n]{1+q} - 1$ decreases continually while the number n increases. For, in the first place, the ratio of 0.900,000 to 0.500,000 is equal to that of 0.378,404 to 0.210,224, and therefore is greater than that of 0.378,404 to 0.224,744; and, in the second place, the ratio of 0.378,404 to 0.224,744 is equal to the ratio of 0.174,054 to 0.103,375, and therefore is greater than that of 0.174,054 to 0.106,681; and, lastly, the ratio of 0.174,054 to 0.106,681 is equal to the ratio of 0.083,538 to 0.051,202, and therefore is greater than that of 0.083,538 to 0.051,989. Or, if we make 1 the common antecedent of these four ratios (by which their decrease while the number n increases will become more apparent), the first of these ratios, to wit, that of 0.900,000 to 0.500,000, will be equal to that of 1 to 0.555,555; the second, to wit, that of 0.378,404 to 0.224,744, will be equal to that of 1 to 0.593,926; the third, to wit, that of 0.174,054 to 0.106,681, will be equal to that of 1 to 0.612,918; and the fourth, to wit, that of 0.083,538 to 0.051,989, will be equal to that of 1 to 0.622,339; so that the said four ratios are equal to the four ratios following, to wit,

That of 1 to 0.555,555,

That of 1 to 0.593,926,

That of 1 to 0.612,918,

And That of 1 to 0.622,339;

of which it is evident, the first is greater than the second, the second than the third, and the third than the fourth.

It appears therefore that the ratio of $\sqrt[n]{1+k} - 1$ to $\sqrt[n]{1+q} - 1$ decreases continually while the number n increases.

But, though this ratio decreases continually while the number n increases, it does not decrease *ad infinitum* so as to become at last equal to nothing, or to a ratio of equality (which is a ratio of no magnitude at all), but approaches only to a certain ratio of majority as its limit. And this ratio of majority, to which it so approaches as its limit, and to which it may, by increasing the number n , be made to come as near as we please, is shewn in lemma 1, article 11, to be the ratio, or proportion, of the ratio of $\sqrt[n]{1+k}$ to 1 to the ratio of $\sqrt[n]{1+q}$ to 1, or the proportion of the n^{th} part of the ratio of $1+k$ to 1 to the n^{th} part of the ratio of $1+q$ to 1, and consequently is equal to the proportion of the whole ratio of $1+k$ to 1, to the whole ratio of $1+q$ to 1; that is, the first ratio, or proportion, of the ratio of $1+k$ to 1 to the ratio of $1+q$ to 1, is the limit of the varying ratio of $\sqrt[n]{1+k} - 1$ to $\sqrt[n]{1+q} - 1$.

But, by lemma 2, coroll. 2, it appears, that, by increasing the number n , the ratio of the quantity $\sqrt[n]{1+k} - 1$ to $\frac{1}{n} \times$ the infinite series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c$ may be made to come as near as we please to a ratio of equality, and the ratio of the quantity $\sqrt[n]{1+q} - 1$ to $\frac{1}{n} \times$ the infinite series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \mathcal{E}c$ may likewise be made to come as near as we please to a ratio of equality. Now, if these two ratios were absolutely ratios of equality,

it would follow that the ratio of $\sqrt[n]{1+k} - 1$ to $\sqrt[n]{1+q} - 1$ would be equal to the ratio of $\frac{1}{n} \times$ the infinite series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c$ to $\frac{1}{n} \times$ the infinite series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \mathcal{E}c$, and consequently to the ratio of n times $\frac{1}{n} \times$ the infinite series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c$ to n times $\frac{1}{n} \times$ the infinite series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \mathcal{E}c$, or to the ratio of the whole series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c$ *ad infinitum* to the whole series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \mathcal{E}c$ *ad infinitum*. Therefore, while the number n increases, the ratio of $\sqrt[n]{1+k} - 1$ to $\sqrt[n]{1+q} - 1$ continually approaches to the ratio of the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c$ *ad infinitum* to the series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \mathcal{E}c$ *ad infinitum*, and may be made to come as near to it as we please; or, in other words, the fixt ratio of the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c$ *ad infinitum* to the series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \mathcal{E}c$ *ad infinitum* is the limit of the varying ratio of $\sqrt[n]{1+k} - 1$ to $\sqrt[n]{1+q} - 1$.

But we before shewed that the fixt ratio, or proportion, of the ratio of $1+k$ to 1 to the ratio of $1+q$ to 1 was also the limit of the varying ratio of $\sqrt[n]{1+k} - 1$ to $\sqrt[n]{1+q} - 1$. It follows therefore that the fixt ratio, or proportion, of the ratio of $1+k$ to 1 to the ratio of $1+q$ to 1 must be equal to the fixt ratio of the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c$ *ad infinitum* to the series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \mathcal{E}c$ *ad infinitum*; or, in other words, the two serieses $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c$ *ad infinitum* and $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \mathcal{E}c$ *ad infinitum* are proportional to, or measures of, the two ratios of $1+k$ to 1 and $1+q$ to 1 . Q. E. D.

The foregoing Demonstration expressed in fewer Words.

105. This reasoning may be expressed more concisely by adopting the notation used in article 103; which may be done as follows.

Let the ratio of $1+k$ to 1 be denoted by the letter E , and the ratio of $1+q$ to 1 be denoted by the letter F ; and let the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c$ *ad infinitum* be denoted by the letter G , and the series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \mathcal{E}c$ *ad infinitum*

$-\frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \mathcal{E}c \text{ ad infinitum}$ be denoted by the letter $\mathfrak{H}$. And let the quantities $1 + k^{\frac{1}{n}} - 1$ and $1 + q^{\frac{1}{n}} - 1$ (which vary continually in their magnitude while the number n increases, and of which the ratio varies at the same time,) be denoted, respectively, by the letters x and y .

Then will the ratio of $1 + k^{\frac{1}{n}}$ to 1 be $= \frac{E}{n}$, and the ratio of $1 + q^{\frac{1}{n}}$ to 1 will be $= \frac{F}{n}$, and $\frac{1}{n} \times$ the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c \text{ ad infinitum}$ will be $= \frac{G}{n}$, and $\frac{1}{n} \times$ the series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \mathcal{E}c \text{ ad infinitum}$ will be $= \frac{H}{n}$.

Now it is shewn in lemma 1, that the ratio of $\frac{E}{n}$ to $\frac{F}{n}$ (which is a fixt, or given, ratio, and is equal to the ratio of E to F) is the limit of the varying ratio of x to y . And consequently the ratio of E to F (which is equal to the ratio of $\frac{E}{n}$ to $\frac{F}{n}$;) will be the limit of the varying ratio of x to y .

And it is shewn in lemma 2, coroll. 2, that the ratio of x to $\frac{G}{n}$ approaches continually, while the number n increases, to a ratio of equality as its limit; and that the ratio of y to $\frac{H}{n}$ approaches continually at the same time to a ratio of equality as its limit. But if x were absolutely equal to $\frac{G}{n}$, and y were absolutely equal to $\frac{H}{n}$, the ratio of x to y would be equal to the ratio of $\frac{G}{n}$ to $\frac{H}{n}$. Therefore, while the number n increases, and the ratio of x to $\frac{G}{n}$ approaches to a ratio of equality as its limit, and the ratio of y to $\frac{H}{n}$ approaches also to a ratio of equality as its limit, the ratio of x to y will approach continually to the ratio of $\frac{G}{n}$ to $\frac{H}{n}$, as its limit. But the ratio of $\frac{G}{n}$ to $\frac{H}{n}$ is equal to the ratio of G to H . Therefore the ratio of x to y will approach to the ratio of G to H as its limit; or the fixt ratio of G to H will be the limit of the varying ratio of x to y .

But it was before shewn, that the fixt ratio of E to F is also the limit of the varying ratio of x to y .

It follows therefore that the fixt ratio of G to H must be equal to the fixt ratio of E to F , or that G will be to H in the same proportion as E to F ; that is, the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c \text{ ad infinitum}$ will be to the series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \mathcal{E}c \text{ ad infinitum}$ in the same proportion as the ratio of $1 + k$ to 1 to the ratio of $1 + q$ to 1 .

Q. E. D.

106. The

106. The reasonings used in demonstrating the other theorem in article 66, (which relates to the series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \mathcal{E}c \text{ ad infinitum}$, invented by Dr. Wallis,) might be illustrated in the same manner as those used in demonstrating the first theorem, concerning Mercator's series. But this, I apprehend, would prove tedious to my readers, after the very ample discussion of the subject in the case of the said first theorem. I shall therefore here put an end to these remarks.

AN
APPENDIX
TO THE
FOREGOING REMARKS

ON THE

Two logarithmick Serieses of Mr. Nicholas Mercator and Dr. John Wallis:

CONTAINING

Investigations of two other infinite Serieses, which were published by Dr. Edmund Halley, and which are related to, and derived from, the two former; and by which we are enabled, when the value of either of the two former Serieses is given, to discover the Ratio to which it belongs, or of which it is the Logarithm.

ARTICLE I.

IN the foregoing remarks it has been shewn, that if k be any quantity not greater than 1, and q be any quantity less than k , the infinite series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c$ will be to the infinite series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \mathcal{E}c$ in the same proportion as the ratio of $1 + k$ to 1 is to the ratio of $1 + q$ to 1, or that the said two serieses are measures, or logarithms, of the said two ratios: and it has also been shewn, that if k be of any magnitude less

less than 1, and q be less than k , the infinite series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c$ will be to the infinite series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \mathcal{E}c$ in the same proportion as the ratio of 1 to $1 - k$ is to the ratio of 1 to $1 - q$, or that the said two serieses are measures, or logarithms, of the said two ratios. Now Dr. Edmund Halley, in his ingenious tract on logarithms, published in the Philosophical Transactions, No. 216, and republished with Sherwin's Mathematical Tables, has given us two additional infinite serieses relating to this subject; by the former of which we may, from the value of the series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c$ *ad infinitum*, derive the value of its first term k , and consequently that of $1 + k$, and thereby discover the ratio of $1 + k$ to 1, of which the said series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c$ *ad infinitum* is the logarithm; and by the latter of which we may, from the value of the series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c$ *ad infinitum*, derive the value of its first term k , and consequently that of $1 - k$, and thereby discover the ratio of 1 to $1 - k$, of which the said series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c$ *ad infinitum* is the logarithm. The former of these serieses, given us by Dr. Halley, is $L + \frac{L^2}{2} + \frac{L^3}{2 \cdot 3} + \frac{L^4}{2 \cdot 3 \cdot 4} + \frac{L^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{L^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \mathcal{E}c$ *ad infinitum*, in which L represents the value of the whole infinite series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c$; and the latter of them is $L - \frac{L^2}{2} + \frac{L^3}{2 \cdot 3} - \frac{L^4}{2 \cdot 3 \cdot 4} + \frac{L^5}{2 \cdot 3 \cdot 4 \cdot 5} - \frac{L^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \mathcal{E}c$ *ad infinitum*, in which L represents the value of the whole infinite series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c$. The former of them had been first invented by Sir Isaac Newton about the year 1669, but was not published till many years after; and the latter was invented by the learned Mr. Leibnitz before the year 1676, but not published till many years after. But Dr. Halley, in his tract on this subject, does not mention either Newton or Leibnitz as the inventor of either of these serieses; and therefore it seems probable that he did not know that those great men had invented them: and, in this case, he must also be considered as an inventor of them, though not as the first inventor. The former of these serieses is mentioned by Newton in his first letter to Mr. Oldenburg, dated June 13, 1676; and the latter series is mentioned by Mr. Leibnitz in his letter to the same person, dated August 27, 1676. See the *Commercium Epistolicum*, pages 139 and 150.

These two serieses consisting of the powers of L may not improperly be called *anti-logarithmick serieses*. The former of them, to wit, $L + \frac{L^2}{2} + \frac{L^3}{2 \cdot 3} + \frac{L^4}{2 \cdot 3 \cdot 4} + \frac{L^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{L^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \mathcal{E}c$ *ad infinitum*, may be derived from Mercator's series

$k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c$ *ad infinitum* by the help of Sir Isaac Newton's binomial theorem, in the first and simplest case of it, or that in which the index of the power to which any binomial quantity, as $1 + x$, is to be raised, is a positive whole number. And the latter of them, to wit, $L - \frac{L^2}{2} + \frac{L^3}{2.3} - \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} - \frac{L^6}{2.3.4.5.6} + \mathcal{E}c$ *ad infinitum*, may be derived in the like manner from Dr. Wallis's series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c$ *ad infinitum*, by the help of Sir Isaac Newton's theorem for finding the powers of a residual quantity, such as $1 - x$, in the first, or simplest, case of it, or that in which the index of the power to which such residual quantity is to be raised, is a positive whole number. The manner in which the former of these anti-logarithmick serieses, to wit, $L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \mathcal{E}c$ *ad infinitum*, may be derived from Mercator's logarithmick series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c$ *ad infinitum*, may be explained as follows.

Of the Anti-logarithmick Infinite Series $L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \mathcal{E}c$.

2. We are therefore now to shew, that, if k be any quantity not greater than 1, and L be equal to the infinite series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c$, or to the logarithm of the ratio of $1 + k$ to 1, the first term k of the said series will be equal to the infinite series $L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \mathcal{E}c$; the terms of which will always decrease with a considerable degree of swiftness, because both the co-efficients of the powers of L , to wit, 1, $\frac{1}{2}$, $\frac{1}{2.3}$, $\frac{1}{2.3.4}$, $\frac{1}{2.3.4.5}$, $\frac{1}{2.3.4.5.6}$, $\mathcal{E}c$, decrease with a very great, and an increasing, degree of swiftness, and the powers of L themselves, to wit, L , L^2 , L^3 , L^4 , L^5 , L^6 , $\mathcal{E}c$, will always decrease in some degree, because the greatest possible magnitude of L is less than 1, being equal to the decimal fraction 0.693,147,180,559,945,304, $\mathcal{E}c$, or to the value of the infinite series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c$ when k is of its greatest possible magnitude, or is equal to 1, that is, to the value of the infinite series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \mathcal{E}c$ *ad infinitum*, or to the logarithm of the ratio of 1 + 1, or 2, to 1, which has been shewn, in article 35 of the foregoing remarks, to be = 0.693,147,180,559,945,304, $\mathcal{E}c$.

ℰc. And, when L is the logarithm of a much smaller ratio than that of 2 to 1, as, for example, of the ratio of 10 to 9, or of the ratio of 11 to 10, (in the former of which cases it is equal to 0.105,360,515,657,826,302, ℰc, and in the latter to 0.095,310,179,804,324,858, ℰc,) the series $L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \text{ℰc ad infinitum}$ will converge with great swiftness.

An Investigation of the said Series.

3. In order to derive this *anti-logarithmick* series $L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \text{ℰc ad infinitum}$, which is equal to the first term k of Mr. Mercator's logarithmick series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \text{ℰc ad infinitum}$, from the said latter series, it will be convenient to premise the following lemma.

L E M M A I.

If n denote any whole number whatsoever, the n^{th} power of the binomial quantity $1 + x$ will be equal to the following series of terms continued to $n + 1$ terms, to wit, $1 + \frac{n}{1} \times x + \frac{n}{1} \times \frac{n-1}{2} \times x^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times x^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times x^4 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times x^5 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6} \times x^6 + \text{ℰc}$, or (if we put the capital letters $A, B, C, D, E, F, G, \text{ℰc}$ for the first term 1, and the co-efficients of $x, x^2, x^3, x^4, x^5, x^6$, and the following powers of x , in the 2nd, 3rd, 4th, 5th, 6th, 7th, and other following terms of the series respectively,) to the series $1 + \frac{n}{1} A x + \frac{n-1}{2} B x^2 + \frac{n-2}{3} C x^3 + \frac{n-3}{4} D x^4 + \frac{n-4}{5} E x^5 + \frac{n-5}{6} F x^6 + \text{ℰc}$ continued to $n + 1$ terms; in which the law of the continuation of the terms is very manifest, to wit, that the numerators $n, n-1, n-2, n-3, n-4, n-5, \text{ℰc}$, of the generating fractions $\frac{n}{1}, \frac{n-1}{2}, \frac{n-2}{3}, \frac{n-3}{4}, \frac{n-4}{5}, \frac{n-5}{6}, \text{ℰc}$ are formed from each other by the continual subtraction of an unit until the number n is exhausted, and the denominators 1, 2, 3, 4, 5, 6, ℰc of the same fractions are formed from each other by the continual addition of an unit.

4. This is the famous binomial theorem of Sir Isaac Newton, in its first and simplest case, or that in which the index of the power to which the binomial quantity

quantity is to be raised, is a positive whole number. Its truth is universally admitted by mathematicians; and some of them have given demonstrations of it; and, amongst the rest, Mr. James Bernouilli, (a very clear and satisfactory writer on mathematical subjects,) in his excellent treatise on the doctrine of chances, intituled *Ars Conjectandi*.—See the said treatise, book 2nd.—On the present occasion I shall take this proposition for granted.

5. Coroll. 1. If x be taken equal to the n^{th} part of any given quantity called b , so that $1 + x$ shall be equal to $1 + \frac{b}{n}$, it will follow that $1 + x^n$, or $1 + \frac{b^n}{n^n}$, will be equal to the series $1 + \frac{n}{1} \times \frac{b}{n} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{b^2}{n^2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{b^3}{n^3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{b^4}{n^4} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{b^5}{n^5} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6} \times \frac{b^6}{n^6} + \text{etc}$ continued to $n + 1$ terms, or to the series $1 + \frac{n}{1} A \times \frac{b}{n} + \frac{n-1}{2} B \times \frac{b^2}{n^2} + \frac{n-2}{3} C \times \frac{b^3}{n^3} + \frac{n-3}{4} D \times \frac{b^4}{n^4} + \frac{n-4}{5} E \times \frac{b^5}{n^5} + \frac{n-5}{6} F \times \frac{b^6}{n^6} + \text{etc}$ continued to $n + 1$ terms.

6. Coroll. 2. Whatever be the magnitude of n , the quantity $1 + \frac{b^n}{n^n}$ must be less than the series $1 + b + \frac{b^2}{2} + \frac{b^3}{2.3} + \frac{b^4}{2.3.4} + \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6} + \text{etc}$ continued to $n + 1$ terms.

For the series $1 + \frac{n}{1} \times \frac{b}{n} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{b^2}{n^2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{b^3}{n^3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{b^4}{n^4} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{b^5}{n^5} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6} \times \frac{b^6}{n^6} + \text{etc}$ continued to $n + 1$ terms, is evidently less than the series $1 + \frac{n}{1} \times \frac{b}{n} + \frac{n}{1} \times \frac{n-0}{2} \times \frac{b^2}{n^2} + \frac{n}{1} \times \frac{n-0}{2} \times \frac{n-0}{3} \times \frac{b^3}{n^3} + \frac{n}{1} \times \frac{n-0}{2} \times \frac{n-0}{3} \times \frac{n-0}{4} \times \frac{b^4}{n^4} + \frac{n}{1} \times \frac{n-0}{2} \times \frac{n-0}{3} \times \frac{n-0}{4} \times \frac{n-0}{5} \times \frac{b^5}{n^5} + \frac{n}{1} \times \frac{n-0}{2} \times \frac{n-0}{3} \times \frac{n-0}{4} \times \frac{n-0}{5} \times \frac{n-0}{6} \times \frac{b^6}{n^6} + \text{etc}$ continued to the same number of terms, or than the series $1 + \frac{n}{1} \times \frac{b}{n} + \frac{n^2}{2} \times \frac{b^2}{n^2} + \frac{n^3}{2.3} \times \frac{b^3}{n^3} + \frac{n^4}{2.3.4} \times \frac{b^4}{n^4} + \frac{n^5}{2.3.4.5} \times \frac{b^5}{n^5} + \frac{n^6}{2.3.4.5.6} \times \frac{b^6}{n^6} + \text{etc}$ continued to the same number of terms, or than the series $1 + \frac{nb}{n} + \frac{n^2 b^2}{2n^2} + \frac{n^3 b^3}{2.3.n^3} + \frac{n^4 b^4}{2.3.4.n^4} + \frac{n^5 b^5}{2.3.4.5.n^5} + \frac{n^6 b^6}{2.3.4.5.6.n^6} + \text{etc}$ continued to the same number of terms, or than the series $1 + b + \frac{b^2}{2} + \frac{b^3}{2.3} + \frac{b^4}{2.3.4} + \text{etc}$

$\frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \mathcal{E}c$ continued to the same number of terms. But by coroll. 1, the series $1 + \frac{n}{1} \times \frac{b}{n} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{b^2}{n^2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{b^3}{n^3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{b^4}{n^4} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{b^5}{n^5} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6} \times \frac{b^6}{n^6} + \mathcal{E}c$ continued to $n+1$ terms is equal to $1 + \frac{b^n}{n}$. Therefore $1 + \frac{b^n}{n}$ must be less than the series $1 + b + \frac{b^2}{2} + \frac{b^3}{2 \cdot 3} + \frac{b^4}{2 \cdot 3 \cdot 4} + \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \mathcal{E}c$ continued to $n+1$ terms.

Q. E. D.

7. Coroll. 3. But though $1 + \frac{b^n}{n}$ be always less than the series $1 + b + \frac{b^2}{2} + \frac{b^3}{2 \cdot 3} + \frac{b^4}{2 \cdot 3 \cdot 4} + \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \mathcal{E}c$ continued to $n+1$ terms, yet the difference between them will continually decrease while the index n increases; and the said index n may be taken of so great a magnitude that the said difference shall become less than any assigned quantity, how small soever.

For the series $1 + \frac{n}{1} \times \frac{b}{n} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{b^2}{n^2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{b^3}{n^3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{b^4}{n^4} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{b^5}{n^5} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6} \times \frac{b^6}{n^6} + \mathcal{E}c$ continued to $n+1$ terms (to which the quantity $1 + \frac{b^n}{n}$ is equal) is equal to the series $1 + \frac{n}{n} \times b + \frac{n^2-n}{n^2} \times \frac{b^2}{2} + \frac{n^3-3n^2+2n}{n^3} \times \frac{b^3}{2 \cdot 3} + \frac{n^4-6n^3+11n^2-6n}{n^4} \times \frac{b^4}{2 \cdot 3 \cdot 4} + \frac{n^5-10n^4+35n^3-50n^2+24n}{n^5} \times \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{n^6-15n^5+85n^4-225n^3+274n^2-120n}{n^6} \times \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \mathcal{E}c$ continued to $n+1$ terms, and consequently to the series $1 + b + 1 - \frac{1}{n} \times \frac{b^2}{2} + 1 - \frac{3}{n} + \frac{2}{n^2} \times \frac{b^3}{2 \cdot 3} + 1 - \frac{6}{n} + \frac{11}{n^2} - \frac{6}{n^3} \times \frac{b^4}{2 \cdot 3 \cdot 4} + 1 - \frac{10}{n} + \frac{35}{n^2} - \frac{50}{n^3} + \frac{24}{n^4} \times \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + 1 - \frac{15}{n} + \frac{85}{n^2} - \frac{225}{n^3} + \frac{274}{n^4} - \frac{120}{n^5} \times \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \mathcal{E}c$ continued to $n+1$ terms, or to the series $1 + b + \frac{b^2}{2} + \frac{b^3}{2 \cdot 3} + \frac{b^4}{2 \cdot 3 \cdot 4} + \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \mathcal{E}c$ continued to $n+1$ terms, — the series $\frac{1}{n} \times \frac{b^2}{2} + \frac{3}{n} - \frac{2}{n^2} \times \frac{b^3}{2 \cdot 3} + \frac{6}{n} - \frac{11}{n^2} + \frac{6}{n^3} \times \frac{b^4}{2 \cdot 3 \cdot 4} + \frac{10}{n} - \frac{35}{n^2} + \frac{50}{n^3} - \frac{24}{n^4} \times \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{15}{n} - \frac{85}{n^2} + \frac{225}{n^3} - \frac{274}{n^4} + \frac{120}{n^5} \times \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \mathcal{E}c$ continued to $n-1$ terms; the difference of which from the series $1 + b + \frac{b^2}{2} + \frac{b^3}{2 \cdot 3} + \frac{b^4}{2 \cdot 3 \cdot 4} +$

$\frac{b^3}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \mathcal{E}c$ continued to $n + 1$ terms, is the series $\frac{1}{n} \times \frac{b^2}{2} +$
 $\frac{3}{n} - \frac{2}{n^2} \times \frac{b^3}{2 \cdot 3} + \frac{6}{n} - \frac{11}{n^2} + \frac{6}{n^3} \times \frac{b^4}{2 \cdot 3 \cdot 4} + \frac{10}{n} - \frac{35}{n^2} + \frac{50}{n^3} - \frac{24}{n^4} \times \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5}$
 $+ \frac{15}{n} - \frac{85}{n^2} + \frac{225}{n^3} - \frac{274}{n^4} + \frac{120}{n^5} \times \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \mathcal{E}c$ continued to $n - 1$ terms.
 Therefore this last series is equal to the difference whereby the series $1 + b +$
 $\frac{b^2}{2} + \frac{b^3}{2 \cdot 3} + \frac{b^4}{2 \cdot 3 \cdot 4} + \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \mathcal{E}c$ continued to $n + 1$ terms, exceeds
 the quantity $1 + \frac{b^n}{n}$. Now it is evident, that the said series $\frac{1}{n} \times \frac{b^2}{2} +$
 $\frac{3}{n} - \frac{2}{n^2} \times \frac{b^3}{2 \cdot 3} + \frac{6}{n} - \frac{11}{n^2} + \frac{6}{n^3} \times \frac{b^4}{2 \cdot 3 \cdot 4} + \frac{10}{n} - \frac{35}{n^2} + \frac{50}{n^3} - \frac{24}{n^4} \times \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5}$
 $+ \frac{15}{n} - \frac{85}{n^2} + \frac{225}{n^3} - \frac{274}{n^4} + \frac{120}{n^5} \times \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \mathcal{E}c$ continued to $n - 1$ terms,
 will continually decrease while n increases, because the powers of n constitute the
 denominators of the several members of the co-efficients of its terms; and n may
 be taken of so great a magnitude that the said series shall be less than any as-
 signed quantity, how small soever. Therefore, while the index n increases, the
 difference whereby the series $1 + b + \frac{b^2}{2} + \frac{b^3}{2 \cdot 3} + \frac{b^4}{2 \cdot 3 \cdot 4} + \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$
 $+ \mathcal{E}c$ continued to $n + 1$ terms, exceeds the quantity $1 + \frac{b^n}{n}$, will decrease
 continually *ad infinitum*, or the index n may be taken of so great a magnitude
 that the said difference shall be less than any assigned quantity, how small soever.
 Q. E. D.

8. Coroll. 4. It follows, from the foregoing corollary, that the infinite series
 $1 + b + \frac{b^2}{2} + \frac{b^3}{2 \cdot 3} + \frac{b^4}{2 \cdot 3 \cdot 4} + \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \mathcal{E}c$ *ad infinitum* is the limit
 of the magnitude of $1 + \frac{b^n}{n}$, or the quantity to which $1 + \frac{b^n}{n}$ may, by in-
 creasing continually the index n , be made to come as near as we please.

For, by increasing n continually, the excess of the series $1 + b + \frac{b^2}{2} + \frac{b^3}{2 \cdot 3}$
 $+ \frac{b^4}{2 \cdot 3 \cdot 4} + \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \mathcal{E}c$ continued to $n + 1$ terms, above the quan-
 tity $1 + \frac{b^n}{n}$ may be made to decrease till it is less than any assigned quantity,
 how small soever, as we have seen in the foregoing corollary. And, by in-
 creasing n continually, it is evident that the excess of the infinite series $1 + b +$
 $\frac{b^2}{2} + \frac{b^3}{2 \cdot 3} + \frac{b^4}{2 \cdot 3 \cdot 4} + \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \mathcal{E}c$ *ad infinitum* above the series $1 +$
 $b + \frac{b^2}{2} + \frac{b^3}{2 \cdot 3} + \frac{b^4}{2 \cdot 3 \cdot 4} + \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \mathcal{E}c$ continued to $n + 1$ terms,
 may be diminished as far as we please, or made to become less than any assigned
 quantity,

quantity, how small soever. It follows therefore, that, by increasing the index n continually, the sum of these two excesses, or the excess of the infinite series $1 + b + \frac{b^2}{2} + \frac{b^3}{2 \cdot 3} + \frac{b^4}{2 \cdot 3 \cdot 4} + \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \mathcal{E}c$ *ad infinitum* above the quantity $1 + \frac{b^n}{n}$, may be diminished as far as we please, or made to become less than any assigned quantity, how small soever. Or, in other words, the infinite series $1 + b + \frac{b^2}{2} + \frac{b^3}{2 \cdot 3} + \frac{b^4}{2 \cdot 3 \cdot 4} + \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \mathcal{E}c$ *ad infinitum* is the limit of the magnitude of the quantity $1 + \frac{b^n}{n}$. Q. E. D.

9. Coroll. 5. If we make use of the language of infinities, (which, though not a correct way of speaking, is often very convenient on account of its brevity, and has therefore been sometimes adopted even by the most careful and elegant writers on mathematical subjects; and, amongst the rest, by the great Mr. Huygens himself, who deserves that character above all other writers;) the substance of the last, or 4th, corollary may be expressed in the manner following.

When the index n of the power to which the binomial quantity $1 + \frac{b}{n}$ is to be raised, becomes infinite, the series $1 + \frac{n}{1} \times \frac{b}{n} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{b^2}{n^2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{b^3}{n^3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{b^4}{n^4} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{b^5}{n^5} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6} \times \frac{b^6}{n^6} + \mathcal{E}c$ continued to $n + 1$ terms, (to which $1 + \frac{b^n}{n}$ is equal) or $1 + \frac{n}{n} \times b + \frac{nn - n}{nn} \times \frac{b^2}{2} + \frac{n^3 - 3n^2 + 2n}{n^3} \times \frac{b^3}{2 \cdot 3} + \frac{n^4 - 6n^3 + 11n^2 - 6n}{n^4} \times \frac{b^4}{2 \cdot 3 \cdot 4} + \frac{n^5 - 10n^4 + 35n^3 - 50n^2 + 24n}{n^5} \times \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{n^6 - 15n^5 + 85n^4 - 225n^3 + 274n^2 - 120n}{n^6} \times \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \mathcal{E}c$ continued to $n + 1$ terms, becomes infinite in the number of its terms, and equal to the series $1 + \frac{n}{n} \times b + \frac{nn}{nn} \times \frac{b^2}{2} + \frac{n^3}{n^3} \times \frac{b^3}{2 \cdot 3} + \frac{n^4}{n^4} \times \frac{b^4}{2 \cdot 3 \cdot 4} + \frac{n^5}{n^5} \times \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{n^6}{n^6} \times \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \mathcal{E}c$ *ad infinitum*, or $1 + b + \frac{b^2}{2} + \frac{b^3}{2 \cdot 3} + \frac{b^4}{2 \cdot 3 \cdot 4} + \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \mathcal{E}c$ *ad infinitum*, because all the members of the several numerators $nn - n$, $n^3 - 3n^2 + 2n$, $n^4 - 6n^3 + 11n^2 - 6n$, $n^5 - 10n^4 + 35n^3 - 50n^2 + 24n$, and $n^6 - 15n^5 + 85n^4 - 225n^3 + 274n^2 - 120n$, $\mathcal{E}c$, after the first members n^2 , n^3 , n^4 , n^5 , n^6 , $\mathcal{E}c$, will vanish, or become infinitely small, in respect of the said first members, which contain a higher power of n than the said following members; and consequently those whole numerators will be equal to their said first members, n^2 , n^3 , n^4 , n^5 , n^6 , $\mathcal{E}c$, respectively. Therefore the quantity $1 + \frac{b^n}{n}$ (which in all cases is equal to the said series $1 +$

$$\frac{n}{n} \times b + \frac{nn-n}{nn} \times \frac{b^2}{2} + \frac{n^3-3n^2+2n}{n^3} \times \frac{b^3}{2.3} + \frac{n^4-6n^3+11n^2-6n}{n^4} \times \frac{b^4}{2.3.4} +$$

$$\frac{n^5-10n^4+35n^3-50n^2+24n}{n^5} \times \frac{b^5}{2.3.4.5} + \frac{n^6-15n^5+85n^4-225n^3+274n^2-120n}{n^6} \times$$

$$\frac{b^6}{2.3.4.5.6} + \mathcal{E}c \text{ continued to } n+1 \text{ terms,)} \text{ will, in this case of the infinite magni-}$$

tude of n , be equal to the infinite series $1 + b + \frac{b^2}{2} + \frac{b^3}{2.3} + \frac{b^4}{2.3.4} + \frac{b^5}{2.3.4.5}$

$$+ \frac{b^6}{2.3.4.5.6} + \mathcal{E}c \text{ ad infinitum.} \quad Q. E. D.$$

10. These things being premised, the investigation of the first anti-logarithmick series $L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \mathcal{E}c \text{ ad infinitum}$, or the derivation of it from Mr. Mercator's logarithmick series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c \text{ ad infinitum}$, may be performed in the manner following.

PROBLEM I.

Let k be any quantity not greater than 1, and L be equal to the infinite series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c \text{ ad infinitum}$; which is the logarithm of the ratio of $1+k$ to 1 in Napier's system of logarithms, and which consequently can never be greater than the logarithm of the ratio of $1+1$, or 2, to 1 in that system, or than the decimal fraction 0.693,147,180,559,945,304, $\mathcal{E}c$. And let L , or the value of this whole series, be supposed to be known, but k , or the value of its first term alone, and consequently the separate values of all its other terms (which contain the powers of k ,) to be unknown. It is required to find from L , or the value of the whole series, the value of its first term k .

SOLUTION.

Let us suppose the letter n to denote some very great number, such as a nonillion, or the ninth power of a million.

Then it will follow, from lemma 2, coroll. 2, art. 17, of the foregoing remarks on Mercator's and Wallis's serieses, that $1 + k^{\frac{1}{n}} - 1$ will be very nearly equal to $\frac{1}{n} \times$ the infinite series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c \text{ ad infinitum}$, that is, (because L is supposed to be equal to the said series,) very nearly equal to $\frac{1}{n} \times L$, or $\frac{L}{n}$. Therefore, if we add 1 to both sides of the equation, we shall have $1 + k^{\frac{1}{n}}$ very nearly equal to $1 + \frac{L}{n}$, and (raising both

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sides of this last equation to the n^{th} power,) $1 + k$ very nearly equal to $1 + \frac{L}{n}$.
 And n may be taken of so great a magnitude that the ratio of $1 + k$ to $1 + \frac{L}{n}$ shall approach as near as we please to a ratio of equality.

But, because the index n is of an immensely-great magnitude, it follows, from the fourth corollary of the foregoing lemma, that $1 + \frac{L}{n}$ will be very nearly equal to the infinite series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \mathcal{E}c \text{ ad infinitum}$, and that n may be taken of so great a magnitude that the difference between $1 + \frac{L}{n}$ and this series shall be less than any assigned quantity, how small soever. Therefore $1 + k$ will be very nearly equal to the said infinite series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \mathcal{E}c \text{ ad infinitum}$, and k will be very nearly equal to the infinite series $L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \mathcal{E}c \text{ ad infinitum}$. And n may be taken of so great a magnitude that the ratio of $1 + k$ to the series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \mathcal{E}c \text{ ad infinitum}$ shall approach as near to a ratio of equality as we please. Therefore $1 + k$ must be not only very nearly, but accurately, equal to the said series, and consequently k will be not only very nearly, but accurately, equal to the infinite series $L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \mathcal{E}c \text{ ad infinitum}$.

Q. E. I.

11. This solution may be expressed with greater brevity in the language of infinites, as follows.

Let us suppose the letter n to denote a number infinitely great.

Then it will follow, from lemma 2, coroll. 2, art. 17, of the foregoing remarks, that $\sqrt[n]{1 + k} - 1$ will be equal to $\frac{1}{n} \times$ the infinite series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c \text{ ad infinitum}$, that is, (because the said series is $= L$,) to $\frac{1}{n} \times L$, or to $\frac{L}{n}$. Therefore (adding 1 to both sides of the equation) we shall have $\sqrt[n]{1 + k} = 1 + \frac{L}{n}$, and consequently (raising both sides of this last equation to the n^{th} power,) $1 + k = 1 + \frac{L}{n}$.

But, because the index n is an infinitely-great number, it follows, from coroll.
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5 of the foregoing lemma, that $1 + \frac{L^n}{n}$ will be equal to the infinite series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \&c \text{ ad infinitum}$. Therefore $1 + k$ will be equal to the same infinite series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \&c \text{ ad infinitum}$, and consequently k will be equal to the infinite series $L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \&c \text{ ad infinitum}$. Q. E. I.

12. Coroll. 1. Since k is equal to the infinite series $L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \&c \text{ ad infinitum}$, it follows that the ratio of $1 + k$ to 1 is equal to the ratio of the infinite series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \&c \text{ ad infinitum}$ to 1 ; or the ratio corresponding to any given logarithm in Napier's system denoted by L (that is not greater than 0.693,147,180,559,945,304, $\&c$, or the logarithm of the ratio of 2 to 1,) is the ratio of the infinite series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \&c \text{ ad infinitum}$ to 1 .

An Example of the foregoing Method of discovering the Ratio corresponding to a given Logarithm L in Napier's System, by Means of the Infinite Series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \&c \text{ ad infinitum}$.

13. Let the given logarithm, of which we are to find the ratio, be $\frac{1}{2}$, or 0.500,000,000.

Then will L^2 be $= \frac{1}{4}$, and L^3 be $= \frac{1}{8}$, and $L^4 = \frac{1}{16}$, and $L^5 = \frac{1}{32}$, and $L^6 = \frac{1}{64}$, and $L^7, L^8, L^9, L^{10}, \&c$ equal to the 7th, 8th, 9th, 10th, and other following powers of $\frac{1}{2}$, and consequently the series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \&c$ will be equal to $1 + \frac{1}{2} + \frac{1}{4.2} + \frac{1}{8.2.3} + \frac{1}{16.2.3.4} + \frac{1}{32.2.3.4.5} + \frac{1}{64.2.3.4.5.6} + \&c$; or if we denote these several terms by the capital letters A, B, C, D, E, F, $\&c$, so that the first term 1 shall be $= A$, and the second term $\frac{1}{2}$ shall be B, and the third term $\frac{1}{4.2}$ shall be $= C$, and the fourth, fifth, sixth, seventh, eighth, and other following terms, shall be equal to D, E, F, G, H, $\&c$, the

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series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \text{Ec}$ will be equal to $1 + \frac{A}{2} + \frac{B}{2.2} + \frac{C}{2.3} + \frac{D}{2.4} + \frac{E}{2.5} + \frac{F}{2.6} + \frac{G}{2.7} + \frac{H}{2.8} + \frac{I}{2.9} + \frac{K}{2.10} + \text{Ec ad infinitum}$. These terms may be computed in the manner following.

$$A = 1.000,000,000;$$

$$B (= \frac{A}{2} = \frac{1.000,000,000}{2}) = 0.500,000,000;$$

$$C (= \frac{B}{2.2} = \frac{B}{4} = \frac{0.500,000,000}{4}) = 0.125,000,000;$$

$$D (= \frac{C}{2.3} = \frac{C}{6} = \frac{0.125,000,000}{6}) = 0.020,833,333;$$

$$E (= \frac{D}{2.4} = \frac{D}{8} = \frac{0.020,833,333}{8}) = 0.002,604,166;$$

$$F (= \frac{E}{2.5} = \frac{E}{10} = \frac{0.002,604,166}{10}) = 0.000,260,416;$$

$$G (= \frac{F}{2.6} = \frac{F}{12} = \frac{0.000,260,416}{12}) = 0.000,021,701;$$

$$H (= \frac{G}{2.7} = \frac{G}{14} = \frac{0.000,021,701}{14}) = 0.000,001,550;$$

$$I (= \frac{H}{2.8} = \frac{H}{16} = \frac{0.000,001,550}{16}) = 0.000,000,096;$$

$$K (= \frac{I}{2.9} = \frac{I}{18} = \frac{0.000,000,096}{18}) = 0.000,000,005;$$

$$\text{And } L (= \frac{K}{2.10} = \frac{K}{20} = \frac{0.000,000,005}{20}) = 0.000,000,000.$$

Therefore the series $A + B + C + D + E + F + G + H + I + K + L + \text{Ec}$, or $1 + \frac{A}{2} + \frac{B}{2.2} + \frac{C}{2.3} + \frac{D}{2.4} + \frac{E}{2.5} + \frac{F}{2.6} + \frac{G}{2.7} + \frac{H}{2.8} + \frac{I}{2.9} + \frac{K}{2.10} + \text{Ec}$ will be equal to $1.000,000,000 + 0.500,000,000 + 0.125,000,000 + 0.020,833,333 + 0.002,604,166 + 0.000,260,416 + 0.000,021,701 + 0.000,001,550 + 0.000,000,096 + 0.000,000,005 + 0.000,000,000 + \text{Ec} = 1.648,721,267, \text{Ec}$; and consequently the ratio sought, or that of which the fraction $\frac{1}{2}$, or $0.500,000,000$, is the logarithm in Napier's system, is the ratio of $1.648,721,267, \text{Ec}$ to 1 . Q. E. I.

14. The ratio of the square of this number $1.648,721,267, \text{Ec}$ to 1 is double of the ratio of this number itself to 1 ; and consequently the logarithm of the ratio of the square of this number to 1 will be double of the logarithm of the ratio of this number itself to 1 , or will be double of $\frac{1}{2}$, or $0.500,000,000$, or equal to 1 . But the square of the number $1.648,621,267$ is $2.718,281,816, \text{Ec}$. Therefore the ratio corresponding to the logarithm 1 , or of which 1 is the logarithm in Napier's system, is that of $2.718,281,816, \text{Ec}$ to 1 .

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15. In the solution of the foregoing problem the letter L was put for the value of the whole series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} \mathcal{E}c \text{ ad infinitum}$, in which k can never be greater than 1, and the value of which can never therefore be greater than that of the infinite series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \mathcal{E}c \text{ ad infinitum}$, or than the logarithm of the ratio of 1 + 1, or 2, to 1, or than the decimal fraction 0.693,147,180,559,945,304, $\mathcal{E}c$. And therefore it cannot be concluded from the reasonings used in the said solution, (which are grounded on the said supposition, "that L is equal to the said series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c$, and consequently is subject to the said restriction,") that, when L is greater than Napier's logarithm of the ratio of 2 to 1, or than the decimal fraction 0.693,147,180,559,945,304, $\mathcal{E}c$, the ratio corresponding to it, or of which it is the logarithm in Napier's system, is the ratio of the series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \mathcal{E}c \text{ ad infinitum}$ to 1. Nevertheless this proposition is true in these cases as well as in the former, though it does not follow from the foregoing premisses. But it may be shewn from other premisses, that, whatever be the magnitude of L , even though it should be much greater than the decimal fraction 0.693,147,180,559,945,304, $\mathcal{E}c$, and even than 1, as, for example, equal to 10, or 100, or 1000, or any greater number, yet the ratio corresponding to the logarithm L , or of which L is the logarithm in Napier's system, will still be the ratio of the series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \mathcal{E}c \text{ ad infinitum}$ to 1.

For though, when L is greater than 1, the powers of L , to wit, $L, L^2, L^3, L^4, L^5, L^6, \mathcal{E}c$, which form the numerators of the terms of the said series, will be a series of increasing quantities, every one of which will be greater than that which immediately precedes it in the proportion of L to 1, yet the increase of the whole terms $L, \frac{L^2}{2}, \frac{L^3}{2.3}, \frac{L^4}{2.3.4}, \frac{L^5}{2.3.4.5}, \frac{L^6}{2.3.4.5.6}, \mathcal{E}c \text{ ad infinitum}$, arising from such increase of their numerators $L, L^2, L^3, L^4, L^5, L^6, \mathcal{E}c$, will, in some part or other of the series, be more than counter-balanced by the decrease of them arising from the increase of their denominators 2, 2.3, 2.3.4, 2.3.4.5, 2.3.4.5.6, $\mathcal{E}c$; and consequently the said whole terms (though they will have diverged, or increased, continually from the beginning of the series to that term at which the said over-balancing takes place) will afterwards converge, or decrease, continually, and that with an increasing velocity, or faster than in a continued geometrical proportion; and therefore the value, or sum of all the terms, of the said series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \mathcal{E}c$ will (notwithstanding the divergency, or increase of its first terms,) be only of a finite magnitude.

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This convergency of the terms of this series, in some part or other of it, even when L is much greater than 1, may be easily deduced from an attentive consideration of the manner in which the denominators of its terms increase: and, further, it may be demonstrated from other premises, that, whatever be the magnitude of L , the ratio corresponding to it, or of which it is the logarithm in Napier's system, will be the ratio of the series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \text{\&c ad infinitum}$ to 1. See below, in articles 39, 40, and 41. But this is a speculation of more curiosity than use, on account of the great labour of computing a sufficient number of the terms of this series, when L is much greater than 1, to obtain its value to any considerable degree of exactness. It may not, however, be amiss to give the reader an instance of the truth of this proposition in one remarkable case of it, that lies out of the limits of the supposition made in the foregoing problem, by computing the value of the series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \text{\&c ad infinitum}$, when L is equal to 1, and thereby determining (in case this proposition holds true in the extent we have just now assigned to it,) the ratio corresponding to 1, or of which 1 is the logarithm in Napier's system; which we shall find to be that of the number 2.718,281,816, &c to 1, as it has been already shewn to be in article 14. The computation of the series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \text{\&c ad infinitum}$ in this case, in which it becomes equal to the series $1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \frac{1}{2.3.4.5} + \frac{1}{2.3.4.5.6} + \text{\&c ad infinitum}$, may be performed as follows.

The Computation of the Series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \text{\&c ad infinitum}$, when L is = 1.

16. Let the first term 1 of the series $1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \frac{1}{2.3.4.5} + \frac{1}{2.3.4.5.6} + \text{\&c ad infinitum}$, to which the series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \text{\&c}$ is in this case equal, be called A, and the second term 1 be called B, and the third term $\frac{1}{2}$ be called C, and the fourth term $\frac{1}{2.3}$ be called D, and the fifth term $\frac{1}{2.3.4}$ be called E, and the sixth, seventh, eighth, ninth, and other following terms, or $\frac{1}{2.3.4.5}$, $\frac{1}{2.3.4.5.6}$, $\frac{1}{2.3.4.5.6.7}$, $\frac{1}{2.3.4.5.6.7.8}$, &c, be denoted by F, G, H, I, and the following capital letters of the alphabet. And the foregoing series $1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \frac{1}{2.3.4.5} + \frac{1}{2.3.4.5.6} + \frac{1}{2.3.4.5.6.7}$

$+ \frac{1}{2.3.4.5.6.7.8} + \mathcal{E}c$ will become equal to the series $1 + \frac{A}{1} + \frac{B}{2} + \frac{C}{3} + \frac{D}{4} + \frac{E}{5} + \frac{F}{6} + \frac{G}{7} + \frac{H}{8} + \mathcal{E}c$. These terms, when reduced into decimal fractions, will be as follows.

$$A = 1.000,000,000,000,000,000,000,000;$$

$$B = \frac{A}{1} = 1.000,000,000,000,000,000,000,000;$$

$$C = \frac{B}{2} = 0.500,000,000,000,000,000,000,000;$$

$$D = \frac{C}{3} = 0.166,666,666,666,666,666,666,666;$$

$$E = \frac{D}{4} = 0.041,666,666,666,666,666,666,666;$$

$$F = \frac{E}{5} = 0.008,333,333,333,333,333,333,333;$$

$$G = \frac{F}{6} = 0.001,388,888,888,888,888,888,888;$$

$$H = \frac{G}{7} = 0.000,198,412,698,412,698,412,698;$$

$$I = \frac{H}{8} = 0.000,024,801,587,301,587,301,587;$$

$$K = \frac{I}{9} = 0.000,002,755,731,922,398,589,065;$$

$$L = \frac{K}{10} = 0.000,000,275,573,192,239,858,906;$$

$$M = \frac{L}{11} = 0.000,000,025,052,108,385,441,718;$$

$$N = \frac{M}{12} = 0.000,000,002,087,675,698,786,809;$$

$$O = \frac{N}{13} = 0.000,000,000,160,590,438,368,216;$$

$$P = \frac{O}{14} = 0.000,000,000,011,470,745,597,729;$$

$$Q = \frac{P}{15} = 0.000,000,000,000,764,716,373,181;$$

$$R = \frac{Q}{16} = 0.000,000,000,000,047,794,773,323;$$

$$S = \frac{R}{17} = 0.000,000,000,000,002,811,457,254;$$

$$T = \frac{S}{18} = 0.000,000,000,000,000,156,192,069;$$

$$V = \frac{T}{19} = 0.000,000,000,000,000,008,220,635;$$

$$W = \frac{V}{20} = 0.000,000,000,000,000,000,411,031;$$

$$X = \frac{W}{21} = 0.000,000,000,000,000,000,019,572;$$

$$Y = \frac{X}{22} = 0.000,000,000,000,000,000,000,889;$$

$$Z = \frac{Y}{23} = 0.000,000,000,000,000,000,000,038;$$

$$A = \frac{Z}{24} = 0.000,000,000,000,000,000,000,001.$$

$$2.718,281,828,459,045,235,360,274.$$

Therefore, when L is $= 1$, the series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \text{\textit{\textcircled{E}}c}$ is $= 2.718,281,828,459,045,235,360,274, \text{\textit{\textcircled{E}}c}$; and the ratio corresponding to 1, or of which 1 is the logarithm in Napier's system, is the ratio of $2.718,281,828,459,045,235,360,274, \text{\textit{\textcircled{E}}c}$ to 1; which is the same with the ratio assigned above in article 14 for the ratio corresponding to the same logarithm 1, to wit, the ratio of $2.718,281,816, \text{\textit{\textcircled{E}}c}$ to 1, except that it is expressed with more exactness, or to a greater number of figures.

17. Coroll. 2. If B be the logarithm of the same ratio in Briggs's system of logarithms of which L is the logarithm in Napier's system, B will be less than L in the proportion of Briggs's logarithm of the ratio of 10 to 1 to Napier's logarithm of the same ratio, that is, in the proportion of 1 to $2.302,585,092,994,045,668, \text{\textit{\textcircled{E}}c}$; or L will be $= 2.302,585,092,994,045,668, \text{\textit{\textcircled{E}}c} \times B$. Therefore, if B be any logarithm, in Briggs's system, of which we wish to find the correspondent ratio, we must first multiply B by the number $2.302,585,092,994,045,668, \text{\textit{\textcircled{E}}c}$, whereby we shall obtain the value of L , or of Napier's logarithm of the same ratio; and then we must compute as many terms of the series $1 + L + \frac{L^2}{2} = \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \text{\textit{\textcircled{E}}c}$ *ad infinitum* as may be necessary to express the ratio we are seeking to the proposed degree of exactness. For the proportion of the said series, or of the sum of so many terms of it as we shall have computed, to 1 will be the ratio sought.

An Example of this Method of discovering the Ratio corresponding to a given Logarithm in Briggs's System of Logarithms.

18. Let it be required to find the ratio corresponding to the logarithm $\frac{1}{100}$, or 0.01, in Briggs's system of logarithms.

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The logarithm of this ratio in Napier's system is $= 2.302,585,092,994,045,668, \text{ \&c} \times 0.01 = 0.023,025,850,929,940,456,68, \text{ \&c}$. Call this logarithm L . Then will L^2 be equal to $0.023,025,850,929,940,456,68, \text{ \&c}^2$, or (neglecting the last eleven figures, as inconsiderable,) equal to $0.023,025,850^2 = 0.000,530,189$; and L^3 will be $= 0.000,012,208$; and L^4 will be $= 0.000,000,281$; and L^5 will be $= 0.000,000,006$. Therefore $\frac{L^2}{2}$ will be $= \frac{0.000,530,189}{2} = 0.000,265,094$; and $\frac{L^3}{2.3}$ will be $= \frac{0.000,012,208}{6} = 0.000,002,034$; and $\frac{L^4}{2.3.4}$ will be $= \frac{0.000,000,281}{24} = 0.000,000,011$; and $\frac{L^5}{2.3.4.5}$ will be $= \frac{0.000,000,006}{120} = 0.000,000,000$; and consequently the series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \text{ \&c}$ will be $= 1.000,000,000 + 0.023,025,850 + 0.000,265,094 + 0.000,002,034 + 0.000,000,011 + 0.000,000,000 + \text{ \&c} = 1.023,292,989, \text{ \&c}$. Therefore the ratio corresponding to L , or $0.023,025,850$, or of which $0.023,025,850$ is the logarithm in Napier's system of logarithms, or of which 0.01 , or $\frac{1}{100}$, is the logarithm in Briggs's system of logarithms, is the ratio of $1.023,292,989, \text{ \&c}$ to 1 . Q. E. I.

19. This number $1.023,292,989, \text{ \&c}$ is the hundredth root of 10 . For the ratio of the 100^{th} root of 10 to 1 is the 100^{th} part of the ratio of 10 to 1 ; and consequently the logarithm of the ratio of the hundredth root of 10 to 1 is the hundredth part of the logarithm of the ratio of 10 to 1 . But in Briggs's system of logarithms 1 is the logarithm of the ratio of 10 to 1 . Therefore $\frac{1}{100}$, or 0.01 , is the logarithm of the ratio of the hundredth root of 10 to 1 . But it has been shewn, that 0.01 is the logarithm of the ratio of $1.023,292,989, \text{ \&c}$ to 1 in Briggs's system of logarithms. Therefore the ratio of the hundredth root of 10 to 1 is equal to the ratio of $1.023,292,989, \text{ \&c}$ to 1 , they having both the same logarithm. Therefore (by El. 5, 9) the hundredth root of 10 must be equal to $1.023,292,989, \text{ \&c}$. Q. E. D.

CONCERNING THE
SECOND ANTI-LOGARITHMICK SERIES

PUBLISHED BY DR. HALLEY,

TO WIT,

The Series $L - \frac{L^2}{2} + \frac{L^3}{2.3} - \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} - \frac{L^6}{2.3.4.5.6} + \mathcal{E}c \text{ ad infinitum}.$

20. **W**E come now to consider the second anti-logarithmick series invented by Dr. Halley, to wit, the series $L - \frac{L^2}{2} + \frac{L^3}{2.3} - \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} - \frac{L^6}{2.3.4.5.6} + \mathcal{E}c \text{ ad infinitum},$ in which L denotes the value of the infinite series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c \text{ ad infinitum},$ invented by Dr. Wallis, or the logarithm of the ratio of 1 to $1 - k$. Now, in order to derive the former of these serieses from the latter, it will be convenient to premise the following lemma.

L E M M A II.

21. If x be any quantity less than 1 , and n denote any whole number whatsoever, the n^{th} power of the residual quantity $1 - x$, or of the excess of 1 above x , will be equal to the following series of terms, to wit, $1 - \frac{n}{1} \times x + \frac{n}{1} \times$

$\frac{n-1}{2} \times x^2 - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times x^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times x^4$
 $- \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times x^5 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times$
 $\frac{n-4}{5} \times \frac{n-5}{6} \times x^6 - \mathcal{E}c$, continued to $n+1$ terms, or (if we put the capital letters A, B, C, D, E, F, G, $\mathcal{E}c$ for the first term 1, and the several co-efficients of $x, x^2, x^3, x^4, x^5, x^6$, and the following powers of x respectively) to the series $1 -$
 $\frac{n}{1} A x + \frac{n-1}{2} B x^2 - \frac{n-2}{3} C x^3 + \frac{n-3}{4} D x^4 - \frac{n-4}{5} E x^5 + \frac{n-5}{6} F x^6$
 $- \mathcal{E}c$; in which the law of continuation of the terms is very manifest, to wit, that the terms are alternately positive and negative, or that the 3rd, 5th, 7th, and all the following odd terms, are to be added to the first term 1, and the 2nd, 4th, 6th, and other following even terms, are to be subtracted from it; and, with respect to the co-efficients of $x, x^2, x^3, x^4, x^5, x^6, \mathcal{E}c$, that they are generated from the first term 1 by the continual multiplication of the generating fractions $\frac{n}{1}, \frac{n-1}{2}, \frac{n-2}{3}, \frac{n-3}{4}, \frac{n-4}{5}, \frac{n-5}{6}, \mathcal{E}c$, of which the numerators $n, n-1, n-2, n-3, n-4, n-5, \mathcal{E}c$, are formed from each other by the continual subtraction of an unit, and the denominators 1, 2, 3, 4, 5, 6, $\mathcal{E}c$, are formed from each other by the continual addition of an unit.

22. This proposition may be demonstrated in the same manner as the celebrated binomial theorem, of which it is usually considered as a branch. On the present occasion I shall take it for granted, as I did the binomial theorem itself in article 3.

23. Coroll. 1. If x be equal to the n^{th} part of any given quantity called b , so that $1 - x$ shall be equal to $1 - \frac{b}{n}$, it will follow that $\overline{1 - x^n}$, or $\overline{1 - \frac{b^n}{n^n}}$, will be equal to the series $1 - \frac{n}{1} \times \frac{b}{n} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{b^2}{n^2} - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{b^3}{n^3}$
 $+ \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{b^4}{n^4} - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{b^5}{n^5}$
 $+ \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6} \times \frac{b^6}{n^6} - \mathcal{E}c$, continued to $n+1$ terms, or to the series $1 - \frac{n}{1} A \times \frac{b}{n} + \frac{n-1}{2} B \times \frac{b^2}{n^2} -$
 $\frac{n-2}{3} C \times \frac{b^3}{n^3} + \frac{n-3}{4} D \times \frac{b^4}{n^4} - \frac{n-4}{5} E \times \frac{b^5}{n^5} + \frac{n-5}{6} F \times \frac{b^6}{n^6} - \mathcal{E}c$, continued to $n+1$ terms.

24. Coroll. 2. Whatever be the magnitude of n , the quantity $\overline{1 - \frac{b^n}{n^n}}$ will be greater than the series $1 - b + \frac{b^2}{2} - \frac{b^3}{2 \cdot 3} + \frac{b^4}{2 \cdot 3 \cdot 4} - \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \mathcal{E}c$, continued to $n+1$ terms.

For, by the foregoing corollary, we have $1 - \frac{b}{n} = 1 - \frac{n}{1} \times \frac{b}{n} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{b^2}{n^2} - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{b^3}{n^3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{b^4}{n^4} - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{b^5}{n^5} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6} \times \frac{b^6}{n^6} - \mathcal{E}c$, continued to $n+1$ terms; which is =

$$1 - b + \frac{n^2 - n}{n^2} \times \frac{b^2}{2} - \frac{n^3 - 3n^2 + 2n}{n^3} \times \frac{b^3}{2.3} + \frac{n^4 - 6n^3 + 11n^2 - 6n}{n^4} \times \frac{b^4}{2.3.4} - \frac{n^5 - 10n^4 + 35n^3 - 50n^2 + 24n}{n^5} \times \frac{b^5}{2.3.4.5} + \frac{n^6 - 15n^5 + 85n^4 - 225n^3 + 274n^2 - 120n}{n^6} \times \frac{b^6}{2.3.4.5.6} - \mathcal{E}c$$
, continued to $n+1$ terms, or to $1 - b + 1 - \frac{1}{n} \times \frac{b^2}{2} - \left[1 - \frac{3}{n} + \frac{2}{n^2}\right] \times \frac{b^3}{2.3} + \left[1 - \frac{6}{n} + \frac{11}{n^2} - \frac{6}{n^3}\right] \times \frac{b^4}{2.3.4} - \left[1 - \frac{10}{n} + \frac{35}{n^2} - \frac{50}{n^3} + \frac{24}{n^4}\right] \times \frac{b^5}{2.3.4.5} + \left[1 - \frac{15}{n} + \frac{85}{n^2} - \frac{225}{n^3} + \frac{274}{n^4} - \frac{120}{n^5}\right] \times \frac{b^6}{2.3.4.5.6} + \mathcal{E}c$, continued to $n+1$ terms, or to the series $1 - b + \frac{b^2}{2} - \frac{b^3}{2.3} + \frac{b^4}{2.3.4} - \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6} - \mathcal{E}c$, continued to $n+1$ terms, — the series $\frac{1}{n} \times \frac{b^2}{2} - \left[\frac{3}{n} - \frac{2}{n^2}\right] \times \frac{b^3}{2.3} + \left[\frac{6}{n} - \frac{11}{n^2} + \frac{6}{n^3}\right] \times \frac{b^4}{2.3.4} - \left[\frac{10}{n} - \frac{35}{n^2} + \frac{50}{n^3} - \frac{24}{n^4}\right] \times \frac{b^5}{2.3.4.5} + \left[\frac{15}{n} - \frac{85}{n^2} + \frac{225}{n^3} - \frac{274}{n^4} + \frac{120}{n^5}\right] \times \frac{b^6}{2.3.4.5.6} - \mathcal{E}c$, continued to $n-1$ terms.

Therefore $1 - \frac{b}{n}$ is greater than the series $1 - b + \frac{b^2}{2} - \frac{b^3}{2.3} + \frac{b^4}{2.3.4} - \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6} - \mathcal{E}c$, continued to $n+1$ terms, by an excess, or difference, equal to the series $\frac{1}{n} \times \frac{b^2}{2} - \left[\frac{3}{n} - \frac{2}{n^2}\right] \times \frac{b^3}{2.3} + \left[\frac{6}{n} - \frac{11}{n^2} + \frac{6}{n^3}\right] \times \frac{b^4}{2.3.4} - \left[\frac{10}{n} - \frac{35}{n^2} + \frac{50}{n^3} - \frac{24}{n^4}\right] \times \frac{b^5}{2.3.4.5} + \left[\frac{15}{n} - \frac{85}{n^2} + \frac{225}{n^3} - \frac{274}{n^4} + \frac{120}{n^5}\right] \times \frac{b^6}{2.3.4.5.6} - \mathcal{E}c$, continued to $n-1$ terms. Q. E. D.

25. Coroll. 3. But, though the quantity $1 - \frac{b}{n}$ be always greater than the series $1 - b + \frac{b^2}{2} - \frac{b^3}{2.3} + \frac{b^4}{2.3.4} - \frac{b^5}{2.3.4.5} + \frac{b^6}{2.3.4.5.6} - \mathcal{E}c$, continued to $n+1$ terms, yet the difference between them will continually decrease while the index n increases; and the said index n may be taken of so great a magnitude that the said difference shall become less than any assigned quantity, how small soever.

For it has been shewn in the preceding corollary, that this difference is equal to the series $\frac{1}{n} \times \frac{b^2}{2} - \left[\frac{3}{n} - \frac{2}{n^2}\right] \times \frac{b^3}{2.3} + \left[\frac{6}{n} - \frac{11}{n^2} + \frac{6}{n^3}\right] \times \frac{b^4}{2.3.4} -$

$\left[\frac{10}{n} - \frac{35}{n^2} + \frac{50}{n^3} - \frac{24}{n^4} \right] \times \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \left[\frac{15}{n} - \frac{85}{n^2} + \frac{225}{n^3} - \frac{274}{n^4} + \frac{120}{n^5} \right] \times \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$
 — $\mathcal{E}c$, continued to $n - 1$ terms. Now it is evident, that this series will continually decrease while n increases; because the powers of n constitute the denominators of the several members of the co-efficients of its terms; and n may be taken of so great a magnitude that the said series shall be less than any assigned quantity, how small soever. Therefore, while the index n increases, the difference whereby the quantity $1 - \frac{b^n}{n}$ exceeds the series $1 - b + \frac{b^2}{2} - \frac{b^3}{2 \cdot 3} + \frac{b^4}{2 \cdot 3 \cdot 4} - \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \mathcal{E}c$, continued to $n + 1$ terms, will decrease continually *ad infinitum*, or the index n may be taken of so great a magnitude that the said difference shall be less than any assigned quantity, how small soever.
 Q. E. D.

26. Coroll. 4. It follows, from the foregoing corollary, that the infinite series $1 - b + \frac{b^2}{2} - \frac{b^3}{2 \cdot 3} + \frac{b^4}{2 \cdot 3 \cdot 4} - \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \mathcal{E}c$ *ad infinitum*, is the limit of the magnitude of $1 - \frac{b^n}{n}$, or the quantity to which $1 - \frac{b^n}{n}$ may, by increasing continually the index n , be made to come as near as we please.

For, by increasing the index n continually, the excess of the quantity $1 - \frac{b^n}{n}$ above the series $1 - b + \frac{b^2}{2} - \frac{b^3}{2 \cdot 3} + \frac{b^4}{2 \cdot 3 \cdot 4} - \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \mathcal{E}c$, continued to $n + 1$ terms, may be made to decrease till it is less than any assigned quantity, how small soever, as we have seen in the foregoing corollary. And, by increasing the said index n continually, it is evident that the difference between the said series $1 - b + \frac{b^2}{2} - \frac{b^3}{2 \cdot 3} + \frac{b^4}{2 \cdot 3 \cdot 4} - \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \mathcal{E}c$, continued to $n + 1$ terms, and the same series, continued *ad infinitum*, may be diminished as far as we please, or made to become less than any assigned quantity, how small soever. It follows therefore, that by increasing the index n continually, the difference of the quantity $1 - \frac{b^n}{n}$ and the infinite series $1 - b + \frac{b^2}{2} - \frac{b^3}{2 \cdot 3} + \frac{b^4}{2 \cdot 3 \cdot 4} - \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \mathcal{E}c$ *ad infinitum* (which will be equal to the sum of the said excess of $1 - \frac{b^n}{n}$ above the series $1 - b + \frac{b^2}{2} - \frac{b^3}{2 \cdot 3} + \frac{b^4}{2 \cdot 3 \cdot 4} - \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \mathcal{E}c$, continued to $n + 1$ terms, and of the excess of the said finite series above the same series continued *ad infinitum*, in case the said finite series is greater than the same series continued *ad infinitum*; but otherwise will be equal to the difference between the said excess of $1 - \frac{b^n}{n}$ above the said finite series and the excess of the said infinite series above the said finite series,) may be diminished as far as we please, or made to become less than any assigned quantity, how small soever; or, in other words, the infinite series $1 - b +$

$b + \frac{b^2}{2} - \frac{b^3}{2 \cdot 3} + \frac{b^4}{2 \cdot 3 \cdot 4} - \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \mathcal{E}c \text{ ad infinitum}$, is the limit of the magnitude of the quantity $1 - \frac{b}{n}$. Q. E. D.

27. Coroll. 5. If we make use of the language of infinites, the substance of the last, or 4th corollary may be expressed in the manner following.

When the index n of the power to which the residual quantity $1 - \frac{b}{n}$ is to be raised, becomes infinite, the series $1 - \frac{n}{1} \times \frac{b}{n} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{b^2}{n^2} - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{b^3}{n^3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{b^4}{n^4} - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{b^5}{n^5} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6} \times \frac{b^6}{n^6} - \mathcal{E}c$, continued to $n + 1$ terms (to which the quantity $1 - \frac{b}{n}$ is equal,) or its equal, the series $1 - \frac{n}{n} \times b + \frac{nn-n}{nn} \times \frac{b^2}{2} - \frac{n^3-3n^2+2n}{n^3} \times \frac{b^3}{2 \cdot 3} + \frac{n^4-6n^3+11n^2-6n}{n^4} \times \frac{b^4}{2 \cdot 3 \cdot 4} - \frac{n^5-10n^4+35n^3-50n^2+24n}{n^5} \times \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{n^6-15n^5+85n^4-225n^3+274n^2-120n}{n^6} \times \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \mathcal{E}c$, continued to $n + 1$ terms, will become infinite in the number of its terms, and will be equal to the series $1 - \frac{n}{n} \times b + \frac{n^2}{n^2} \times \frac{b^2}{2} - \frac{n^3}{n^3} \times \frac{b^3}{2 \cdot 3} + \frac{n^4}{n^4} \times \frac{b^4}{2 \cdot 3 \cdot 4} - \frac{n^5}{n^5} \times \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{n^6}{n^6} \times \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \mathcal{E}c \text{ ad infinitum}$, or to the series $1 - b + \frac{b^2}{2} - \frac{b^3}{2 \cdot 3} + \frac{b^4}{2 \cdot 3 \cdot 4} - \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \mathcal{E}c \text{ ad infinitum}$; because all the members of the several numerators $nn - n$, $n^3 - 3n^2 + 2n$, $n^4 - 6n^3 + 11n^2 - 6n$, $n^5 - 10n^4 + 35n^3 - 50n^2 + 24n$, and $n^6 - 15n^5 + 85n^4 - 225n^3 + 274n^2 - 120n$, $\mathcal{E}c$, after the first members n^2 , n^3 , n^4 , n^5 , n^6 , $\mathcal{E}c$, will vanish, or become infinitely small, in respect of the said first members, which contain a higher power of n than the said following members, and consequently those whole numerators will be equal to their said first members n^2 , n^3 , n^4 , n^5 , n^6 , $\mathcal{E}c$, respectively. Therefore the quantity $1 - \frac{b}{n}$ (which is in all cases equal to the said series $1 - \frac{n}{n} \times b + \frac{n^2-n}{n^2} \times \frac{b^2}{2} - \frac{n^3-3n^2+2n}{n^3} \times \frac{b^3}{2 \cdot 3} + \frac{n^4-6n^3+11n^2-6n}{n^4} \times \frac{b^4}{2 \cdot 3 \cdot 4} - \frac{n^5-10n^4+35n^3-50n^2+24n}{n^5} \times \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{n^6-15n^5+85n^4-225n^3+274n^2-120n}{n^6} \times \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \mathcal{E}c$, continued to $n + 1$ terms,) will, in this case of the infinite magnitude of n , be equal to the infinite series $1 - b + \frac{b^2}{2} - \frac{b^3}{2 \cdot 3} + \frac{b^4}{2 \cdot 3 \cdot 4} - \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{b^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \mathcal{E}c \text{ ad infinitum}$. Q. E. D.

28. These

28. These things being premised, the investigation of the second anti-logarithmick series $L = \frac{L^2}{2} + \frac{L^3}{2 \cdot 3} - \frac{L^4}{2 \cdot 3 \cdot 4} + \frac{L^5}{2 \cdot 3 \cdot 4 \cdot 5} - \frac{L^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \mathcal{E}c \text{ ad infinitum}$, or the derivation of it from Dr. Wallis's logarithmick series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c \text{ ad infinitum}$, may be performed in the manner following.

PROBLEM II.

Let k be any quantity less than 1, and let L be equal to the infinite series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c \text{ ad infinitum}$; which is the logarithm of the ratio of 1 to $1 - k$ in Napier's system of logarithms. And let L , or the value of this whole series, be supposed to be known, but k , or the value of its first term alone, and consequently the separate values of all its other terms (which contain the powers of k), to be unknown. It is required to find from L , or the value of the whole series, the value of its first term k .

SOLUTION.

Let us suppose the letter n to denote some very great number, such as a nonillion, or the ninth power of a million.

Then it will follow, from lemma 3, coroll. 2, art. 65, of the foregoing remarks on Mercator's and Wallis's serieses, that $1 - \sqrt[n]{1 - k}$ will be very nearly equal to $\frac{1}{n} \times$ the infinite series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c \text{ ad infinitum}$, that is, (because L is supposed to be equal to the said series,) very nearly equal to $\frac{1}{n} \times L$, or to $\frac{L}{n}$. Therefore, if we add $\sqrt[n]{1 - k}$ to both sides, we shall have 1 very nearly $= \frac{L}{n} + \sqrt[n]{1 - k}$, and (subtracting $\frac{L}{n}$ from both sides) $\sqrt[n]{1 - k}$ very nearly $= 1 - \frac{L}{n}$, and (raising both sides of this last equation to the n^{th} power) $1 - k$ very nearly equal $1 - \frac{L^n}{n^n}$. And n may be taken of so great a magnitude that the ratio of $1 - k$ to $1 - \frac{L^n}{n^n}$ shall approach as near as we please to a ratio of equality.

But, because the index n is of an immensely-great magnitude, it follows, from the fourth corollary of the foregoing lemma, that $1 - \frac{L^n}{n^n}$ will be very nearly equal to the infinite series $1 - L + \frac{L^2}{2} - \frac{L^3}{2 \cdot 3} + \frac{L^4}{2 \cdot 3 \cdot 4} - \frac{L^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{L^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \mathcal{E}c \text{ ad infinitum}$, and that n may be taken of so great a magnitude that the difference

ference between $1 - \frac{L}{n}$ and the said series shall be less than any assigned quantity, how small soever. Therefore $1 - k$ will be very nearly equal to the said infinite series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \mathcal{E}c$ *ad infinitum*, and consequently (adding k to both sides) 1 will be very nearly equal to $k + 1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \mathcal{E}c$ *ad infinitum*, and (adding the series $L + \frac{L^3}{2.3} + \frac{L^5}{2.3.4.5} + \mathcal{E}c$ *ad infinitum* to both sides) $1 + L + \frac{L^3}{2.3} + \frac{L^5}{2.3.4.5} + \mathcal{E}c$ *ad infinitum*, will be very nearly equal to $k + 1 + \frac{L^2}{2} + \frac{L^4}{2.3.4} + \frac{L^6}{2.3.4.5.6} + \mathcal{E}c$ *ad infinitum*, and (subtracting 1 from both sides) $L + \frac{L^3}{2.3} + \frac{L^5}{2.3.4.5} + \mathcal{E}c$ *ad infinitum*, will be very nearly equal to $k + \frac{L^2}{2} + \frac{L^4}{2.3.4} + \frac{L^6}{2.3.4.5.6} + \mathcal{E}c$ *ad infinitum*, and, lastly, (subtracting the infinite series $\frac{L^2}{2} + \frac{L^4}{2.3.4} + \frac{L^6}{2.3.4.5.6} + \mathcal{E}c$ *ad infinitum* from both sides,) k will be very nearly equal to the infinite series $L - \frac{L^2}{2} + \frac{L^3}{2.3} - \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} - \frac{L^6}{2.3.4.5.6} + \mathcal{E}c$ *ad infinitum*. And since, by increasing the magnitude of n continually, the quantity $1 - k$ may be made to approach as near as we please to an equality with the infinite series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \mathcal{E}c$ *ad infinitum*, it follows, that it is not only very nearly, but accurately, equal to the said infinite series, and consequently that k is not only very nearly, but accurately, equal to the infinite series $L - \frac{L^2}{2} + \frac{L^3}{2.3} - \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} - \frac{L^6}{2.3.4.5.6} + \mathcal{E}c$ *ad infinitum*. Q. E. I.

11. This solution may be expressed with greater brevity in the language of infinities, as follows.

Let us suppose the letter n to denote a number infinitely great.

Then it will follow, from lemma 3, coroll. 2, art. 65, of the foregoing remarks on Mercator's and Wallis's serieses, that $1 - \left(1 - \frac{k}{n}\right)^{\frac{1}{n}}$ will be equal to $\frac{1}{n} \times$ the infinite series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c$ *ad infinitum*, that is, (because the said series is $= L$), to $\frac{1}{n} \times L$, or to $\frac{L}{n}$. Therefore (adding $\left(1 - \frac{k}{n}\right)^{\frac{1}{n}}$ to both sides) we shall have $1 = \frac{L}{n} + \left(1 - \frac{k}{n}\right)^{\frac{1}{n}}$, and (subtracting $\frac{L}{n}$ from

from both sides) $1 - \frac{L}{n} = \sqrt[n]{1 - k}$, or $\sqrt[n]{1 - k} = 1 - \frac{L}{n}$, and consequently (raising both sides of this last equation to the n^{th} power,) $1 - k = \left(1 - \frac{L}{n}\right)^n$.

But, because the index n is an infinitely-great number, it follows, from coroll. 5 of the foregoing lemma, that $\sqrt[n]{1 - k}$ will be equal to the infinite series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \mathcal{E}c \text{ ad infinitum}$. Therefore $1 - k$ will be equal to the same infinite series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \mathcal{E}c \text{ ad infinitum}$; and consequently (adding k to both sides of the equation) 1 will be $= k + 1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \mathcal{E}c \text{ ad infinitum}$; and (adding $L + \frac{L^3}{2.3} + \frac{L^5}{2.3.4.5} + \mathcal{E}c \text{ ad infinitum}$ to both sides,) $1 + L + \frac{L^3}{2.3} + \frac{L^5}{2.3.4.5} + \mathcal{E}c \text{ ad infinitum}$ will be $= k + 1 + \frac{L^2}{2} + \frac{L^4}{2.3.4} + \frac{L^6}{2.3.4.5.6} + \mathcal{E}c \text{ ad infinitum}$; and (subtracting 1 from both sides) $L + \frac{L^3}{2.3} + \frac{L^5}{2.3.4.5} + \mathcal{E}c \text{ ad infinitum}$ will be $= k + \frac{L^2}{2} + \frac{L^4}{2.3.4} + \frac{L^6}{2.3.4.5.6} + \mathcal{E}c \text{ ad infinitum}$; and, lastly, (subtracting the infinite series $\frac{L^2}{2} + \frac{L^4}{2.3.4} + \frac{L^6}{2.3.4.5.6} + \mathcal{E}c \text{ ad infinitum}$ from both sides,) k will be $=$ the infinite series $L - \frac{L^2}{2} + \frac{L^3}{2.3} - \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} - \frac{L^6}{2.3.4.5.6} + \mathcal{E}c \text{ ad infinitum}$. Q. E. I.

30. Coroll. 1. Since k is equal to the infinite series $L - \frac{L^2}{2} + \frac{L^3}{2.3} - \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} - \frac{L^6}{2.3.4.5.6} + \mathcal{E}c \text{ ad infinitum}$, it follows that the ratio of 1 to $1 - k$ will be equal to the ratio of 1 to $1 -$ the infinite series $L - \frac{L^2}{2} + \frac{L^3}{2.3} - \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} - \frac{L^6}{2.3.4.5.6} + \mathcal{E}c \text{ ad infinitum}$, or of 1 to the infinite series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \mathcal{E}c \text{ ad infinitum}$; or the ratio corresponding to any given logarithm in Napier's system denoted by L is the ratio of 1 to the infinite series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \mathcal{E}c \text{ ad infinitum}$.

An Example of the foregoing Method of discovering the Ratio corresponding to a given Logarithm L in Napier's System, by Means of the Infinite Series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \&c$ ad infinitum.

31. Let the given logarithm, of which we are to find the ratio, be $\frac{1}{2}$, or 0.500,000,000.

Then will L^2 be $= \frac{1}{4}$, $L^3 = \frac{1}{8}$, $L^4 = \frac{1}{16}$, $L^5 = \frac{1}{32}$, $L^6 = \frac{1}{64}$, and $L^7, L^8, L^9, L^{10}, \&c$ will be equal to the 7th, 8th, 9th, 10th, and other following powers of $\frac{1}{2}$, as in the example given above in article 13. Therefore the series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \&c$ ad infinitum will in this case be equal to 1.000,000,000 $-$ 0.500,000,000 $+$ 0.125,000,000 $-$ 0.020,833,333 $+$ 0.002,604,166 $-$ 0.000,260,416 $+$ 0.000,021,701 $-$ 0.000,001,550 $+$ 0.000,000,096 $-$ 0.000,000,005 $+$ 0.000,000,000 $- \&c = 1.127,625,963 - 0.521,095,304 = 0.606,530,659$. And consequently the ratio sought, or that of which $\frac{1}{2}$, or 0.500,000,000, is the logarithm in Napier's system, is that of 1 to 0.606,530,659. Q. E. I.

32. This ratio of 1 to 0.606,530,659 is the same with that which was found above in article 13 for the ratio corresponding to the same logarithm $\frac{1}{2}$, or 0.500,000,000, to wit, the ratio of 1.648,721,267 to 1.

For 1.648,721,267 is to 1 as 1 is to $\frac{1.000,000,000,000,000,000,0}{1.648,721,267} = 0.606,530,661$; which exceeds 0.606,530,659 by only 0.000,000,002, or $\frac{2}{1.000,000,000}$, or $\frac{1}{500,000,000}$, or one five-hundred-millionth part of an unit.

33. This equality of the two ratios which are found to be correspondent to the same logarithm $\frac{1}{2}$, or 0.500,000,000, by means of the two serieses $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \&c$ ad infinitum, and $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \&c$ ad infinitum, which were derived from the two logarithmick serieses $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$ ad infinitum, and $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \&c$ ad infinitum, invented by Mr. Mercator and Dr. Wallis, affords an additional proof of the truth of what is asserted in theorem 3, article 83, of the foregoing remarks on those two serieses,

serieses, to wit, that the logarithms exhibited by the said two serieses of Mercator and Wallis belong to the same system.

Another Example of the foregoing Method of discovering the Ratio corresponding to a given Logarithm L in Napier's System by Means of the Infinite Series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \&c$ ad infinitum.

34. Let the given logarithm L be equal to 1. It is required to find the ratio of which it is the logarithm by means of the series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \&c$ ad infinitum.

Now, when L is = 1, we shall have $L^2 = 1$, and $L^3 = 1$, and $L^4 = 1$, and every following power of $L = 1$; and consequently the series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \&c$ ad infinitum, will in this case become equal to the series $1 - 1 + \frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3.4} - \frac{1}{2.3.4.5} + \frac{1}{2.3.4.5.6} - \&c$ ad infinitum, or $\frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3.4} - \frac{1}{2.3.4.5} + \frac{1}{2.3.4.5.6} - \&c$ ad infinitum, the several terms of which series have been already found, in article 16, to be as follows.

$$C = \frac{B}{2} = 0.500,000,000,000,000,000,000,000;$$

$$D = \frac{C}{3} = 0.166,666,666,666,666,666,666,666;$$

$$E = \frac{D}{4} = 0.041,666,666,666,666,666,666,666;$$

$$F = \frac{E}{5} = 0.008,333,333,333,333,333,333,333;$$

$$G = \frac{F}{6} = 0.001,388,888,888,888,888,888,888;$$

$$H = \frac{G}{7} = 0.000,198,412,698,412,698,412,698;$$

$$I = \frac{H}{8} = 0.000,024,801,587,301,587,301,587;$$

$$K = \frac{I}{9} = 0.000,002,755,731,922,398,589,065;$$

$$L = \frac{K}{10} = 0.000,000,275,573,192,239,858,906;$$

$$M = \frac{L}{11} = 0.000,000,025,052,108,385,441,718;$$

$$N = \frac{M}{12} = 0.000,000,002,087,675,698,786,809;$$

$$O = \frac{N}{13} = 0.000,000,000,160,590,438,368,216;$$

$$P = \frac{O}{14} = 0.000,000,000,011,470,745,597,729;$$

$$Q = \frac{P}{15} = 0.000,000,000,000,764,716,373,181;$$

$$R = \frac{Q}{16} = 0.000,000,000,000,047,794,773,323;$$

$$S = \frac{R}{17} = 0.000,000,000,000,002,811,457,254;$$

$$T = \frac{S}{18} = 0.000,000,000,000,000,156,192,069;$$

$$V = \frac{T}{19} = 0.000,000,000,000,000,008,220,635;$$

$$W = \frac{V}{20} = 0.000,000,000,000,000,000,411,031;$$

$$X = \frac{W}{21} = 0.000,000,000,000,000,000,019,572;$$

$$Y = \frac{X}{22} = 0.000,000,000,000,000,000,000,889;$$

$$Z = \frac{Y}{23} = 0.000,000,000,000,000,000,000,038;$$

$$A' = \frac{Z}{24} = 0.000,000,000,000,000,000,000,001.$$

Therefore $\frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3.4} - \frac{1}{2.3.4.5} + \frac{1}{2.3.4.5.6} - \mathcal{E}c \text{ ad infinitum, or } \frac{1}{2}$
 $- \frac{C}{3} + \frac{D}{4} - \frac{E}{5} + \frac{F}{6} - \frac{G}{7} + \frac{H}{8} - \frac{I}{9} + \frac{K}{10} - \frac{L}{11} + \frac{M}{12} - \frac{N}{13} + \frac{O}{14} - \frac{P}{15}$
 $+ \frac{Q}{16} - \frac{R}{17} + \frac{S}{18} - \frac{T}{19} + \frac{V}{20} - \frac{W}{21} + \frac{X}{22} - \frac{Y}{23} + \frac{Z}{24} - \mathcal{E}c \text{ ad infinitum, or}$
 $\frac{1}{2} + \frac{D}{4} + \frac{F}{6} + \frac{H}{8} + \frac{K}{10} + \frac{M}{12} + \frac{O}{14} + \frac{Q}{16} + \frac{S}{18} + \frac{V}{20} + \frac{X}{22} + \frac{Z}{24} + \mathcal{E}c -$
 $\frac{C}{3} - \frac{E}{5} - \frac{G}{7} - \frac{I}{9} - \frac{L}{11} - \frac{N}{13} - \frac{P}{15} - \frac{R}{17} - \frac{T}{19} - \frac{W}{21} - \frac{Y}{23} - \mathcal{E}c, \text{ will be } =$

$$\begin{aligned} & 0;500,000,000,000,000,000,000,000, \\ & + ;.41,666,666,666,666,666,666,666, \\ & + ;..1,388,888,888,888,888,888,888, \\ & + ;...24,801,587,301,587,301,587, \\ & + ;....275,573,192,239,858,906, \\ & + ;.....2,087,675,698,786,809, \\ & + ;.....11,470,745,597,729, \\ & + ;.....47,794,773,323, \\ & + ;.....156,192,069, \\ & + ;.....411,031, \\ & + ;.....889, \\ & + ;.....1, \\ & + \mathcal{E}c. \end{aligned}$$

[illegible]

0.543,080,634,815,243,778,477,898, ‰ — 0.175,201,193,643,801,456,882,376, ‰ = 0.367,879,441,171,442,321,595,522, ‰ . Therefore the ratio sought, or that of which 1 is the logarithm in Napier's system, is that of 1 to the decimal fraction 0.367,879,441,171,442,321,595,522, ‰ . Q. E. I.

35. This ratio is equal to the ratio of 2.718,281,828,459,045,235,360,274, &c to 1; which was found above, in article 16, to be the ratio of which 1 was the logarithm in the system of logarithms exhibited by Mercator's series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$ ad infinitum. For 2.718,281,828,459,045,235,360,274 is to 1 as 1 is to $\frac{1.000,000,000,000,000,000,000,000,000,000, \&c}{2.718,281,828,459,045,235,360,274} = 0.367,879,441,171,442,321,595,525$; which agrees with the foregoing number 0.367,879,441,171,442,321,595,522, in all the figures but the last.

The operation of this division of 1.000,000,000,000,000,000,000,000,000,000,
Éc by the long number 2.718,281,828,459,045,235,360,274, is as follows.

In order to avoid mistakes in performing so long and laborious a division, it will be expedient, before we enter upon it, to find all the products that can arise by multiplying the divisor 2.718,281,828,459,045,235,360,274 into the eight first numbers 2, 3, 4, 5, 6, 7, 8, and 9, by the more simple operation of addition, in which it is hardly possible that we should make any slips. This may be done in the following manner, the said divisor 2.718,281,828,459,045,235,360,274 being (for brevity's sake) denoted by the capital letter A.

$$\begin{array}{r}
 2.718,281,828,459,045,235,360,274 = A. \\
 2.718,281,828,459,045,235,360,274 \\
 \hline
 2 A = 5.436,563,656,918,090,470,720,548 \\
 2.718,281,828,459,045,235,360,274 \\
 \hline
 3 A = 8.154,845,485,377,135,706,080,822 \\
 2.718,281,828,459,045,235,360,274 \\
 \hline
 4 A = 10.873,127,313,836,180,941,441,096 \\
 2.718,281,828,459,045,235,360,274 \\
 \hline
 5 A = 13.591,409,142,295,226,176,801,370 \\
 2.718,281,828,459,045,235,360,274
 \end{array}$$

$$\begin{aligned}
 6 A &= 16.309,690,970,754,271,412,161,644 \\
 &\quad 2.718,281,828,459,045,235,360,274 \\
 7 A &= 19.027,972,799,213,316,647,521,918 \\
 &\quad 2.718,281,828,459,045,235,360,274 \\
 8 A &= 21.746,254,627,672,361,882,882,192 \\
 &\quad 2.718,281,828,459,045,235,360,274 \\
 9 A &= 24.464,536,456,131,407,118,242,466
 \end{aligned}$$

These several products being thus previously obtained in a manner hardly liable to error, the operation of dividing 1, or 1.000,000,000,000,000,000,000,000,000,000,000,000, &c, by the long number 2.718,281,828,459,045,235,360,274, may be performed as follows.

The Division of 1, or 1.000,000,000,000,000,000,000,000,000,000,000,000, &c, by 2.718,281,828,459,045,235,360,274.

The divisor. The quotient.
 2.718,281,828,459,045,235,360,274) (0.367,879,441,171,442,321,595,525

The dividend.

$$\begin{aligned}
 &1.000,000,000,000,000,000,000,000,000,000,000,000, \&c. \\
 &8154845485377135706080822 \\
 &\cdot 18451545146228642939191780 \\
 &16309690970754271412161644 \\
 &\cdot 21418541754743715270301360 \\
 &19027972799213316647521918 \\
 &\cdot 23905689555303986227794420 \\
 &21746254627672361882882192 \\
 &\cdot 21594349276316243449122280 \\
 &19027972799213316647521918 \\
 &\cdot 25663764771029268016003620 \\
 &24464536456131407118242466 \\
 &\cdot 11992283148978608977611540 \\
 &10873127313836180941441096 \\
 &\cdot 11191558351424280361704440 \\
 &10873127313836180941441096 \\
 &\cdot 3184310375880994202633440 \\
 &2718281828459045235360274 \\
 &\cdot 4660285474219489672731660 \\
 &2718281828459045235360274 \\
 &\cdot 19420036457604444373713860 \\
 &19027972799213316647521918 \\
 &\cdot 3920636583911277261919420 \\
 &2718281828459045235360274 \\
 &\cdot 12023547554522320265591460 \\
 &10873127313836180941441096 \\
 &1.1504202406861393241503640
 \end{aligned}$$

$$\begin{array}{r}
 .11504202406861393241503640 \\
 10873127313836180941441096 \\
 \hline
 ..6310750930252123000625440 \\
 5436563656918090470720548 \\
 \hline
 .8741872733340325299048920 \\
 8154845485377135706080822 \\
 \hline
 .5870272479631895929680980 \\
 5436563656918090470720548 \\
 \hline
 .4337088227138054589604320 \\
 2718281828459045235360274 \\
 \hline
 16188063986790093542440460 \\
 13591409142295226176801370 \\
 \hline
 .25966548444948673656390900 \\
 24464536456131407118242466 \\
 \hline
 .15020119888172665381484340 \\
 13591409142295226176801370 \\
 \hline
 .14287107458774392046829700 \\
 13591409142295226176801370 \\
 \hline
 ..6956983164791658700283300 \\
 5436563656918090470720548 \\
 \hline
 15204195078735682295627520 \\
 13591409142295226176801370 \\
 \hline
 .1612785936440456118826150
 \end{array}$$

36. This equality of these two ratios, of which 1 is the logarithm in the two systems of logarithms exhibited by the two serieses $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \text{\&c ad infinitum}$, and $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \text{\&c ad infinitum}$, invented by Mr. Mercator and Dr. Wallis, affords another proof of the truth of what is asserted in theorem 3, article 83, of the foregoing remarks on those two serieses, to wit, that the logarithms exhibited by both serieses belong to one and the same system.

A general Demonstration of the Equality between the Ratio of the Infinite Series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \text{\&c}$ to 1, and the Ratio of 1 to the Infinite Series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \text{\&c}$.

37. But the equality of the two ratios that are found, by means of the two infinite serieses $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \text{\&c ad infinitum}$, and $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \text{\&c ad infinitum}$, to belong to the same logarithm L, or the equality of the ratio of 1 to the infinite series

series $1 - L + \frac{L^2}{2} - \frac{L^3}{2 \cdot 3} + \frac{L^4}{2 \cdot 3 \cdot 4} - \frac{L^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{L^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \text{Ec ad infinitum}$ to the ratio of the infinite series $1 + L + \frac{L^2}{2} + \frac{L^3}{2 \cdot 3} + \frac{L^4}{2 \cdot 3 \cdot 4} + \frac{L^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{L^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \text{Ec ad infinitum}$ to 1, will appear in the clearest manner by dividing 1 by the infinite series $1 - L + \frac{L^2}{2} - \frac{L^3}{2 \cdot 3} + \frac{L^4}{2 \cdot 3 \cdot 4} - \frac{L^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{L^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \text{Ec}$. For it will be found that the quotient of this division will be the other series $1 + L + \frac{L^2}{2} + \frac{L^3}{2 \cdot 3} + \frac{L^4}{2 \cdot 3 \cdot 4} + \frac{L^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{L^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \text{Ec}$; from whence it follows, that the ratio of 1 to the former series, (which is the divisor of this operation of division,) will be equal to the ratio of the latter series (which is the quotient of the same division) to 1. This division may be performed in the manner following.

The divisor.

$$1 - L + \frac{L^2}{2} - \frac{L^3}{2 \cdot 3} + \frac{L^4}{2 \cdot 3 \cdot 4} - \frac{L^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{L^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \text{Ec}.$$

The quotient.

$$1 + L + \frac{L^2}{2} + \frac{L^3}{2 \cdot 3} + \frac{L^4}{2 \cdot 3 \cdot 4} + \frac{L^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{L^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \text{Ec}.$$

The dividend.

1	*	*	*	*	*	*	*
1	-	L	+	$\frac{L^2}{2}$	-	$\frac{L^3}{2 \cdot 3}$	+
1	-	L	+	$\frac{L^2}{2}$	-	$\frac{L^3}{2 \cdot 3}$	+
*	+	L	-	$\frac{L^2}{2}$	+	$\frac{L^3}{2 \cdot 3}$	-
*	+	L	-	$\frac{L^2}{2}$	+	$\frac{L^3}{2 \cdot 3}$	-
*	+	$\frac{L^2}{2}$	-	$\frac{2L^3}{2 \cdot 3}$	+	$\frac{3L^4}{2 \cdot 3 \cdot 4}$	-
*	+	$\frac{L^2}{2}$	-	$\frac{L^3}{2}$	+	$\frac{L^4}{2 \cdot 2}$	-
*	+	$\frac{L^3}{2 \cdot 3}$	-	$\frac{3L^4}{2 \cdot 3 \cdot 4}$	+	$\frac{6L^5}{2 \cdot 3 \cdot 4 \cdot 5}$	-
*	+	$\frac{L^3}{2 \cdot 3}$	-	$\frac{L^4}{2 \cdot 3}$	+	$\frac{L^5}{2 \cdot 3 \cdot 2}$	-
*	+	$\frac{L^4}{2 \cdot 3 \cdot 4}$	-	$\frac{4L^5}{2 \cdot 3 \cdot 4 \cdot 5}$	+	$\frac{10L^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$	-
*	+	$\frac{L^4}{2 \cdot 3 \cdot 4}$	-	$\frac{L^5}{2 \cdot 3 \cdot 4}$	+	$\frac{L^6}{2 \cdot 3 \cdot 4 \cdot 2}$	-
*	+	$\frac{L^5}{2 \cdot 3 \cdot 4 \cdot 5}$	-	$\frac{5L^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$	+	Ec	-
*	+	$\frac{L^5}{2 \cdot 3 \cdot 4 \cdot 5}$	-	$\frac{L^6}{2 \cdot 3 \cdot 4 \cdot 5}$	+	Ec	-
*	+	$\frac{L^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$	-	Ec	+	Ec	-
*	+	$\frac{L^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$	-	Ec	+	Ec	-

38. This is a very remarkable property of these two serieses, “ that the division of unity by one of them should produce the other, or that the quotient of such division should be a series consisting of exactly the same terms as the divisor, only with different signs prefixed to the terms that fill the even places of the series, or that involve the odd powers of L .” I do not recollect any other such instance of similarity between the divisor and the quotient of a division in all the operations of this kind that I have seen in Algebra.

39. From this similarity between the two serieses $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \mathcal{E}c \text{ ad infinitum}$, and $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \mathcal{E}c \text{ ad infinitum}$, it follows universally, that, whatever be the magnitude of L , the ratio of the series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \mathcal{E}c \text{ ad infinitum}$ (which is the quotient of the foregoing division of unity by the other series) to 1 will be equal to the ratio of 1 to the said other series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \mathcal{E}c \text{ ad infinitum}$. Therefore any quantity denoted by L will be the logarithm of the same ratio in the system of logarithms exhibited by Mr. Mercator's series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c \text{ ad infinitum}$, from which the series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \mathcal{E}c$ was derived, as in the system of logarithms exhibited by Dr. Wallis's series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c \text{ ad infinitum}$, from which the series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \mathcal{E}c$ was derived; or, in other words, the logarithms exhibited by Mr. Mercator's infinite series $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \mathcal{E}c \text{ ad infinitum}$ belong to the same system as the logarithms exhibited by Dr. Wallis's infinite series $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \mathcal{E}c \text{ ad infinitum}$, agreeably to theorem 3, article 83, of the foregoing remarks on those serieses of Mercator and Wallis.

A SCHOLIUM.

40. When L is less than 1, the numerators of the terms of the two serieses $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \mathcal{E}c \text{ ad infinitum}$, and $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \mathcal{E}c \text{ ad infinitum}$, will converge, or decrease, continually; and consequently the whole terms will converge, or decrease, with considerable swiftness, because the denominators of the terms, to wit, 2, 2.3, 2.3.4, $\mathcal{E}c$, increase continually, and thereby

contribute to the decrease of the whole terms. Of this decrease of the terms of these serieses we have had examples above, in articles 13 and 31, in which L was supposed to be $= \frac{1}{2}$, or 0.500,000,000.

And when L is equal to 1, and consequently the numerators of the terms of these two serieses, to wit, 1, L , L^2 , L^3 , L^4 , &c, are all equal to 1, the whole terms of them, to wit, 1, L , $\frac{L^2}{2}$, $\frac{L^3}{2.3}$, $\frac{L^4}{2.3.4}$, &c, or 1, 1, $\frac{1}{2}$, $\frac{1}{2.3}$, $\frac{1}{2.3.4}$, &c after the first term 1, will still converge, or decrease, and with a considerable degree of swiftness, on account of the continual increase of their denominators 2, 2.3, 2.3.4, &c. Of the decrease of the terms of these serieses in this case we have had examples above in articles 16 and 34, in which we have computed the values of these serieses exact to no fewer than 23 places of figures.

But, when L is greater than 1, the numerators of the terms of these two serieses, to wit, 1, L , L^2 , L^3 , L^4 , &c, will continually diverge, or increase, in the proportion of L to 1, and thereby will tend to make the whole terms increase. And the second term L will be greater than the first term 1; and if L be considerably greater than 1, as, for instance, if it be equal to 10, there will be several of the terms in the beginning of the series, besides the second term, which will be greater than the terms that immediately precede them: or the serieses will, for a while, be diverging serieses. Yet such is the tendency of the terms of these serieses to converge, or decrease, arising from the continual increase of their denominators 2, 2.3, 2.3.4, 2.3.4.5, 2.3.4.5.6, 2.3.4.5.6.7, 2.3.4.5.6.7.8, &c, that, whatever be the magnitude of L , and the consequent increase, or divergency, of the numerators 1, L , L^2 , L^3 , L^4 , L^5 , L^6 , L^7 , L^8 , &c, the diminution of the terms arising from the increase of their denominators will, in some part or other of the said serieses, more than counterbalance the said divergency, or increase, of them arising from the increase of their numerators, and make the whole terms, after the term in which such over-balancing first takes place, continually converge, or decrease; and this with an increasing velocity, or more swiftly than in a continued geometrical proportion.

For, since the denominators of the terms of these serieses, to wit, 2, 2.3, 2.3.4, 2.3.4.5, 2.3.4.5.6, 2.3.4.5.6.7, 2.3.4.5.6.7.8, 2.3.4.5.6.7.8.9, 2.3.4.5.6.7.8.9.10, &c, increase by the continual multiplication of the natural numbers 3, 4, 5, 6, 7, 8, 9, 10, &c *ad infinitum*, it follows, that, however great we may suppose L to be, we shall, in some part or other of the series, come to a term of which the denominator will be derived from the denominator of the next preceding term, by the multiplication of a number still greater than L ; and this term will be less than the said next preceding term, because its denominator will be greater than the denominator of the said next preceding term in a greater proportion than that in which its numerator will exceed the numerator of the said preceding term, which is that of L to 1. Thus, for example, if L is equal to 1000, it is evident that the thousand and first term of either of these serieses will have for its numerator the 1000th power of L , or L^{1000} , and for its denominator the product 2.3.4.5.6.7.8.9.10, &c, continued to 1000; and the thousand and second term will have for its numerator the 1001st power of L , or L^{1001} , and for its denominator

nator the product 2.3.4.5.6.7.8.9.10, &c, continued to 1001. Therefore the 1002nd term will be equal to the 1001st term multiplied into the fraction $\frac{L}{1001}$, or $\frac{1000}{1001}$, and consequently will be less than the 1001st term. And, in like manner, the 1003rd term will be derived from the 1002nd term, by multiplying it into $\frac{L}{1002}$, or $\frac{1000}{1002}$; and the 1004th term will be derived from the 1003rd term, by multiplying it into $\frac{L}{1003}$, or $\frac{1000}{1003}$; and every following term will be derived from the term immediately preceding it, by multiplying it into a fraction that is less than $\frac{1000}{1003}$, and also than the multiplying, or generating, fraction next before it. And consequently, after the 1001st term, the several terms of the series will continually decrease, and that with an increasing velocity, or more swiftly than in a continued geometrical proportion. Q. E. D.

41. And hence it follows, that the two serieses $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \text{\textit{\&c ad infinitum}}$, and $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \text{\textit{\&c ad infinitum}}$, will always be of finite magnitudes, whatever be the magnitude of L ; agreeably to what was asserted above in the scholium in article 15. For, when L is of a very great magnitude, as, for example, when it is equal to 1000, the first part of each of these serieses, though it will consist of terms that continually diverge, or increase, yet will be of a finite magnitude, because the number of its terms is finite; and the second part of the series, from the part where it begins to converge, will converge with an increasing degree of velocity, or more swiftly than in a continued geometrical proportion, and therefore will be of a finite magnitude, notwithstanding the infinite number of its terms, because every infinite series of terms that decrease in a continued geometrical proportion is of a finite magnitude. Therefore the sum of the first and second parts, or of the diverging and converging parts, of each of these serieses, that is, the whole of each of these serieses, will be of a finite magnitude. Q. E. D.

42. But these serieses can never, as I conceive, be applied to any useful purpose when L is greater than 1; nor, indeed, will they be of much use in the computation of numbers from the logarithms that belong to them, even when L is less than 1, but nearly equal to it. But when L is much less than 1, as, for example, when it is equal to $\frac{1}{10}$, or any lesser number, these serieses will converge with sufficient swiftness to be of great use in making such computations.

43 In the foregoing corollary to problem 2, we have shewn how to find the ratio corresponding to a given logarithm of Napier's system: it remains that we shew how, when a logarithm of Briggs's system is given, we may find the ratio which corresponds to it. This will be the subject of a second corollary.

Coroll. 2. If B be the logarithm of any ratio in Briggs's system of logarithms, and L be the logarithm of the same ratio in Napier's system, L will be greater than B in the same proportion in which Napier's logarithm of the ratio of 10 to 1 is greater than Briggs's logarithm of the ratio of 10 to 1, that is, in the proportion of 2.302,585,092,994,045,668, &c to 1; or L will be $= 2.302,585,092,994,045,668, \&c \times B$.

Therefore, if B be any logarithm in Briggs's system, and we are required to find the ratio corresponding to it, or of which it is the logarithm, by means of the foregoing series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \&c$ ad infinitum, we must first multiply B by the number 2.302,585,092,994,045,668, &c; whereby we shall obtain the value of L , or of Napier's logarithm of the same ratio: and then we must compute as many terms of the said series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \&c$ ad infinitum as may be necessary to express the ratio we are seeking to the proposed degree of exactness. For the proportion of 1 to the said series, or to so many terms of it as we shall have computed, will be the ratio sought.

An Example of this Method of discovering the Ratio corresponding to a given Logarithm in Briggs's System of Logarithms.

44. Let it be required to find the ratio corresponding to the logarithm $\frac{1}{100}$, or 0.01, in Briggs's system of logarithms.

The logarithm of this ratio in Napier's system of logarithms is $= 2.302,585,092,994,045,668, \&c \times 0.01 = 0.023,025,850,929,940,456,68, \&c$, or (neglecting the last eleven figures, as inconsiderable) 0.023,025,850. Call this logarithm L .

Then will L^2 be $= 0.023,025,850^2 = 0.000,530,189$, and L^3 will be $= 0.000,012,208$, and L^4 will be $= 0.000,000,281$, and L^5 will be $= 0.000,000,006$. Therefore $\frac{L^2}{2}$ will be $= \frac{0.000,530,189}{2} = 0.000,265,094$, and $\frac{L^3}{2.3}$ will be $= \frac{0.000,012,208}{6} = 0.000,002,034$, and $\frac{L^4}{2.3.4}$ will be $= \frac{0.000,000,281}{24} = 0.000,000,011$; and $\frac{L^5}{2.3.4.5}$ will be $= \frac{0.000,000,006}{120} = 0.000,000,000$, and consequently the series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \&c$ will be $= 1.000,000,000 - 0.023,025,850 + 0.000,265,094 - 0.000,002,034 + 0.000,000,011 - 0.000,000,000 + \&c = 1.000,265,105 - 0.023,027,884 = 0.977,237,221$. Therefore the ratio corresponding to L , or 0.023,025,850, or of which 0.023,025,850 is the logarithm in Napier's system of logarithms, or of which 0.01, or $\frac{1}{100}$, is the logarithm in Briggs's system of logarithms, is the ratio of 1 to 0.977,237,221.

Q. E. I.

45. This

45. This ratio of 1 to 0.977,237,221, which is here found by means of the infinite series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \mathcal{E}c$ to be the ratio of which 0.01, or $\frac{1}{100}$, is the logarithm in Briggs's system, is equal to the ratio of 1.023,292,989, $\mathcal{E}c$ to 1, which was found above in article 18, by means of the infinite series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \mathcal{E}c$, to be the ratio corresponding to the same logarithm. For 1.023,292,989 is to 1 as 1 is to $\frac{1.000,000,000,000, \mathcal{E}c}{1.023,292,989} = 0.977,237,224$; which agrees with the number 0.977,237,221 in the first eight figures, and exceeds it by only 0.000,000,003, or $\frac{3}{1000,000,000}$, or three thousand-millionth parts of an unit.

Another Example of this Method of discovering the Ratio corresponding to a given Logarithm in Briggs's System of Logarithms.

46. Suppose it were required to find the 365th root of $1 + \frac{1}{20}$, or $1 + \frac{5}{100}$, or 1.05, the logarithm of the ratio of 1.05 to 1 in Briggs's system being given.

The logarithm of the ratio of 1.05 to 1 in Briggs's system is $= \log. \frac{105}{100} = \log. \frac{21}{20} = \log. \frac{21}{1} - \log. \frac{20}{1} =$ (according to the computations in the foregoing remarks, article 98) $1.322,219,294,733,919,356 - 1.301,029,995,663,981,308 = 0.021,189,299,069,938,048$. Therefore the logarithm of the ratio of $\sqrt[365]{1.05}$, or $\sqrt[365]{1.05}^{\frac{1}{365}}$, to 1, will be $= \frac{0.021,189,299,069,938,048}{365} = 0.000,058,052,874,164,213$. This is Briggs's logarithm of the ratio of $\sqrt[365]{1.05}^{\frac{1}{365}}$ to 1, from which we are now to discover that ratio itself, or the value of $\sqrt[365]{1.05}^{\frac{1}{365}}$.

Now, since Briggs's logarithm of this ratio is 0.000,058,052,874,164,213, or (neglecting the last nine figures, as inconsiderable,) 0.000,058,052, it follows that Napier's logarithm of the same ratio will be $= 2.302,585,092, \mathcal{E}c \times 0.000,058,052 = 0.000,133,669, \mathcal{E}c$; that is, L will be $= 0.000,133,669, \mathcal{E}c$. We shall therefore have $L^2 (= 0.000,133,669^2) = 0.000,000,017$, and $L^3 (= L^2 \times L = 0.000,000,017 \times 0.000,133,669) = 0.000,000,000$; and consequently $\frac{L^2}{2} (= \frac{0.000,000,017}{2}) = 0.000,000,008$, and $L - \frac{L^2}{2} (= 0.000,133,669 - 0.000,000,008) = 0.000,133,661$, and $1 - L + \frac{L^2}{2} = 1.000,000,000 - 0.000,133,661 = 0.999,866,339$; that is, the series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \mathcal{E}c$ will be $= 0.999,866,339$. Therefore the ratio

corre-

corresponding to L , or $0.000,133,669$, $\mathcal{E}c$, or of which $0.000,133,669$, $\mathcal{E}c$ is the logarithm in Napier's system of logarithms, or of which $0.000,058,052$, $\mathcal{E}c$ is the logarithm in Briggs's system of logarithms, is the ratio of 1 to $0.999,866,339$; and consequently $\sqrt[365]{1.05}$ is to 1 as 1 is to $0.999,866,339$, and $\sqrt[365]{1.05}$ is $= \frac{1.000,000,000}{0.999,866,339} = 1.000,133,678$, or the 365^{th} root of 1.05 is $1.000,133,678$.

Q. E. I.

47. The ratio corresponding to L is equal to that of the series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \mathcal{E}c$ to 1 as well as to the ratio of 1 to the series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \mathcal{E}c$. Now the series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \mathcal{E}c$, is, in this case, equal to $1 + 0.000,133,669 + 0.000,000,008 + \mathcal{E}c = 1.000,133,677$, $\mathcal{E}c$. Therefore the ratio corresponding to L , or $0.000,133,669$, $\mathcal{E}c$, or of which $0.000,133,669$, $\mathcal{E}c$ is the logarithm in Napier's system of logarithms, or of which $0.000,058,052$, $\mathcal{E}c$ is the logarithm in Briggs's system of logarithms, is the ratio of $1.000,133,677$, $\mathcal{E}c$ to 1; and consequently the ratio of $\sqrt[365]{1.05}$ to 1 is equal to the ratio of $1.000,133,677$, $\mathcal{E}c$ to 1, and therefore $\sqrt[365]{1.05}$ is $= 1.000,133,677$, $\mathcal{E}c$.

Q. E. I.

This value of $\sqrt[365]{1.05}$ agrees with the former value of it, to wit, $1.000,133,678$, (which was obtained by means of the series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \mathcal{E}c$), in all the figures but the last, which differs from the last figure of the said former value only by an unit.

48. This number $1.000,133,677$, which is equal to the 365^{th} root of 1.05, is the quantity to which one pound sterling, or any other original sum of money denoted by 1, would increase by being lent out at interest for the space of only one day, if the terms of the loan were such that the money was to be paid back to the lender every night, with the interest due upon it, and then immediately to be lent out again, together with the interest due upon it added to it, and yet that the whole interest that should be gained upon it in the course of a whole year, or 365 days, by these 365 repeated loans, should amount only to one twentieth part, or five hundredth parts, of one pound, or the other original sum, whatever it was, which was denoted by 1. It is a very little less than $1 + \frac{.05}{365}$, or $1 + 0.000,136,986$, or $1.000,136,986$, to which the sum 1 would increase in the space of one day, if the interest to be paid for it for one day was the 365^{th} part of $\frac{5}{100}$, or of the interest paid for it for a whole year, when lent for a year at a time, at the interest of 5 per cent. per annum.

49. Before I put an end to this appendix, I think it necessary to declare, that the investigations of the two anti-logarithmick infinite serieses $L + \frac{L^2}{2} + \frac{L^3}{2 \cdot 3} + \frac{L^4}{2 \cdot 3 \cdot 4} + \frac{L^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{L^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \mathcal{E}c$, and $L - \frac{L^2}{2} + \frac{L^3}{2 \cdot 3} - \frac{L^4}{2 \cdot 3 \cdot 4} + \frac{L^5}{2 \cdot 3 \cdot 4 \cdot 5} - \frac{L^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \mathcal{E}c$, which have been given above in the preceding lemmas and problems, were suggested to me by reading Dr. Halley's discourse on logarithms above-mentioned, and that I therefore suppose they are in substance the same with his investigations of the same serieses contained, or, rather, hinted at, in the said discourse. But this I am not quite sure of, because, from the extreme conciseness with which he has there treated the subject, I find myself unable to understand him so thoroughly as I could wish. However, whether the investigations which I have here given of these serieses are the same with his or not, I hope they will be found sufficient to establish his conclusions, and that they have been explained in such a manner as to give my readers as little trouble as possible in understanding them.

THE END OF VOL. I.



Maseres, Francis, 1731-1824

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